STABILITY OF THE HULL(S) OF THE N-SPHERE

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A dissertation submitted to the

School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Xiaojun Huang

And approved by

New Brunswick, New Jersey

May, 2020

ABSTRACT OF THE DISSERTATION

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For a particular natural embedding of the real n-sphere in $\mathbb{C}^n$, the CR singularities are elliptic and nondegenerate and form an $(n-2)$-sphere on the equator. In particular, for $n \geq 3$, these singularities are non-isolated. This distinguishes the difficulty of this problem from the well-studied case of $n = 2$. It can easily be seen that the n-sphere can be filled by an $(n-1)$-parameter family of attached holomorphic discs foliating towards the singularities. This family of discs forms a real $(n+1)$-dimensional ball, which is the holomorphic and polynomial hull of the n-sphere. This dissertation investigates whether these properties are stable under C^3-small perturbations and what regularity can be expected from the resulting manifold. We find that under such perturbations, the local and global structure of the set of singularities remains the same. We then solve a Riemann-Hilbert problem, modifying a construction by Alexander, to obtain an $(n-1)$-parameter family of holomorphic discs attached to the perturbed sphere, away from the set of singularities. We then use the theory of multi-indices for attached holomorphic discs and nonlinear functional analysis to study the regularity of the resulting manifold. We find that in the case that the perturbation is $C^{k+2,\alpha}$, the construction yields a $C^{k,\alpha}$ manifold. In the case that the perturbation is $\mathcal{C}^\infty$ smooth or real analytic we show that the regularity of the manifold matches the regularity of the perturbation. We then patch this construction with small discs constructed by Kënig, Webster, and Huang near nondegenerate elliptic singularities to obtain a complete filling of the perturbed sphere by attached holomorphic discs, with an additional loss of regularity near the CR singularities. This filled sphere is diffeomorphic to the $(n+1)$-dimensional ball and is clearly contained in the hull of holomorphy. Finally, we show that if the perturbation is real analytic and admits a uniform lower bound on its radius of convergence, this perturbed ball is in fact exactly the polynomial (and holomorphic) hull of the perturbed sphere.

Acknowledgements

This thesis represents joint work with Purvi Gupta, which has been submitted for publication.

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Chapter 1

Introduction

1.1 Motivation and statement of result

Given a prescribed surface S in $\mathbb{C}^2$, the problem of finding a hypersurface M in $\mathbb{C}^2$ such that $\partial M = S$ and M is *Levi-flat*, i.e., locally foliated by analytic curves, is called the *Levi-flat plateau problem.* This problem has been extensively studied for 2-spheres in $\mathbb{C}^2$. The first foundational result in this direction was due to Bishop (in [6]), whose construction of discs produces a local solution near any nondegenerate elliptic complex point of S. The regularity (up to the boundary) and the uniqueness of Bishop's local solution were settled much later in the works of Kenig-Webster ([14]; the smooth case), and Moser-Webster, Moser, and Huang-Krantz ([17], [16] and [13]; the real-analytic case).

The global problem for spheres was studied in a series of papers in the 80's and 90's, starting with Bedford-Gaveau (see [4]), who proved the existence and uniqueness of the global solution for S in graph form (and with 2 elliptic complex points), followed by Bedford ([3]), who established the stability of the solution. The CR-geometric and regularity conditions on S were later substantially weakened via geometric methods, as in the works of Bedford-Klingenberg ([5]) and Chirka-Shcherbina ([8]), as well as via PDE techniques, as in the work of Slodkowski-Tomassini ([19]). In the case when S is either $\mathcal{C}^\infty$-smooth or real-analytic, and has only elliptic complex points, the regularity of the global solution follows from the local results discussed above.

We note that, in all the results cited above, the uniqueness of the solution M follows from the fact that M is the envelope of holomorphy (and, in some cases, the polynomially convex hull) of S. Given a compact set $K \subset \mathbb{C}^n$, its *polynomially convex hull* is the set $\widehat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \sup_{x\in K}|p(x)|\,,\ \text{for all holomorphic polynomials } p\}$. In terms of function algebras, $\widehat{K}$ can be identified with the maximal ideal space of $\mathcal{P}(K)$ — the closure in $\mathcal{C}(K)$ of the set of holomorphic polynomials on $\mathbb{C}^n$ restricted to K. The *envelope of holomorphy* $\widetilde{K}$ of K, can be analogously defined as the maximal ideal space of $\mathcal{H}(K)$ — the closure in $\mathcal{C}(K)$ of the set $\mathcal{O}(K) = \{f|_K : f$ is holomorphic in some neighborhood of $K\}$. When $\widetilde{K}$ is *schlicht*, i.e., representable as a subset in $\mathbb{C}^n$, it is the maximal set in $\mathbb{C}^n$ such that every $f \in \mathcal{O}(K)$ analytically extends to some $\widetilde{f} \in \mathcal{O}(\widetilde{K})$. It is of fundamental interest in complex analysis to determine these hulls for any given set, and study their analytic structures. Furthermore, if $\widetilde{K} = \widehat{K}$, then $\mathcal{H}(K) = \mathcal{P}(K)$. In $\mathbb{C}$, this equality holds if (and only if) K is simply-connected, but this is far from true in $\mathbb{C}^n$, $n \geq 2$.

In higher dimensions ($n \geq 3$), the corresponding problem for n-spheres in $\mathbb{C}^n$ is not as well understood. The Levi-Flat plateau problem has been studied in $\mathbb{C}^n$, $n \geq 3$, but all the known results consider boundaries that are $(2n-2)$-dimensional spheres in $\mathbb{C}^n$. From the point of view of computing polynomial hulls, it is more natural to consider n-dimensional manifolds in $\mathbb{C}^n$. In this setting, part of the challenge stems from the fact that the CR-singularities of such a manifold are not generically isolated when $n \geq 3$. Moreover, even when a "filling" by attached analytic discs is possible, the resulting manifold has high codimension, thus making it hard to establish its holomorphic or polynomial convexity. Thus, there is a lack of global results, even for n-spheres in $\mathbb{C}^n$.

In this paper, we study the hulls of small perturbations of the following natural embedding of the n-sphere in $\mathbb{C}^n$.

$$S^n = \left\{(z, z') \in \mathbb{C} \times \mathbb{C}^{n-1} : |z|^2 + \left\|z'\right\|^2 = 1,\ \mathrm{Im}\, z' = 0\right\}.$$

We let $\mathbf{B}^{n+1}$ denote the $(n+1)$-ball bound by S^n in $\mathbb{C} \times \mathbb{R}^{n-1}$, and note that $\mathbf{B}^{n+1}$ is both the envelope of holomorphy and the polynomially convex hull of S^n, and is trivially foliated by analytic discs. We establish the following stability result in the style of Bedford ([2]) and Alexander ([1]).

Theorem 1.1. *Let $\rho > 0$ and $\delta > 0$. Then, there is an $\varepsilon > 0$ such that, for $k >> 1$, if $\psi \in \mathcal{C}^{3k+7}(S^n; \mathbb{C}^n)$ with $\|\psi\|_{\mathcal{C}^4(S^n;\mathbb{C}^n)} < \varepsilon$, then there is a $\mathcal{C}^k$-smooth $(n+1)$-dimensional* *submanifold with boundary,* $M \subset \mathbb{C}^n$*, such that*

1. $\partial M = \Psi(S^n)$*, where* $\Psi = \mathbf{I} + \psi$ *on* S^n.

2. M *is foliated by an* $(n-1)$*-parameter family of embedded analytic discs attached to* $\Psi(S^n)$.

3. *There is a* $\mathcal{C}^k$*-smooth diffeomorphism* $j : \mathbf{B}^{n+1} \to M$ *with* $\|j - \mathbf{I}\|_{\mathcal{C}^2(\mathbf{B}^{n+1};\mathbb{C}^n)} < \delta$.

4. *If* ψ *is* $\mathcal{C}^\infty$*-smooth, then* M *is* $\mathcal{C}^\infty$*-smooth up to its boundary.*

5. *If* ψ *is real-analytic, then* M *is real-analytic up to its boundary.*

6. *If* ψ *is real-analytic and the complexified map* $\psi_{\mathbb{C}}$ *extends holomorphically to*

$$\mathcal{N}_\rho S^n_{\mathbb{C}} = \{\xi \in \mathbb{C}^{2n} : \operatorname{dist}(\xi, S^n_{\mathbb{C}}) < \rho\},$$

where $S^n_{\mathbb{C}} = \{(z, \overline{z}) \in \mathbb{C}^{2n} : z \in S^n\}$*, and* $\sup_{\overline{\mathcal{N}_r S^n_{\mathbb{C}}}} |\psi_{\mathbb{C}}| < \varepsilon$*, then* $M = \widetilde{\Psi(S^n)} = \widehat{\Psi(S^n)}$.

In order to construct M, we need to consider the CR-structure of $\Psi(S^n)$. First, we note that the set of CR-singularities of $\Psi(S^n)$ forms an $(n-2)$-sphere consisting only of *nondegenerate elliptic* CR-singularities (see Lemma 2.1). A point p in an n-manifold $X \subset \mathbb{C}^n$ is a nondegenerate elliptic CR-singularity of X if, after a local holomorphic change of coordinates, X near $p = 0$ is given by

$$\begin{aligned} z_n &= |z_1|^2 + 2\lambda \operatorname{Re}(z_1^2) + O(|z|^3); \\ y_j &= O(|z|^3), \quad j = 2, ..., n-1, \end{aligned}$$

where $\lambda \in [0, \frac{1}{2})$. The local hull of holomorphy of a smooth (real-analytic) X at such a p is a smooth (real analytic) $(n+1)$-dimensional manifold that is foliated by Bishop discs attached to X. As discussed earlier, when $n = 2$, this follows from the works of Bishop, Kenig-Webster, Moser-Webster, Moser and Huang-Krantz. In higher dimensions, this problem was settled by Kenig-Webster ([15]) and Huang ([20]) (see Theorem 4.2).

Away from the set of CR-singularities of $\Psi(S^n)$, we solve a Riemann-Hilbert problem to produce the necessary attached discs. We note that such a construction was done by Alexander in [1], and his technique can be used to show that for any k large enough, there is an $\varepsilon_k > 0$ such that every ε_k-small $\mathcal{C}^{k+2}$-perturbation of S^n contains the boundary of a $\mathcal{C}^k$-smooth manifold foliated by attached holomorphic discs. However, ε_k may shrink to zero as k increases, and thus we need a different approach for $\mathcal{C}^\infty$-smooth perturbations. For this, we fix a sufficiently small perturbation Ψ, construct the ($\mathcal{C}^1$-smooth) foliation attached to $\Psi(S^n)$ à la Alexander, and then, use the Forstnerič-Globevnik theory ([10], [11]) of multi-indices of discs attached to totally real manifolds to smoothly reparametrize the foliation near each leaf.

Finally, to establish the polynomial convexity of M, we globally flatten M to a domain in $\mathbb{C} \times \mathbb{R}^{n-1}$, and use a trick due to Bedford for Levi-flat graphs of hypersurface type. In order to carry out this flattening, we must assume that our perturbation is real-analytic with a uniformly bounded below radius of convergence on S^n. Hence, the assumptions stated in (6) in Theorem 1.1. It is not clear whether these assumptions can be done away with.

1.2 Plan of the Thesis

The proof of our main result is organized as follows. In Chapter 2, we discuss the CR structure of the perturbed sphere, including the local and global structure of its singularities. In Chapter 3, we establish the stability of the holomorphic discs whose boundaries in S^n lie outside a neighborhood of its CR-singularities and in its subsections 3.3 and 3.4, we show that the regularity is maintained in the real analytic and $\mathcal{C}^\infty$ cases, respectively. Next, in Chapter 4, we complete the proof of claims (1) to (5) in Theorem 1.1 by patching up the construction in Chapter 3 with the local hulls of holomorphy of the perturbed sphere near its CR-singularities. Finally, in Chapter 5, we establish the polynomial convexity of the constructed manifold under the stated assumptions.

1.3 Notation and Setup

We will use the following notation throughout this paper.

- The unit disc and its boundary in $\mathbb{C}$ are denoted by Δ and $\partial\Delta$, respectively.
- The open Euclidean ball centered at the origin and of radius $r > 0$ in $\mathbb{R}^k$ is denoted by $D^k(r)$.
- Bold small letters such as $\mathbf{t}$ and $\mathbf{s}$ denote vectors in $\mathbb{R}^{n-1}$. For the sake of convenience, we index the components of these vectors from 2 to n, i.e., $\mathbf{t} = (t_2, ..., t_n)$.
- We will denote the identity map by $\mathbf{I}$, where the domain will depend on the context.
- Given any normed function space $\left(\mathcal{F}(K), \|\cdot\|_{\mathcal{F}}\right)$ on a set $K \subset \mathbb{C}^n$, we let
  - $\mathcal{F}(K;\mathbb{R}) = \{f \in \mathcal{F}(K) : f \text{ is } \mathbb{R}\text{-valued}\}$, with the same norm.
  - $\mathcal{F}(K;\mathbb{R}^n) = \{(f_1, ..., f_n) : K \to \mathbb{R}^n : f_j \in \mathcal{F}(K;\mathbb{R})\}$, with $\|(f_j)\|_{\mathcal{F}} = \sup_j \|f_j\|_{\mathcal{F}}$.
  - $\mathcal{F}(K;\mathbb{C}^n) = \{(f_1, ..., f_n) : K \to \mathbb{C}^n : f_j \in \mathcal{F}(K)\}$, with $\|(f_j)\|_{\mathcal{F}} = \sup_j \|f_j\|_{\mathcal{F}}$.
- For any n-dimensional submanifold $M \subset \mathbb{C}^n$, we denote the set of CR-singularities of M by $\mathrm{Sing}(M)$.

We now make some preliminary observations on the perturbations considered in this article. Let $\mathcal{B}_3$ denote an ε-neighborhood of the origin in $\mathcal{C}^3(S^n;\mathbb{C}^n)$, where $\varepsilon > 0$ will be determined later on. We let $K_s = \{z \in \mathbb{C}^n : \mathrm{dist}(z, S^n) < s\}$, where $s > 0$ is small enough so that there is a smooth retraction ρ of K_s to S^n. We may choose an $\varepsilon > 0$ small enough so that

- there is a $t \in (0, s)$ such that for every $\psi \in \mathcal{B}_3$, the diffeomorphism $\Psi : K_s \to \mathbb{C}^n$ given by $z \mapsto z + \psi(\rho(z))$ satisfies $\Psi(S^n) \subset K_t \subset \Psi(K_s)$; and
- the map $\mathrm{Inv} : \mathcal{B}_3 \to \mathcal{C}^3(K_t;\mathbb{C}^n)$ given by $\psi \mapsto (\Psi^{-1} - \mathbf{I})|_{K_t}$ is well-defined and $\mathcal{C}^2$-smooth.

We denote $\Psi^{-1}|_{K_t}$ by Φ and $\mathrm{Inv}(\psi) = \Phi - \mathbf{I}$ by ϕ. For $\phi \in \mathrm{Inv}(\mathcal{B}_3)$, we let

$$S^n_\phi = \Psi(S^n),$$

where the $\phi = \mathrm{Inv}(\psi)$. Thus, $z \in K = K_t$ satisfies $z \in S^n_\phi$ if and only if $z - \phi(z) \in S^n$.

Chapter 2

CR Structure of Perturbed Sphere

2.1 CR Dimension and Singularities

Let X be an m-dimensional manifold embedded in $\mathbb{C}^n$, defined by

$$X = \{z \in \mathbb{C}^n : \rho_1(z) = \cdots = \rho_m(z) = 0\}$$

where $\rho_j : \mathbb{C}^n \to \mathbb{R}$ are at least $\mathcal{C}^1$ and $d\rho_1 \wedge \cdots \wedge d\rho_m \neq 0$ on X. Note that the ideas in this section can be generalized to non-embedable manifolds, but these generalizations are not needed here.

Definition 2.1. Let $p \in X$. Let T_pX denote the tangent space to X at p. Then, define the *CR dimension* of X at p, which we denote $CR\dim_p X$, to be the (complex) dimension of the largest complex subspace of T_pX.

We say that a point $p \in X$ is in the *totally real part* of X if $CR\dim_p X = 0$.

Example 2.2. Here are four illustrative examples.

1. Let $X = \{z \in \mathbb{C}^n : \rho(z) = 0\}$ be any real hypersurface in $\mathbb{C}^n$. Then, for all $p \in X$, $CR\dim_p X = n - 1$.

2. Let $X = \{(z_1, z_2) \in \mathbb{C}^2 : x_2 = y_2 = 0\}$. Then, $T_pX = \operatorname{span}_{\mathbb{R}}\{(0, 1), (0, i)\} = \{0\} \times \mathbb{C}$. So, for all $p \in X$, $CR\dim_p X = 1$.

3. Let $X = \{(z_1, z_2) \in \mathbb{C}^2 : x_1 = x_2 = 0\}$. Then, $T_pX = \operatorname{span}_{\mathbb{R}}\{(1, 0), (0, 1)\} = \mathbb{R} \times \mathbb{R}$. So, for all $p \in X$, $CR\dim_p X = 0$. In other words, X is totally real.

4. Consider the embedding $S^2 = \{(z_1, z_2) \in \mathbb{C}^2 : y_2 = 0, |z_1|^2 + |z_2|^2 = 1\}$. Then, $T_pS^2 = \operatorname{span}_{\mathbb{R}}\{(\overline{p_1}, \overline{p_2}), (0, i)\}$. Then, for $p \neq (\pm 1, 0)$, $CR\dim_p S^2 = 0$ and $CR\dim_{(\pm 1,0)} S^2 = 1$. In other words, S^2 is totally real except at the poles $(\pm 1, 0)$.

Definition 2.3. In the last example above, the two points $(\pm 1, 0)$ where the CR dimension jumps are called *CR singularities* and we denote the set of CR singularities for a manifold X by $\mathrm{Sing}(X)$.

An important example to this document is the real n-sphere in $\mathbb{C}^n$:

$$S^n = \{z \in \mathbb{C}^n : \|z\|^2 = 1, y_2 = y_3 = \cdots = y_n = 0\}.$$

Similar to the 2 dimensional version, one can easily compute that $\mathrm{Sing}(S^n) = \{z \in S^n : z_1 = 0\}$, which is an $n-2$-dimensional sphere. In fact, this property is stable under $\mathcal{C}^3$-small perturbations, as shown in the following

Given $\eta > 0$, there is a $\tau \in (0,1)$ such that for any $\psi \in \tau\mathcal{B}_3$, there exists a $\mathcal{C}^2$-smooth diffeomorphism $\iota : S^n \to S^n$ such that $(\Psi \circ \iota)(\mathrm{Sing} S^n) = \mathrm{Sing}(S^n_\phi)$, and $||\Psi \circ \iota - \mathbf{I}||_{\mathcal{C}^2(S^n;\mathbb{C}^n)} < \eta$. In particular, $\mathrm{Sing}(S^n_\phi)$ is an $(n-2)$-dimensional sphere.

Proof. We first parametrize S^n by $\Theta : \overline{D^2(1)} \times [0, 2\pi]^{n-2} \to \mathbb{C}^n$ as follows

$$\Theta : (a, b, \theta_1, ..., \theta_{n-2}) \mapsto \left(a + ib, \mathfrak{S}\left(\sqrt{1 - a^2 - b^2}, \theta_1, ..., \theta_{n-2}\right)\right),$$

where $\mathfrak{S}(r, \theta_1, ..., \theta_{n-2})$ is a point in $\mathbb{R}^{n-1}$ with spherical coordinates $r, \theta_1, ..., \theta_{n-2}$. Note that $\Theta^{-1}(\mathrm{Sing} S^n) = \{(0, \boldsymbol{\theta}) : \boldsymbol{\theta} \in [0, 2\pi]^{n-2}\}$.

Let $R : \mathbb{C}^n \to \mathbb{R}^n$ be given by $(z_1, z_2, ..., z_n) \mapsto \big(|z_1|^2 + \cdots + |z_n|^2 - 1, \mathrm{Im}(z_2), ..., \mathrm{Im}(z_n)\big)$. We note that since $\mathrm{Sing}(S^n) = \{z \in S^n : \mathrm{rank}\, \mathrm{Jac}_{\mathbb{C}}\, R(z) < n\}$, and

$$\mathrm{Jac}_{\mathbb{C}}\, R(z_1, ..., z_n) = \begin{pmatrix} \overline{z_1} & \overline{z_2} & \cdots & \overline{z_n} \\ 0 & \frac{1}{2i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \frac{1}{2i} \end{pmatrix},$$

we have that $\mathrm{Sing}(S^n) = \{z \in S^n : \det \mathrm{Jac}_{\mathbb{C}}\, R(z) = 0\}$. Now, let $J : \mathcal{B}_3 \times \overline{D^2(1)} \times [0, 2\pi]^{n-2} \to \mathbb{R}^2$ be given by $(\psi, a, b, \boldsymbol{\theta}) \mapsto \det \mathrm{Jac}_{\mathbb{C}}(R \circ \Phi)(\Psi \circ \Theta(a, b, \boldsymbol{\theta}))$, where $\boldsymbol{\theta} = (\theta_1, ..., \theta_{n-2})$, and Ψ and Φ are related to ψ as discussed above. Note that J is a $\mathcal{C}^2$-smooth map such that

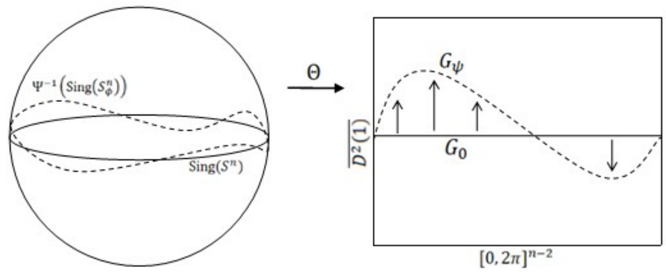

Figure 2.1: The preimage of $\mathrm{Sing}(S^n_\phi)$ under Ψ in the parameter space is a graph.

- $\Theta(a, b, \boldsymbol{\theta}) \in \Psi^{-1}(\mathrm{Sing}(S^n_\phi))$ if and only if $J(\psi, a, b, \boldsymbol{\theta}) = 0$ (after possibly shrinking $\mathcal{B}_3$);
- For any $\boldsymbol{\theta} \in [0, 2\pi]^{n-2}$, $J(0, 0, \boldsymbol{\theta}) = 0$ and $D_{a,b}J(0, 0, \boldsymbol{\theta}) = (\frac{1}{2i})^{n-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Thus, by the implicit function theorem (and the compactness of $[0, 2\pi]^{n-2}$), there is a $\tau \in (0, 1)$, a neighborhood U of 0 in $\mathbb{C}$, and a $\mathcal{C}^2$-smooth map $\Gamma : \tau\mathcal{B}_3 \times [0, 2\pi]^{n-2} \to \mathbb{C}$ such that $J(\psi, z_1, \boldsymbol{\theta}) = 0$ if and only if $z_1 = \Gamma(\psi, \boldsymbol{\theta})$, for any $(\psi, z_1, \boldsymbol{\theta}) \in \tau\mathcal{B}_3 \times U \times [0, 2\pi]^{n-2}$.

Thus, in the parameter space $\overline{D^2(1)} \times [0, 2\pi]^{n-2}$, $\Psi^{-1}(\mathrm{Sing}(S^n_\phi))$ pulls back to the $\mathcal{C}^2$-smooth graph $G_\psi = (\Gamma(\psi, \boldsymbol{\theta}), \boldsymbol{\theta})$. By shrinking τ further, we may assume that G_ψ lies in a thin neighborhood N of G_0. As both G_0 and G_ψ are graphs over $[0, 2\pi]^{n-2}$, there is a diffeomorphism $\tilde{\iota}$ of $\overline{D^2(1)} \times [0, 2\pi]^{n-2}$ that is $\mathcal{C}^2$-close to identity, maps G_0 to G_ψ and is identity outside N. Setting $\iota = \Theta \circ \tilde{\iota} \circ \Theta^{-1}$, we obtain the necessary map. $\square$

2.2 Moser Webster Normal Form and Elliptic Singularities

For a point on a real n manifold X in $\mathbb{C}^n$ which is at least $\mathcal{C}^3$ smooth, we consider a point $p \in X$ with $CR\dim_p X = 1$. At this point, because the tangent space contains a 1-dimensional complex subspace, after linear transformation which sends p to 0, a relabeling of the variables, and applying implicit function theorem, we have X defined locally near 0 as a graph of $(z_1, \overline{z_1}, x_1)$.

In other words, X is defined near 0 as

$$z_n = F(z_1, \overline{z_1}, x)$$
$$y_\alpha = f_\alpha(z_1, \overline{z_1}, x) = \overline{f_\alpha(z_1, \overline{z_1}, x)} \qquad \alpha = 2, \ldots, n-1$$

where

$$F = q + q_1 + q_2 + O(|z|^3)$$
$$f_\alpha = q_\alpha + q_{1\alpha} + q_{2\alpha} + O(|z|^3) \qquad 2 \le \alpha \le n-1$$

and

$$q = az_1^2 + bz_1\overline{z_1} + c\overline{z_1}^2$$
$$q_1 = \sum_\ell x_\ell(a_\ell z_1 + b_\ell \overline{z_1})$$
$$q_2 = \sum_{\ell,k} c_{\ell k} x_\ell x_k$$
$$q_\alpha = a_\alpha z_1^2 + b_\alpha z_1\overline{z_1} + \overline{a_\alpha z_1}^2$$
$$q_{1\alpha} = 2Re \sum_\ell c_{\alpha\ell} x_\ell z_1$$
$$q_2 = \sum_{\ell,k} c_{\alpha\ell k} x_\ell x_k$$

where $b_\alpha, c_{\alpha\ell k} \in \mathbb{R}$. Note that in this section, β, ℓ, and α will always range from 2 to $n-1$.

Definition 2.4. We say that a singularity is *nondegenerate* if, after the above transformation, we have $b \neq 0$.

In [17], Moser and Webster constructed a biholomorphic change of coordinates which reduce the functions F and f_α above to a useful form, which is often referred to as the *Moser Webster normal form.* In particular, they proved the following

Theorem 2.5. *Let X be a real n-dimensional $\mathcal{C}^3$-smooth manifold in $\mathbb{C}^n$ such that $0 \in X$ is a CR singularity with $CR \dim_p X = 1$. Then, after a holomorphic change of coordinates, X can be written locally near 0 as a graph*

$$z_n = \gamma(z_1^2 + \overline{z_1}^2) + z_1\overline{z_1} + O(|z|^3)$$

$$y_\alpha = O(|z|^3) \qquad\qquad 2 \le \alpha \le n-1$$

where $0 \le \gamma < \infty$*.*

Proof. We start with the definitions of F, f_α as above. As we are assuming that the singularity is nondegenerate, we can replace $z_n \mapsto \frac{1}{b}(z_n - (a-c)z_1^2)$. This only affects q, which becomes

$$q = \gamma(z_1^2 + \overline{z_1}^2) + z_1\overline{z_1}$$

where $\gamma = a = c$. Now, replace $z_1 \mapsto z_1 + \sum A_\beta z_\beta$ where A_β solves

$$2\gamma\overline{A_\beta} + A_\beta = -b_\beta.$$

At this point, we assume $\gamma \neq \frac{1}{2}$. Now, we get

$$\begin{aligned}
q &= \gamma\left(z_1^2 + 2z_1\sum A_\beta z_\beta + (\sum A_\beta z_\beta)^2 + \overline{z_1}^2 + 2\overline{z_1}\sum \overline{Z_\beta}\overline{z_\beta} + (\sum \overline{A_\beta}\overline{z_\beta})^2\right)\\
&+ z_1\overline{z_1} + z_1\sum \overline{a_\beta z_\beta} + \overline{z_1}\sum A_\beta z_\beta + (\sum A_\beta z_\beta)(\sum \overline{z_\beta z_\beta})\\
&= z_1\left(-\sum \overline{b_\beta}x_\beta + i(2\gamma\sum A_\beta y_\beta - \sum \overline{A_\beta}y_\beta)\right)\\
&+ \overline{z_1}\left(-\sum b_\beta x_\beta + i(-2\gamma\sum \overline{A_\beta}y_\beta + \sum A_\beta y_\beta)\right)\\
&+ \gamma\left(z_1^2 + (\sum A_\beta z_\beta)^2 + \overline{a_1}^2 + (\sum \overline{a_\beta z_\beta})^2\right) + z_1\overline{z_1} + (\sum A_\beta z_\beta)(\sum \overline{A_\beta}\overline{z_\beta}),
\end{aligned}$$

$$q_1 = \sum_\ell a_\ell x_\ell z_1 + \sum_{\ell,\beta} a_\ell x_\ell A_\beta z_\beta + \sum_\ell b_\ell x_\ell \overline{z_1} + \sum_{\ell,\beta} b_\ell x_\ell \overline{A_\beta}\overline{z_\beta},$$

and q_2 remains unchanged. Therefore, collecting and cancelling terms appropriately, we have

$$\begin{aligned}
F &= \sum(a_\ell - \overline{b_\ell})x_\ell z_1 + i(2\gamma\sum A_\beta y_\beta - \sum\overline{A_\beta}y_\beta)z_1 + i(\sum A_\beta y_\beta - 2\gamma\sum\overline{A_\beta}y_\beta)\overline{z_1}\\
&+ \gamma(\overline{z_1}^2 + (\sum A_\beta z_\beta)^2 + \overline{z_1}^2 + (\sum \overline{Z_\beta}\overline{z_\beta})^2) + z_1\overline{z_1} + (\sum A_\beta z_\beta)(\sum\overline{A_\beta}\overline{z_\beta})\\
&+ \sum_{\ell,\beta}(a_\ell A_\beta z_\beta + b_\ell\overline{A_\beta}\overline{z_\beta})x_\ell + \sum_{\ell,k} c_{\ell k}x_\ell x_k + O(3)
\end{aligned}$$

Now, replace $z_n \mapsto z_n - \sum(z_\ell z_\ell z_1 + c_{\ell\beta} z_\ell z_\beta)$, so F becomes

$$
\begin{aligned}
F &= \gamma(z_1^2 + \overline{z_2}^2) + z_1\overline{z_1} - \sum(\overline{b_\ell} - a_\ell)x_\ell z_1 \\
&+ \gamma \sum A_\beta A_\ell z_\beta z_\ell + \gamma \sum \overline{A_\beta A_\ell}\,\overline{z_\beta z_\ell} + \sum A_\beta \overline{A_\ell} z_\beta \overline{z_\ell} \\
&+ \sum a_\ell A_\beta z_\beta x_\ell + \sum b_\ell \overline{A_\beta}\,\overline{z_\beta} x_\ell + \sum c_{\ell\beta} x_\ell x_\beta \\
&+ i \sum 2\gamma A_\beta y_\beta z_1 - i \sum \overline{A_\beta} y_\beta z_1 \\
&+ i \sum A_\beta y_\beta \overline{z_1} - i \sum 2\gamma \overline{A_\beta} y_\beta \overline{z_1} + O(3)
\end{aligned}
$$

Since the y_β are $O(2)$, the last few terms are order 3, so we can simplify this to

$$
\begin{aligned}
F &= \gamma(z_1^2 + \overline{z_2}^2) + z_1\overline{z_1} - \sum(\overline{b_\ell} - a_\ell)x_\ell z_1 \\
&+ \gamma \sum A_\beta A_\ell z_\beta z_\ell + \gamma \sum \overline{A_\beta A_\ell}\,\overline{z_\beta z_\ell} + \sum A_\beta \overline{A_\ell} z_\beta \overline{z_\ell} \qquad (2.1) \\
&+ \sum a_\ell A_\beta z_\beta x_\ell + \sum b_\ell \overline{A_\beta}\,\overline{z_\beta} x_\ell + \sum c_{\ell\beta} x_\ell x_\beta + O(3)
\end{aligned}
$$

At this point, we focus on the second and third line of equation (2.1), separate $z_\beta = x_\beta + iy_\beta$, and define

$$
\begin{aligned}
a'_\ell &= a_\ell - \overline{b_\ell} \\
c'_{\ell\beta} &= \gamma A_\beta A_\ell + \gamma \overline{A_\beta A_\ell} + A_\beta \overline{A_\ell} + a_\ell A_\beta + b_\ell \overline{A_\beta} c_{\ell\beta}
\end{aligned}
$$

Then, again grouping any terms of order 3 or higher into the O(3), we have

$$F = \gamma(z_1^2 + \overline{z_2}^2) + z_1\overline{z_1} + \sum a'_\ell x_\ell z_1 + \sum c'_{\ell\beta} x_\ell x_\beta + O(3)$$

Lastly, transforming $z_n \mapsto z_n - \sum a'_\ell z_\ell z_1 - \sum c'_{\ell\beta} z_\ell z_\beta$ gives the desired form

$$F = \gamma(z_1^2 + \overline{z_2}^2) + z_1\overline{z_1} + O(3)$$

Now, it remains to simplify the f_α. First, recall that $f_\alpha = q_\alpha + q_{1\alpha} + 2_{2\alpha}$ where

$$q_\alpha = a_\alpha z_1^2 + b_\alpha z_1 \overline{z_1} + \overline{a_\alpha z_1}^2$$
$$q_{1\alpha} = 2Re \sum_\ell c_{\alpha\ell} x_\ell z_1$$
$$q_2 = \sum_{\ell,k} c_{\alpha\ell k} x_\ell x_k$$

and $b_\alpha, c_{\alpha\ell k} \in \mathbb{R}$. The $b_\alpha z_1 \overline{z_1}$ term is removed by the transformation $z_\alpha \mapsto z_\alpha + i b_\alpha z_n$. Then, all remaining terms in $q_\alpha, q_{1\alpha}$, and $q_{2\alpha}$ are removed by the transformation

$$z_\alpha \mapsto z_\alpha + 2i \sum_\beta c_{\alpha\beta} z_\beta z_\alpha + i \sum_{\beta,\ell} c_{\alpha\beta\ell} z_\beta z_\ell + 2i a_\alpha z_n$$

□

Definition 2.6. If $0 \leq \gamma < \frac{1}{2}$ in the Moser Webster normal form, then we say that the singularity is *elliptic*.

For example, one can easily see that each singularity on S^n is elliptic and nonegenerate. In fact, we have the following

If $\|\psi\|_{\mathcal{C}^2}$ is small enough, then the singularites of S^n_ϕ are elliptic and nondegenerate.

Proof. Because both b and γ only depend on the defining functions and their first two derivatives, the lemma follows from the fact that the conditions $b \neq 0$ and $0 \leq \gamma < \frac{1}{2}$ are open conditions. □

Chapter 3

Attached Discs and Smoothness Away from Singularities

3.1 Preliminaries and Infinite Dimensional Calculus

We begin with some function spaces on the unit circle $\partial\Delta$. Given $0 < \alpha < 1$, let

$$\mathcal{C}^{0,\alpha}(\partial\Delta) = \left\{ f \in \mathcal{C}(\partial\Delta;\mathbb{C}) : \|f\|_\alpha = \|f\|_\infty + \sup_{\substack{x,y\in\partial\Delta \\ x\neq y}} \frac{\left|f(x)-f(y)\right|}{|x-y|^\alpha} < \infty \right\},$$

where $\|f\|_\infty = \sup_{x\in\partial\Delta}\left\|f(x)\right\|$. For $k \in \mathbb{N}$, let

$$\mathcal{C}^{k,\alpha}(\partial\Delta) = \left\{ f \in \mathcal{C}^k(\partial\Delta;\mathbb{C}) : \|f\|_{k,\alpha} = \sum_{j=0}^{k} \left\|D^j f\right\|_\alpha < \infty \right\}.$$

Note that we use notation $\mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{R})$, $\mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{R}^n)$ and $\mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{C}^n)$ according to the convention establish in Section 1.3. We will use the notation $B_{k,\alpha}(f,r)$ to denote the ball of radius r centered at f in the Banach space $\mathcal{C}^{k,\alpha}(\partial\Delta)$ (or in $\mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{C}^n)$, depending on the context).

We also work with the Banach space

$$\mathcal{A}^{k,\alpha}(\partial\Delta) = \{ f \in \mathcal{C}^{k,\alpha}(\partial\Delta) : \exists\ \widetilde{f} \in \mathcal{O}(\Delta) \cap \mathcal{C}^{k,\alpha}(\overline{\Delta}) \text{ such that } \widetilde{f}|_{\partial\Delta} = f \} \tag{3.1}$$

with the same norm as that on $\mathcal{C}^{k,\alpha}(\partial\Delta)$. It is known that if f and $\widetilde{f}$ are as above, then $\left\|\widetilde{f}\right\|_{\mathcal{C}^{k,\alpha}(\overline{\Delta})} \lesssim \|f\|_{k,\alpha}$.

In this section, we let E, F, G denote Banach spaces with norms $\|\cdot\|_E$, $\|\cdot\|_F$, $\|\cdot\|_G$, respectively. We let $\mathcal{L}(E,F)$ denote the space of bounded linear maps from E to F.

Definition 3.1. For a map $T : E \to F$, the *Fréchet derivative* of T, denoted DT is a map from E to $\mathcal{L}(E,F)$ such that for each point $x \in E$,

$$\lim_{\|h\|_E \to 0} \frac{\left\|T(x+h) - T(x) - DT(x)h\right\|_F}{\|h\|_E} = 0$$

Example 3.2. Here are some examples which will be useful for later .

1. If T is bounded and linear, then $DT(x) = T$ for all $x \in E$.

2. If $\phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and E is a space of continuous functions on some subset of $\mathbb{R}^n$, then for $f \in E$ with appropriate range, we can define the map

$$\mathrm{ev}_\phi : f \mapsto \phi \circ f$$

Then $D\,\mathrm{ev}_\phi(f)h = \phi'(f)h$

Definition 3.3. If E, F are complex Banach spaces and $DT(x)$ is complex linear, then we say that T is *holomorphic* at x.

Example 3.4. If ϕ in the example above is holomorphic, then so is ev_ϕ.

We can also define partial derivatives and higher-order derivatives as usual:

Definition 3.5. Let $T : E \times F \to G$. Then at each point $(x, y) \in E \times F$, the derivative of T with respect to $x \in E$, denoted DT_E or DT_x, is in $\mathcal{L}(E, G)$ satisfying

$$\lim_{\|h\|_E \to 0} \frac{\left\|T\left((x,y) + (h,0)\right) - T(x,y) - D_E T(x,y)h\right\|_G}{\|h\|_E} = 0$$

Definition 3.6. Let $n \in \mathbb{N}$. Let $\mathcal{L}(^nE, F)$ denote the space of n-linear maps from E^n to F. Then, $D^nT : E \to \mathcal{L}(^nE, F)$ satisfying

$$\lim_{\|h\|_E \to 0} \frac{\left\|D^{n-1}T(x+h)\sigma - D^{n-1}T(x)\sigma - D^nT(x)(\sigma, h)\right\|_F}{\|h\|_E} = 0$$

Given these definitions, we can state two theorems which will be helpful throughout this paper:

Theorem 3.7 (Inverse Function Theorem)**.** *Let $T : E \to E$ be differentiable at a point $x \in E$. Suppose further that $T(x) = 0$ and $DT(x)$ is invertible. Then, there are open sets $x \in U$ and $0 \in V$ such that T is a diffeomorphism from U to V.*

Theorem 3.8 (Implicit Function Theorem)**.** *Let $T : E \times F \to F$ have a partial derivative with respect to F at a point $(x, y) \in E \times F$. Suppose further that $T(x, y) = 0$ and $DT_F(x, y)$ is invertible. Then, there are open neighborhoods U of y in F and V of x in E and there is a map $g : E \to F$ such that $T(x, g(x)) = 0$ for $x \in V$. Furthermore, g inherits the regularity of T.*

For proofs and more details of this nonlinear functional analysis, see [9]. We will additionally need the following

An infinitely differentiable map T between Banach spaces is analytic at a point a in its domain if and only if there exists a neighborhood V_a of a and constants c, ρ such that

$$\left\| D^j T(x) \right\| \leq c \frac{j!}{\rho^j}$$

for all $x \in V_a$. In this case, $T(a + h) = \sum D^j T(a)(h^j)$ for all h small enough, where h^j denotes $(h, \dots, h)$. In particular, this implies that the composition or product of analytic maps is again analytic.

Example 3.9. Suppose K is some neighborhood of S^n and $\phi : K \to \mathbb{C}^n$ is real analytic. Then the map $\mathrm{ev}_\phi : \mathcal{C}^{1,\alpha}(\partial\Delta) \to \mathcal{C}^{1,\alpha}(\partial\Delta)$ is real analytic. This follows from the fact that

$$D^n \,\mathrm{ev}_\phi(f)(h)(\zeta) = \phi^{(n)}(f(\zeta))h(\zeta)$$

With these tools we prove the following lemma, which will prove useful later.

For any $k \in \mathbb{N}$, the map $\mathrm{ev} : \overline{\Delta} \times \mathcal{A}^{k,\alpha}(\partial\Delta; \mathbb{C}^n) \to \mathbb{C}^n$ given by $\mathrm{ev}(\xi, f) = f(\xi)$ is $\mathcal{C}^k$-smooth on $\overline{\Delta} \times \mathcal{A}^{k,\alpha}(\partial\Delta; \mathbb{C}^n)$ and real-analytic on $\Delta \times \mathcal{A}^{k,\alpha}(\partial\Delta; \mathbb{C}^n)$.

Proof. We note that $f \mapsto f$ is a bounded linear transformation. Now, we have that

$$D^j \operatorname{ev}(\xi, f)(\zeta_1, h_1)\cdots(\zeta_j, h_j) = f^{(j)}(\xi)\zeta_1\cdots\zeta_j + \sum_{\ell=1}^{j} h_\ell{}^{(j-1)}(\xi)\frac{\zeta_1\cdots\zeta_j}{\zeta_\ell}.$$

Since all the derivatives of f up to order k satisfy a Hölder condition of the form

$$\left|f^{(j)}(\xi_1) - f^{(j)}(\xi_2)\right| \leq \|f\|_{k,\alpha}|\xi_1 - \xi_2|^\alpha, \qquad \xi_1, \xi_2 \in \overline{\Delta},$$

the continuity of $D^j e$ for $j \leq k$ follows. Thus, we obtain the first part of the claim.

Next, we observe that for any $(\xi, f) \in \Delta \times \mathcal{A}^{k,\alpha}(\partial\Delta; \mathbb{C}^n)$, we may write

$$\operatorname{ev}\big((\xi, f) + (\zeta, h)\big) = \operatorname{ev}(\xi, f) + \sum_{j\geq 1} A_j(\underbrace{(\zeta, h)\cdots(\zeta, h)}_{j \text{ times}})$$

whenever $f, h \in \mathcal{A}^{k,\alpha}(\partial\Delta; \mathbb{C}^n)$ and $|\zeta - \xi| < 1 - |\xi|$, where A_j is the symmetric j-linear map

$$\big((\zeta_1, h_1), ..., (\zeta_n, h_n)\big) \mapsto \frac{f^{(j)}(\xi)}{j!}\zeta_1\cdots\zeta_n + \sum_{\ell=0}^{k} \frac{h_\ell^{(j-1)}(\xi)}{(j-1)!}\frac{\zeta_1\cdots\zeta_n}{\zeta_\ell}.$$

By Cauchy's estimates, we have that $\|A_j\| \leq (1+\|f\|_{k,\alpha})$, $j \in \mathbb{N}$. Thus, $\sum_{j\in\mathbb{N}}\|A_j\| r^j < \infty$ for any $r < 1$, which establishes the real-analyticity of ev at (ξ, f). □

Remark 1. Here onwards, we will identify f and $\widetilde{f}$, i.e., for $f \in \mathcal{A}^{k,\alpha}(\partial\Delta; \mathbb{C}^n)$ and $\xi \in \Delta$, we will denote $\widetilde{f}(\xi)$ simply by $f(\xi)$.

Next, given $f \in \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R})$, we let $\mathcal{H}(f)$ be given by

$$f = a_0 + \sum_{n=1}^{\infty} a_n e^{in\theta} + \overline{a_n}e^{-in\theta} \mapsto \mathcal{H}(f) = \sum_{n=1}^{\infty} -ia_n e^{in\theta} + i\overline{a_n}e^{-in\theta} \tag{3.2}$$

Note that $\mathcal{H}$ is the *standard Hilbert transform.* It is well known that $\mathcal{H}$ is a bounded linear transformation from $\mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R})$ to itself. We then define $\mathcal{J} : \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}) \to \mathcal{C}^{k,\alpha}(\partial\Delta)$ as

$$\mathcal{J} : f \mapsto f + i\mathcal{H}(f).$$

Clearly, $\mathcal{J}$ is also a bounded linear transformation with $\mathcal{J}(\mathcal{C}^{k,\alpha}(\partial\Delta)) \subset \mathcal{A}^{k,\alpha}(\partial\Delta)$. Note that if f is as in (3.2), then $\mathcal{J}(f)(0) = a_0$. In an abuse of notation, the component-wise application of $\mathcal{H}$ and $\mathcal{J}$ on elements in $\mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{R}^n)$ is also denoted by $\mathcal{H}$ and $\mathcal{J}$, respectively.

Lastly, we fix a parametrization for the holomorphic discs that foliate the hull of S^n. For any $(\xi, \mathbf{t}) \in \overline{\Delta} \times D^{n-1}(1)$, we let $\mathfrak{g}_{\mathbf{t}}(\xi) = \left(\sqrt{1-\|\mathbf{t}\|^2}\xi, \mathbf{t}\right)$. The perturbed sphere will be shown to be foliated by boundaries of discs that are perturbations of $\mathfrak{g}_{\mathbf{t}}$. As discussed in Remark 1, we also use $\mathfrak{g}_{\mathbf{t}}$ to denote $\mathfrak{g}_{\mathbf{t}}|_{\partial\Delta}$.

3.2 Construction of Discs

In this section, we follow Alexander's approach (see [1]) to construct a $\mathcal{C}^1$-smooth manifold $M_{\text{TR}} \subset \mathbb{C}^n$ that is foliated by holomorphic discs whose boundaries are attached to the totally real part of S^n_ϕ. For this, we first solve the following nonlinear Riemann-Hilbert problem: find a function $f : \overline{\Delta} \to \mathbb{C}$ that is holomorphic on Δ and whose boundary values on $\partial\Delta$ satisfy $\left|f(z) - \gamma(z)\right| = \sigma(z)$, where $\gamma(z)$ is close to 0 (in some appropriate norm) and σ is a positive function on $\partial\Delta$. The solutions to the above problem give analytic discs attached to the torus $\left\{|z_1| = 1, \left|z_2 - \gamma(z_1)\right| = \sigma(z_1)\right\}$ in $\mathbb{C}^2$.

Let $\alpha \in (0,1)$. There is an open set $\Omega \subset \mathcal{C}^{1,\alpha}(\partial\Delta) \oplus \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R})$ such that

$$\{(0,\sigma) : \sigma > 0\} \subset \Omega \subset \{(\gamma,\sigma) : \sigma > 0\},$$

and there is an analytic map $E : \Omega \to \mathcal{A}^{1,\alpha}(\partial\Delta)$ such that

(*i*) if $(\gamma,\sigma) \in \Omega$ and $E(\gamma,\sigma) = f$, then $|f - \gamma| = \sigma$ on $\partial\Delta$, $f(0) = 0$, and $f'(0) > 0$;

(*ii*) $E(0,c)(\xi) \equiv c\,\xi$ for $\xi \in \partial\Delta$, when c is a positive constant function.

Proof of Lemma 3.2. The idea of the proof is as follows. Given $(\gamma,\sigma) \in \mathcal{C}^{1,\alpha}(\partial\Delta) \oplus \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R})$ with $\sigma > 0$, if there is an $\eta \in \mathcal{C}^{1,\alpha}(\partial\Delta)$ that satisfies

$$\gamma = \eta e^{\mathcal{J}(\log\sigma) - \mathcal{J}(\log|\mathfrak{g}-\eta|)}, \tag{3.3}$$

where $\mathfrak{g}(\xi) = \xi$, $\xi \in \partial\Delta$, and $\mathcal{J} : \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R}) \to \mathcal{A}^{1,\alpha}(\partial\Delta)$ is the operator defined in Section 3.1, then, setting $E(\gamma,\sigma) = f = \mathfrak{g}e^{\mathcal{J}(\log\sigma)}e^{-\mathcal{J}(\log|\mathfrak{g}-\eta|)}$, we have that

$$|f-\gamma| = \left|\mathfrak{g}e^{\mathcal{J}(\log\sigma)}e^{-\mathcal{J}(\log|\mathfrak{g}-\eta|)} - \eta e^{\mathcal{J}(\log\sigma)}e^{-\mathcal{J}(\log|\mathfrak{g}-\eta|)}\right| = e^{\log\sigma}|\mathfrak{g}-\eta|\, e^{-\log|\mathfrak{g}-\eta|} = \sigma. \tag{3.4}$$

Moreover, $f(0) = 0$ and $f'(0) = e^{(\mathcal{J}\log(\sigma/|\mathfrak{g}-\eta|))(0)} > 0$. So, we must solve for η in (3.3) for (γ, σ) close to $(0, \sigma)$ when $\sigma > 0$. But any solution of (3.3) corresponding to (γ, σ) is also a solution corresponding to $(\gamma e^{-\mathcal{J}(\log\sigma)}, 1)$. Thus, it suffices to establish the solvability of (3.3) near $(0,1) \in \mathcal{C}^{1,\alpha}(\partial\Delta) \oplus \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R})$.

Let $U = \{\eta \in \mathcal{C}^{1,\alpha}(\partial\Delta) : \|\eta\|_\infty < 1\}$, which is an open set in $\mathcal{C}^{1,\alpha}(\partial\Delta)$. For $\eta \in U$, let $A(\eta) = e^{-\mathcal{J}(\log|\mathfrak{g}-\eta|)}$. We claim that

$$A : U \to \mathcal{A}^{1,\alpha}(\partial\Delta) \text{ is an analtyic map with } A(0) = 1. \tag{3.5}$$

Further, letting $Q(\eta) = \eta \cdot A(\eta)$, we claim that

$$Q : U \to \mathcal{C}^{1,\alpha}(\partial\Delta) \text{ is an analytic map with } Q(0) = 0 \text{ and } Q'(0) = \mathbf{I}. \tag{3.6}$$

Assuming (3.5) and (3.6) for now, we can apply the inverse function theorem for Banach spaces to Q to obtain open neighborhoods $\mathcal{U} \subseteq U$ and V of 0 in $\mathcal{C}^{1,\alpha}(\partial\Delta)$ such that Q is an analtyic diffeomorphism from $\mathcal{U}$ onto V. Set

$$\Omega = \{(\gamma,\sigma) \in \mathcal{C}^{1,\alpha}(\partial\Delta) \oplus \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R}) : \sigma > 0 \text{ and } \gamma e^{-\mathcal{J}(\log\sigma)} \in V\}$$

and observe that $\eta = Q^{-1}(\gamma e^{-\mathcal{J}(\log\sigma)})$ solves (3.3) for every $(\gamma,\sigma) \in \Omega$.

Now set $E_\pm : \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R}_{>0}) \to \mathcal{A}^{1,\alpha}(\partial\Delta)$ by $E_\pm(\sigma) = e^{\pm J(\log\sigma)}$. The proof of (3.5) below can be imitated to check that $E_\pm$ are analytic maps. Further, $M_\mathfrak{g} : \mathcal{A}^{1,\alpha}(\partial\Delta) \to \mathcal{A}^{1,\alpha}(\partial\Delta)$ defined by $M_\mathfrak{g}(h) = \mathfrak{g}h$ is also analytic since it is a bounded linear transformation. Thus, the map $E : \Omega \to \mathcal{A}^{1,\alpha}(\partial\Delta)$ given by

$$E(\gamma,\sigma) = E_+(\sigma)\left(M_\mathfrak{g} \circ A \circ Q^{-1}\right)(\gamma E_-(\sigma))$$

is analytic. As shown in (3.4), it satisfies (i). Also, $E(0,c) = E_+(c)M_\mathfrak{g}(1) = c\mathfrak{g}$, for $c > 0$.

We must now prove (3.5) and (3.6). For (3.5), we first consider the map $L : \eta \mapsto \log|\mathfrak{g} - \eta|$. We use the fact that if $f \in \mathcal{C}^{1,\alpha}(\partial\Delta)$ and $g \in \mathcal{C}^2(f(\partial\Delta))$, then $g \circ f \in \mathcal{C}^{1,\alpha}(\partial\Delta)$. We apply this fact to $f = \mathfrak{g} - \eta$ for $\eta \in U$, and $g(\cdot) = \log(|\cdot|)$ to obtain that $L(U) \subset \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R})$. Now, for a fixed $\eta \in U$ and a small $h \in \mathcal{C}^{1,\alpha}(\partial\Delta)$, we have that

$$
\begin{aligned}
L(\eta + h) - L(\eta) &= \log|\mathfrak{g} - \eta - h| - \log|\mathfrak{g} - \eta| \\
&= \log\left|1 - \frac{h}{\mathfrak{g} - \eta}\right| \\
&= \frac{1}{2}\log\left(1 - \frac{h}{\mathfrak{g} - \eta}\right) + \frac{1}{2}\log\left(1 - \frac{\bar{h}}{\bar{\mathfrak{g}} - \bar{\eta}}\right) \\
&= \frac{1}{2}\left(-2\operatorname{Re}\left(\frac{h}{\mathfrak{g} - \eta}\right) + O\left(\|h\|_{1,\alpha}^2\right)\right) \qquad \text{as } \|h\|_{1,\alpha} \to 0,
\end{aligned}
$$

where we are using the Taylor series expansion of $\log(1 - z)$ and the submultiplicative property of $\|\cdot\|_{1,\alpha}$ in the last step. Thus, L is differentiable at η and $DL(\eta)(h) = -\operatorname{Re}\left(\frac{h}{\mathfrak{g}-\eta}\right)$. Continuing in this way, we obtain that $D^jL : U \to \mathcal{L}^j(\mathcal{C}^{1,\alpha}(\partial\Delta), \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R}))$ exists and is given by $D^jL(\eta)(h_1, ..., h_j) = -(j-1)!\operatorname{Re}\left(\frac{h_1 \cdots h_j}{(\mathfrak{g}-\eta)^j}\right)$, where

$$
\mathcal{L}^j(\mathcal{C}^{1,\alpha}(\partial\Delta), \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R}))
$$

is the space of bounded j-linear maps from $\mathcal{C}^{1,\alpha}(\partial\Delta)^j$ to $\mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R})$. Thus, for any $j \geq 1$, D^jL is continuous on U when $\mathcal{L}^j(\mathcal{C}^{1,\alpha}(\partial\Delta), \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R}))$ is given the standard norm topology. Finally, observe that

$$
\left\|D^jL(\eta)\right\| = \sup_{\|(h_1,\ldots,h_j)\| \leq 1}\left\|D^jL(\eta)(h_1,\ldots,h_j)\right\| = (j-1)!\left\|\operatorname{Re}\left(\frac{h_1\cdots h_j}{(\mathfrak{g}-\eta)^j}\right)\right\| \leq \frac{j!}{\|\mathfrak{g}-\eta\|^j} \tag{3.7}
$$

Hence, L is analytic. Now, the maps $\mathcal{J}$ and $u \mapsto e^{-u}$ are both analytic on $\mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R})$, since the former is a bounded linear transformation, and the latter has continuous derivatives of all orders of the following form $(h_1, ..., h_j) \mapsto e^{-u}h_1 \cdots h_j$ at any $u \in \mathcal{C}^{1,\alpha}(\partial\Delta;\mathbb{R})$. Thus, A being the composition of analytic maps, is itself analytic. Further, as $L(0) = \log|\mathfrak{g}| = 0$, $A(0) = 1$.

Now, recall that $Q(\eta) = \eta \cdot A(\eta)$. So, $Q(0) = 0$. Being the product of two analytic maps, Q is analytic at any $\eta \in U$. Now, since $DQ(\eta)(h) = \eta DA(\eta)(h) + hA(\eta)$, we have that $DQ(0)(h) \equiv h$. This gives (3.6) and concludes our proof. □

We now apply Lemma 3.2 to solve a nonlinear Riemann-Hilbert problem in n functions. Note that the same problem will be solved using a different technique in Section 3.4, where we will improve the regularity of the manifold constructed here.

Let $\alpha \in (0, 1)$. There is an open neighborhood $\widetilde{\Omega}$ of $D^{n-1}(1) \times \{0\}$ in $D^{n-1}(1) \times \mathcal{C}^3(K; \mathbb{C}^n)$ and a $\mathcal{C}^1$-smooth map $F : \widetilde{\Omega} \to \mathcal{A}^{1,\alpha}(\partial\Delta; \mathbb{C}^n)$ such that $F(\mathbf{t}, 0) = \mathfrak{g}_{\mathbf{t}}$, and if $F(\mathbf{t}, \phi) = f = (f_1, ..., f_n)$ for $(\mathbf{t}, \phi) \in \widetilde{\Omega}$, then $f(\partial\Delta) \subset S^n_\phi$, $f_1(0) = 0$ and $f_1'(0) > 0$.

Proof. Recall that from Lemma 3.2, there exists an open set $\Omega \subset \mathcal{C}^{1,\alpha}(\partial\Delta) \oplus \mathcal{C}^{1,\alpha}(\partial\Delta; \mathbb{R})$ so that the solution operator E is smoothly defined on Ω.

Now, for $(\mathbf{t}, \phi, f) \in D^{n-1}(1) \times \mathcal{C}^3(K; \mathbb{C}^n) \times \mathcal{A}^{1,\alpha}(\partial\Delta; \mathbb{C}^n)$, consider the map

$$P : (\mathbf{t}, \phi, f) \mapsto \left(\phi_1(f), \sqrt{1 - \Sigma(\mathbf{t}, \phi, f)}\right),$$

where $\Sigma(\mathbf{t}, \phi, f) = \sum_{j=2}^n \left(t_j + H(\operatorname{Im}\phi_j(f)) - \operatorname{Re}\phi_j(f)\right)^2$. Then, P is a $\mathcal{C}^1$-smooth map from W into $\mathcal{C}^{1,\alpha}(\partial\Delta) \oplus \mathcal{C}^{1,\alpha}(\partial\Delta; \mathbb{R})$, where $W = \{(\mathbf{t}, \phi, f) : f(\partial\Delta) \subset K$ and $\left|\Sigma(\mathbf{t}, \phi, f)(\xi)\right| < 1$ for all $\xi \in \partial\Delta\}$. This is a consequence of the following observations.

1. P is clearly $\mathcal{C}^\infty$-smooth in the $\mathbf{t}$ variable.

2. Since H and $f \mapsto f^2$ are $\mathcal{C}^\infty$-smooth from $\mathcal{C}^{1,\alpha}(\partial\Delta)$ to $\mathcal{C}^{1,\alpha}(\partial\Delta)$, and $f \mapsto \sqrt{f}$ is $\mathcal{C}^\infty$-smooth from $\mathcal{C}^{1,\alpha}(\partial\Delta; \mathbb{R}_{>0})$ to $\mathcal{C}^{1,\alpha}(\partial\Delta; \mathbb{R})$, our claim reduces to (3) below.

3. If $\omega = \{(\varphi, f) \subset \mathcal{C}^3(B) \times \mathcal{C}^{1,\alpha}(\partial\Delta) : f(\partial\Delta) \subset \operatorname{dom}(\varphi)\}$, where $B \subset \mathbb{C}$ is some closed ball, then the map $(\varphi, f) \mapsto \varphi(f)$ is $\mathcal{C}^1$-smooth from $(\omega, \|\cdot\|_3 \oplus \|\cdot\|_{1,\alpha})$ to $(\mathcal{C}^{1,\alpha}(\partial\Delta), \|\cdot\|_{1,\alpha})$.

Next, we note that when $\mathbf{t} \in D^{n-1}(1)$, $(\mathbf{t}, 0, \mathfrak{g}_{\mathbf{t}}) \in W$ and $P(\mathbf{t}, 0, \mathfrak{g}_{\mathbf{t}}) = (0, \sqrt{1 - \|\mathbf{t}\|^2}) \in \Omega$. So, there exists an open set $\mathcal{W} \subset \mathbb{R}^{n-1} \oplus \mathcal{C}^3(K; \mathbb{C}^n) \oplus \mathcal{A}^{1,\alpha}(\partial\Delta; \mathbb{C}^n)$ such that

(i) $(\mathbf{t}, 0, \mathfrak{g}_{\mathbf{t}}) \in \mathcal{W}$ for all $\mathbf{t} \in D^{n-1}(1)$, (*ii*) $\mathcal{W} \subseteq W$,

(*iii*) $P(\mathcal{W}) \subseteq \Omega$.

Now, consider the map $R : \mathcal{W} \mapsto \mathcal{A}^{1,\alpha}(\partial\Delta; \mathbb{C}^n)$ given by

$$R(\mathbf{t}, \phi, f) = f - \big(E \circ P(\mathbf{t}, \phi, f), \mathbf{t} + H(\operatorname{Im} \boldsymbol{\phi}(f)) + i \operatorname{Im} \boldsymbol{\phi}(f)\big)\,, \tag{3.8}$$

where $\boldsymbol{\phi}$ denotes the tuple $(\phi_2, ..., \phi_n)$, and H acts component-wise. The map R is $\mathcal{C}^1$-smooth. Note that $R(\mathbf{t}, 0, \mathfrak{g}_{\mathbf{t}}) = 0$ and $D_3 R(\mathbf{t}, 0, \mathfrak{g}_{\mathbf{t}}) = \mathbf{I}$ on $\mathcal{A}^{1,\alpha}(\partial\Delta; \mathbb{C}^n)$ for all $\mathbf{t} \in D^{n-1}(1)$. So, by the implicit function theorem for Banach spaces, for each $\mathbf{t} \in D^{n-1}(t)$, there exist neighborhoods $U_{\mathbf{t}}$ of $\mathbf{t}$ in $D^{n-1}(1)$, $V_{\mathbf{t}}$ of 0 in $\mathcal{C}^3(K; \mathbb{C}^n)$ and $W_{\mathbf{t}}$ of $\mathfrak{g}_{\mathbf{t}}$ in $\mathcal{A}^{1,\alpha}(\partial\Delta; \mathbb{C}^n)$, and a $\mathcal{C}^1$-smooth map $F_{\mathbf{t}} : U_{\mathbf{t}} \times V_{\mathbf{t}} \to W_{\mathbf{t}}$ such that $F_{\mathbf{t}}(\mathbf{t}, 0) = \mathfrak{g}_{\mathbf{t}}$ and

$$R(\mathbf{s}, \phi, f) = 0 \text{ for } (\mathbf{s}, \phi, f) \in U_{\mathbf{t}} \times V_{\mathbf{t}} \times W_{\mathbf{t}} \text{ if and only if } f = F_{\mathbf{t}}(\mathbf{s}, \phi). \tag{3.9}$$

But, by uniqueness $F_{\mathbf{t}_1} = F_{\mathbf{t}_2}$ whenever the domains overlap. Thus, there exists an open set $\widetilde{\Omega} \subset D^{n-1}(1) \times \mathcal{C}^3(K; \mathbb{C}^n)$ such that $D^{n-1}(1) \times \{0\} \subset \widetilde{\Omega}$, and a $\mathcal{C}^1$-smooth map $F : \widetilde{\Omega} \to \mathcal{A}^{1,\alpha}(\partial\Delta; \mathbb{C}^n)$ such that $F(\mathbf{t}, 0) = \mathfrak{g}_{\mathbf{t}}$ and $R(\mathbf{t}, \phi, F(\mathbf{t}, \phi)) = 0$ for all $(\mathbf{t}, \phi) \in \widetilde{\Omega}$. The latter condition means that if $F(\mathbf{t}, \phi) = f$, then

$$\begin{gathered} \big|f_1 - \phi_1(f)\big|^2 + \sum_{j=2}^{n} \big(\operatorname{Re} f_j - \operatorname{Re} \phi_j(f)\big)^2 = 1, \\ \operatorname{Im}(f_j) = \operatorname{Im} \phi_j(f), \quad j = 2, ..., n. \end{gathered} \tag{3.10}$$

In other words, $f(\partial\Delta) \subset S^n_\phi$. Further, from (*i*) in Lemma 3.2, $f_1(0) = 0$ and $f_1'(0) > 0$. □

We are now ready to construct the manifold M_{TR}.

Theorem 3.10. *Given $t \in (0, 1)$, there is a neighborhood N_t of 0 in $\mathcal{C}^3(K; \mathbb{C}^n)$ such that $\overline{D^{n-1}(t)} \times N_t \subset \widetilde{\Omega}$, and for $\phi \in N_t$, the map $\mathcal{F}_\phi : \overline{\Delta} \times D^{n-1}(t) \to \mathbb{C}^n$ defined by*

$$\mathcal{F}_\phi(\xi, \mathbf{t}) = F(\mathbf{t}, \phi)(\xi)$$

is a $\mathcal{C}^1$-smooth embedding into $\mathbb{C}^n$, with the the image set $M_{TR} = \mathcal{F}_\phi(\overline{\Delta} \times D^{n-1}(t))$ a disjoint union of analytic discs with boundaries in S^n_ϕ. Further, the map $\phi \mapsto \mathcal{F}_\phi$ is a continuous map from N_t into $\mathcal{C}^1(\overline{\Delta} \times D^{n-1}(t); \mathbb{C}^n)$.

Proof. In Lemma 3.2, the open set $\widetilde{\Omega} \subset D^{n-1}(1) \times \mathcal{C}^3(K; \mathbb{C}^n)$ contains $D^{n-1}(1) \times \{0\}$. Thus, by compactness, for any $t \in (0,1)$, there is an open neighborhood N_t of 0 in $\mathcal{C}^3(K; \mathbb{C}^n)$ such that $\overline{D^{n-1}(t)} \times N_t \subset \widetilde{\Omega}$.

Now, for a fixed $\phi \in N_t$, note that $\mathcal{F}_\phi$ is the composition of two $\mathcal{C}^1$-smooth maps:

$$(\xi, \mathbf{t}) \mapsto (\xi, F(\mathbf{t}, \phi));$$
$$(\xi, f) \mapsto \widetilde{f}(\xi).$$

The smoothness of the second map was established in Lemma 3.1. Thus, $\mathcal{F}_\phi$ is a $\mathcal{C}^1$-smooth map. Since, for $\phi \in N_t$, $\phi \mapsto F(\mathbf{t}, \phi)$ is a $\mathcal{C}^1$-smooth map, we have that $D\mathcal{F}_\phi$ depends continuously on ϕ. Quantitatively, this says that for some $C > 0$,

$$\left\| \mathcal{F}_{\phi^1} - \mathcal{F}_{\phi^2} \right\|_1 \leq C \left\| \phi^1 - \phi^2 \right\|_3$$

for $\phi^1, \phi^2 \in N_t$. Thus, shrinking N_t if necessary, we have that $\mathcal{F}_\phi$ is an embedding for all $\phi \in N_t$, since $\mathcal{F}_0$ is an embedding. □

Remark 2. Based on the above results, we call an $f = (f_1, ..., f_n) \in \mathcal{A}^{k,\alpha}(\partial\Delta; \mathbb{C}^n)$ a *normalized analytic disc attached to* S^n_ϕ if $f(\partial\Delta) \subset S^n_\phi$, $f_1(0) = 0$ and $f_1'(0) > 0$. Note that in the construction above, each $F(\mathbf{t}, \phi)$ is a normalized analytic disc attached to S^n_ϕ.

3.3 The Real Analytic Case

In this section, we will show that the manifold M_{TR} constructed in Theorem 3.10 is, in fact, real analytic if ψ is real analytic. To do so, we will take advantage of the deep connection between real analytic maps on $\mathbb{R}^n$ and holomorphic maps in $\mathbb{C}^n$. From lemma 3.1, we can show that $\mathcal{F}_\phi$ is analytic on $\Delta \times D^{n-1}(1)$, however, we must show analyticity up to $\partial\Delta$. For a single attached disc, we can get regularity up to the boundary by applying a reflection principle, since the disc is attached to an analytic totally real manifold. However, to obtain the regularity in both the ξ and $\mathbf{t}$ direction, we apply the Edge-of-the-Wedge theorem, which acts in this context as a strengthening of the reflection principle.

Let $\mathcal{W} \subset D^{n-1}(1) \oplus \mathcal{C}^3(K;\mathbb{C}^n) \oplus \mathcal{A}^{1,\alpha}(\partial\Delta;\mathbb{C}^n)$, R and F be as in the previous section (see (3.8)). Recall that R is a $\mathcal{C}^1$-smooth map and $D_3R(\mathbf{t}, 0, \mathfrak{g}_{\mathbf{t}}) = \mathbf{I}$ on $\mathcal{A}^{1,\alpha}(\partial\Delta;\mathbb{C}^n)$, for all $\mathbf{t} \in D^{n-1}(1)$. Thus, given $t \in (0,1)$, there is an $\varepsilon_t > 0$ such that, if $\|\phi\|_{\mathcal{C}^3} < \varepsilon_t$, then

- $\phi \in N_t$ where $N_t \subset \mathcal{C}^3(K;\mathbb{C}^n)$ is a neighborhood of 0 obtained in Lemma 3.2;
- $D_3R(\mathbf{t}, \phi, F(\mathbf{t}, \phi))$ is an isomorphism on $\mathcal{A}^{1,\alpha}(\partial\Delta;\mathbb{C}^n)$ for all $\mathbf{t} \in D^{n-1}(t)$.

Now, fix a real-analytic $\phi \in \mathcal{C}^3(K;\mathbb{C}^n)$ with $\|\phi\|_{\mathcal{C}^3} < \varepsilon_t$. Let $R_\phi : \mathcal{W}_\phi \to \mathcal{A}^{1,\alpha}(\partial\Delta;\mathbb{C}^n)$ be the map given by

$$R_\phi(\mathbf{t}, f) = R(\mathbf{t}, \phi, f),$$

where $\mathcal{W}_\phi = \{(\mathbf{t}, f) \in \mathbb{R}^{n-1} \oplus \mathcal{A}^{1,\alpha}(\partial\Delta;\mathbb{C}^n) : (\mathbf{t}, \phi, f) \in \mathcal{W}\}$. Note that $R_\phi(\mathbf{t}, F(\mathbf{t}, \phi)) = 0$ and $D_2R_\phi(\mathbf{t}, F(\mathbf{t}, \phi)) \approx \mathbf{I}$, as long as $\mathbf{t} \in D^{n-1}(t)$. Since ϕ is real analytic, R_ϕ is analytic on $\mathcal{W}_\phi$. This follows from the analyticity of E as shown in lemma 3.2, and the fact that the map $f \to \phi(f)$ is analytic for ϕ analytic, as shown in example 3.9.

We apply the analytic implicit function theorem for Banach spaces to conclude that for each $\mathbf{t} \in D^{n-1}(t)$, there exist neighborhoods $U'_{\mathbf{t}} \subset D^{n-1}(t)$ of $\mathbf{t}$ and $W'_{\mathbf{t}} \subset \mathcal{A}^{1,\alpha}(\partial\Delta;\mathbb{C}^n)$ of $F(\mathbf{t}, \phi)$, and an analytic map $F_{\phi,\mathbf{t}} : U'_{\mathbf{t}} \to W'_{\mathbf{t}}$ such that $F_{\phi,\mathbf{t}}(\mathbf{t}) = F(\mathbf{t}, \phi)$ and

$$R_\phi(\mathbf{s}, f) = 0 \text{ for } (\mathbf{s}, f) \in U'_{\mathbf{t}} \times W'_{\mathbf{t}} \text{ if and only if } f = F_{\phi,\mathbf{t}}(\mathbf{s}). \tag{3.11}$$

As before, the $F_{\phi,\mathbf{t}}$'s coincide when their domains overlap. Thus, there is an analytic map $F_\phi : D^{n-1}(t) \to \mathcal{A}^{1,\alpha}(\partial\Delta;\mathbb{C}^n)$ such that $R_\phi(\mathbf{t}, F_\phi(\mathbf{t})) \equiv 0$ on $D^{n-1}(t)$. We set

$$M'_{\mathrm{TR}} = \left\{ F_\phi(\mathbf{t})(\xi) : (\xi, \mathbf{t}) \in \overline{\Delta} \times D^{n-1}(t) \right\}.$$

The uniqueness in (3.9) and (3.11) shows that, in fact, $F_\phi(\cdot) = F(\cdot, \phi)$ and $M'_{\mathrm{TR}} = M_{\mathrm{TR}}$. Thus, we already know that M'_{TR} is a $\mathcal{C}^1$-smooth embedded manifold in $\mathbb{C}^n$. To show that M'_{TR} is in fact a real-analytic manifold, it suffices to show that $\mathcal{F} : (\xi, \mathbf{t}) \mapsto F_\phi(\mathbf{t})(\xi)$ is real-analytic on $\overline{\Delta} \times D^{n-1}(t)$.

Now, since $\mathcal{F}$ is the composition of $(\xi, \mathbf{t}) \mapsto (\xi, F_\phi(\mathbf{t}))$ and the map ev : $(\xi, f) \mapsto \widetilde{f}(\xi)$, $\mathcal{F} \in \mathcal{C}^\omega(\Delta \times D^{n-1}(t))$; see Lemma 3.1. To show that $\mathcal{F} \in \mathcal{C}^\omega(\overline{\Delta} \times D^{n-1}(t))$, we fix $\mathbf{t}_0 \in D^{n-1}(t)$. Since F_ϕ is real-analytic, there is an $\varepsilon > 0$ such that for $\mathbf{t} \in \mathbf{t}_0 + D^{n-1}(\varepsilon)$, $F_\phi(\mathbf{t})(\xi) = \sum_{\beta \in \mathbb{N}^{n-1}} h_\beta(\xi)(\mathbf{t} - \mathbf{t}_0)^\beta$ with $h_\beta \in \mathcal{A}^{1,\alpha}(\partial\Delta; \mathbb{C})$ and $\left\|h_\beta\right\|_{1,\alpha} \lesssim r^{|\beta|}$ for some $r > 0$. Without loss of generality, let $\mathbf{t} = 0$. Now, let $\xi_0 \in \partial\Delta$ and $z_0 = \mathcal{F}(\xi_0, 0)$. Since $T = \{(z_1, ... z_n) \in \mathbb{C}^n : z_1 \in \partial\Delta, z_2, ..., z_n \in \mathbb{R}\}$ is a real-analytic totally real manifold in $\mathbb{C}^n$ there exists a biholomorphism P near ξ_0 that maps an open piece of T biholomorphically into $\mathbb{R}^n$ in $\mathbb{C}^n$, mapping ξ_0 to the origin. Similarly, there exists a biholomorphism Q near z_0 that maps an open piece of S^n_ϕ biholomorphically into $\mathbb{R}^n$ in $\mathbb{C}^n$, mapping z_0 to the origin. Now, we let $Q^*(z_1, z') = Q(\sum_\beta h_\beta(z_1)(z')^\beta)$, where $z' = (z_2, ..., z_n)$. From the analyticity of F_ϕ, we have that $Q^* \in \mathcal{O}(W) \cap \mathcal{C}(W')$, where

$$\begin{aligned} W &= \{z_1 \in \Delta : |z_1 - \xi_0| < \varepsilon\} \times \{z' \in \mathbb{C}^{n-1} : \left\|z'\right\| < \varepsilon\}, \\ W' &= \{z_1 \in \overline{\Delta} : |z_1 - \xi_0| \leq \varepsilon\} \times \{z' \in \mathbb{C}^{n-1} : \left\|z'\right\| < \varepsilon\}. \end{aligned}$$

For $(z_1, ..., z_n)$ close to 0, we define

$$P^*(z_1, z') = \begin{cases} Q^* \circ P^{-1}(z_1, z'), \ \mathrm{Im}\, z_1 > 0, \\ \overline{Q^* \circ P^{-1}\overline{(z_1, z')}}, \ \mathrm{Im}\, z_1 < 0. \end{cases}$$

Then, by the edge of the wedge theorem, P^* extends holomorphically to a neighborhood of $(0, 0)$ in $\mathbb{C}^n$, and thus, $\mathcal{F}$ extends analytically to a neighborhood of ξ_0 in $\overline{\Delta} \times D^{n-1}(t)$. Repeating this argument for every $\mathbf{t} \in D^{n-1}(t)$, we obtain the real-analyticity of M_{TR}.

3.4 The Case of C^{2k+1} and C^∞

The techniques in section 3.2 can be generalized to higher orders to obtain higher regularity for M_{TR} for more regular ψ. However, this poses two problems for our desired result. First, because we would be applying the implict function theorem in a higher order function space, this would require assumptions on the size of additional derivatives. In this section, we use different methods that allow us to increase the regularity on M_{TR} while still only assuming smallness in the $\mathcal{C}^3$ norm. More importantly, in the weaker version there is nothing stopping the neighborhoods in which our solutions lie from tending to zero as the number of derivatives goes to infinity, leaving only a result in the $\mathcal{C}^\infty$ case for the trivial perturbation.

To overcome these hurdles, we take advantage of the existence of the discs already proved in section 3.2. Recall that for a fixed $t \in (0,1)$, Theorem 3.10 yields a neighborhood N_t of 0 in $\mathcal{C}^1(K;\mathbb{C}^n)$ such that, for $\phi \in N_t$, $M_{\mathrm{TR}} = \mathcal{F}_\phi(\overline{\Delta} \times D^{n-1}(t))$ is a $\mathcal{C}^1$-smooth submanifold in $\mathbb{C}^n$. Shrinking N_t further, if necessary, we show in this section that if $\phi \in \mathcal{C}^{2k+1} \cap N_t$, then for each $\mathbf{t} \in D^{n-1}(t)$, there exists a neighborhood of discs around $\mathcal{F}(\overline{\Delta} \times \{\mathbf{t}\})$ attached to S^n_ϕ and that those discs form a $\mathcal{C}^k$-smooth manifold. By the uniqueness proved above, this will show that M_{TR} is in fact $\mathcal{C}^k$ smooth. In the $\mathcal{C}^\infty$ case, although the neighborhoods for each k may shrink to triviality, this result shows that at the starting disc, M_{TR} is smooth. Repeating the argument at each disc gives the desired result. More precisely, we prove the following

Theorem 3.11. *For any $k \in \mathbb{N}$, $\phi \in N_t \cap \mathcal{C}^{2k+1}(K;\mathbb{C}^n)$ and $\mathbf{t} \in D^{n-1}(t)$, there exist neighborhoods $\mathcal{W}_1$, $\mathcal{W}_2 \subset D^{n-1}(t)$ of $\mathbf{t}$, and a $\mathcal{C}^k$-smooth embedding $\mathcal{G}_k : \overline{\Delta} \times \mathcal{W}_1 \to \mathbb{C}^n$ such that $\mathcal{G}_k(\overline{\Delta} \times \mathcal{W}_1) = \mathcal{F}(\overline{\Delta} \times \mathcal{W}_2)$. Thus, M_{TR} is $\mathcal{C}^k$-smooth. In particular, if $\phi \in N_t \cap \mathcal{C}^\infty(K;\mathbb{C}^n)$, then M_{TR} is a $\mathcal{C}^\infty$-smooth manifold.*

Remark 3. Note that when $\phi \in \mathcal{C}^{2k+1}(K;\mathbb{C}^n) \cap N_t$, then $f_{\mathbf{t}} : \xi \mapsto F(\mathbf{t},\phi)(\xi)$ is in $\mathcal{A}^{2k,\alpha}(\partial\Delta;\mathbb{C}^n)$ (for every $0 < \alpha < 1$) for every $\mathbf{t} \in D^{n-1}(t)$. This follows from known regularity results for analytic discs attached to totally real manifolds in $\mathbb{C}^n$ (see [7]). So, it remains to establish the regularity in the direction of the foliation, i.e. in the $\mathbf{t}$-direction.

The strategy of constructing neighborhoods of discs attached to a manifold around a starting one has been used extensively in $\mathbb{C}^2$. For example, Bedford and Gaveau in [4] defined the index of an analytic disc to construct nearby discs and compute the envelope of holomorphy for certain 2-spheres in $\mathbb{C}^2$. Later, Bedford use these indices in [3] to prove the existence of nearby discs on more general manifolds in $\mathbb{C}^2$. His result was later strengthened by Forstnerič in [10] by introducing a different definition of index which differs from the Bedford and Bedford-Gaveau definition by 1. In this section, we will use the theory of multiindices introduced by Globevnik in [11] to generalize Forstnerič's work to $\mathbb{C}^n$. In particular, lemmas 3.4 and 3.4 below are the $\mathcal{C}^{k,\alpha}$-versions of the main results in Section 6 and 7 of [11].

Notation. In this section, we will sometimes express an $n \times n$ matrix over $\mathbb{C}$ as

$$\begin{pmatrix} a & \boldsymbol{v} \\ \boldsymbol{w}^{\mathrm{T}} & A \end{pmatrix},$$

where $a \in \mathbb{C}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{C}^{n-1}$, and A is an $(n-1) \times (n-1)$ matrix over $\mathbb{C}$.

Let M be an n-dimensional totally real manifold in $\mathbb{C}^n$. Suppose $f : \overline{\Delta} \to \mathbb{C}^n$ is an analytic disc with boundary in M, i.e., $f \in \mathcal{C}(\overline{\Delta}) \cap \mathcal{O}(\Delta)$, and $f(\partial\Delta) \subset M$. Further, suppose $A : \partial\Delta \to \mathrm{GL}(n;\mathbb{C})$ is such that the real span of the columns of $A(\xi)$ is the tangent space $T_{f(\xi)}M$ to M at $f(\xi)$, for each $\xi \in \partial\Delta$. Then, owing to the solvability of the Hilbert boundary problem for vector functions of class $\mathcal{C}^\alpha$ (see [11, Sect. 3], also see [18]), it is known that if A is of class $\mathcal{C}^\alpha$ $(0 < \alpha < 1)$, then there exist maps $F^+ : \overline{\Delta} \to \mathrm{GL}(n;\mathbb{C})$ and $F^- : \hat{\mathbb{C}} \setminus \Delta \to \mathrm{GL}(n;\mathbb{C})$, and integers $\kappa_1 \geq \cdots \geq \kappa_n$, such that

- $F^+ \in \mathcal{C}^\alpha(\overline{\Delta}) \cap \mathcal{O}(\Delta)$ and $F^- \in \mathcal{C}^\alpha(\hat{\mathbb{C}} \setminus \Delta) \cap \mathcal{O}(\hat{\mathbb{C}} \setminus \overline{\Delta})$;
- for all $\xi \in \partial\Delta$,

$$A(\xi)\overline{A(\xi)^{-1}} = F^+(\xi) \begin{pmatrix} \xi^{\kappa_1} & 0 & \cdots & 0 \\ 0 & \xi^{\kappa_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi^{\kappa_n} \end{pmatrix} F^-(\xi), \quad \xi \in \partial\Delta. \tag{3.12}$$

Moreover, the integers $\kappa_1 \geq ... \geq \kappa_n$ are the same for all factorizations of the type (3.12). These integers are called the *partial indices of M along f* and their sum is called the *total index of M along f*. Using the factorization above, a normal form for the bundle $\{T_{f(\xi)}M : \xi \in \partial\Delta\}$ is obtained in [11]. In particular, it is shown that if the partial indices of M along f are even, then there is a $\mathcal{C}^\alpha$-map $\Theta : \overline{\Delta} \to \mathrm{GL}(n;\mathbb{C})$, holomorphic on Δ, and such that for every $\xi \in \partial\Delta$, the real span of the columns of the matrix $\Theta(\xi)\Lambda(\xi)$ is $T_{f(\xi)}M$, where $\Lambda(\xi) = \mathrm{Diag}[\xi^{\kappa_1/2}, ..., \xi^{\kappa_n/2}]$. Conversely, suppose,

there is a $\Theta : \overline{\Delta} \to \mathrm{GL}(n;\mathbb{C})$ of class $\mathcal{C}^\alpha$, holomorphic on Δ, such that $\mathrm{Im}(A^{-1}\Theta\Lambda) \equiv 0$ on $\partial\Delta$ or, equivalently, the real span of the columns of $\Theta(\xi)\Lambda(\xi)$ is $T_{f(\xi)}M$.

Then, for $\xi \in \partial\Delta$,

$$A(\xi)\overline{A(\xi)^{-1}} = \Theta(\xi) \begin{pmatrix} \xi^{\kappa_1} & 0 & \cdots & 0 \\ 0 & \xi^{\kappa_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi^{\kappa_n} \end{pmatrix} \overline{\Theta^{-1}(1/\overline{\xi})},$$

which, due to the holomorphicity of Θ on Δ, is a factorization of type (3.12). Thus, we obtain

Remark 4. Suppose f and A are as above. Then, A satisfies (3.13) if and only if $\kappa_1, ..., \kappa_n$ are the partial indices of M along f. Furthermore, if A is of class $\mathcal{C}^{k,\alpha}$, then Θ in (3.13) can be chosen to be of class $\mathcal{C}^{k,\alpha}$.

Example 3.12. We now use remark 4 to compute the partial indices of the disc $\mathfrak{g}_{\mathbf{t}}(\zeta) = (\sqrt{1-\|\mathbf{t}\|^2}\zeta, \mathbf{t})$ for $\mathbf{t} \in D^{n-1}(1)$ on S^n.

Recall that the map $\mathcal{F}_0 : \partial\Delta \times D^{n-1}(t) \to \mathbb{C}^n$ defined by

$$\mathcal{F}_0(\xi, \mathbf{t}) = \left(\sqrt{1-\|\mathbf{t}\|^2}\,\xi, \mathbf{t}\right)$$

gives a parametrization of S^n. Therefore, we have that the real span of the columns of the matrix

$$D_{\xi,\mathbf{t}}\mathcal{F}_0(\xi,\mathbf{t}) = \begin{pmatrix} i\sqrt{1-||\mathbf{t}||^2}\,\xi & -\dfrac{\mathbf{t}\xi}{\sqrt{1-\|\mathbf{t}\|^2}} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{I}_{n-1} \end{pmatrix}$$

is precisely $T_{\mathcal{F}_0(\xi,t)}S^n$. We can factor the above matrix as

$$\begin{pmatrix} i\sqrt{1-||\mathbf{t}||^2} & -\dfrac{\mathbf{t}\xi}{\sqrt{1-\|\mathbf{t}\|^2}} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{I}_{n-1} \end{pmatrix} \begin{pmatrix} \xi & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{I}_{n-1} \end{pmatrix}.$$

Because the factor on the left clearly extends to a holomorphic map (in ξ) from Δ to $\mathrm{GL}(n;\mathbb{C})$, we have that the partial indices of S^n along $\mathfrak{g}_{\mathbf{t}}$ are $2, 0, \ldots, 0$ for all $\mathbf{t} \in D^{n-1}(1)$.

We now use Remark 4 to establish a stability result for partial indices of S^n_ϕ along the disks constructed in Lemma 3.2.

Let $\widetilde{\Omega}$ and F be as in Lemma 3.2. Then, given any $t \in (0,1)$, there exists a neighborhood $N_t \subset \mathcal{C}^3(K;\mathbb{C}^n)$ such that $\overline{D^{n-1}(t)} \times N_t \subset \widetilde{\Omega}$, and for any $(\mathbf{t},\phi) \in D^{n-1}(t) \times N_t$, the partial indices of S^n_ϕ along $f_{\mathbf{t}} : \xi \mapsto F(\mathbf{t},\phi)(\xi)$, $\xi \in \partial\Delta$, are $2, 0, ..., 0$.

Proof. Let $t \in (0,1)$, and $N_t \subset \mathcal{C}^3(K;\mathbb{C}^n)$ be as in Theorem 3.10. Recall that $\mathcal{F}_\phi :$ $(\xi,\mathbf{t}) \mapsto F(\mathbf{t},\phi)(\xi)$ for $(\xi,\mathbf{t}) \in \overline{\Delta} \times D^{n-1}(t)$. Note that $\mathcal{F}_0(\xi,\mathbf{t}) = (\sqrt{1-||\mathbf{t}||^2}\,\xi,\mathbf{t})$ and $D_{\xi,\mathbf{t}}\mathcal{F}_0(\xi,\mathbf{t}) \in \mathrm{GL}(n;\mathbb{C})$ for all $(\xi,\mathbf{t}) \in \overline{\Delta} \times D^{n-1}(t)$.

Let > 0. As in the proof of Theorem 3.10, N_t can be chosen so that for each $\phi \in N_t$,

1. $\mathcal{F}_{\mathrm{bdy}}$ is a $\mathcal{C}^1$-smooth parametrization of an open totally real subset of S^n_ϕ; where $\mathrm{F}_{\mathrm{bdy}} : (\theta,\mathbf{t}) \mapsto F(\mathbf{t},\phi)(e^{i\theta}), \qquad (e^{i\theta},\mathbf{t}) \in \partial\Delta \times D^{n-1}(t),$

2. $||D_{\xi,\mathbf{t}}\mathcal{F}_\phi - D_{\xi,\mathbf{t}}\mathcal{F}_0||_\infty <$.

Now, we fix a $\phi \in N_t$ and let $\mathcal{F} = \mathcal{F}_\phi$. Since $\dfrac{\partial}{\partial\theta} = i\xi\dfrac{\partial}{\partial\xi}$ when $\xi = e^{i\theta}$, we have that

$$(D_{\theta,\mathbf{t}}\mathcal{F}_{\text{bdy}})(\theta,\mathbf{t}) = \Theta_{\mathbf{t}}(\xi)\begin{pmatrix} \xi & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{I}_{n-1} \end{pmatrix} \quad \text{on } \partial\Delta, \tag{3.13}$$

where

$$\Theta_{\mathbf{t}}(\xi) = (D_{\xi,\mathbf{t}}\mathcal{F})(\xi,\mathbf{t})\begin{pmatrix} i & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{I}_{n-1} \end{pmatrix}.$$

Owing to (1), the real span of the columns of the matrix $A_{\mathbf{t}}(e^{i\theta}) = (D_{\theta,\mathbf{t}}\mathcal{F}_{\text{bdy}})(\theta,\mathbf{t})$ is the tangent space to S^n_ϕ at $f_{\mathbf{t}}(e^{i\theta})$. By (2), if $\varepsilon > 0$ is sufficiently small, then $\Theta_{\mathbf{t}} : \overline{\Delta} \to \mathrm{GL}(n;\mathbb{C})$ since $\mathcal{D}_{\xi,\mathbf{t}}\mathcal{F}_0(\xi,\mathbf{t}) \in \mathrm{GL}(n;\mathbb{C})$ for all $(\xi,\mathbf{t}) \in \overline{\Delta} \times D^{n-1}(t)$. Thus, in order to apply Remark 4 to $f = f_{\mathbf{t}}$ and $A = A_{\mathbf{t}}$, we must show that A is of class $\mathcal{C}^\alpha$, and $\Theta_{\mathbf{t}}$ extends holomorphically to Δ. We will, in fact, show that the entries of $(D_{\xi,\mathbf{t}}\mathcal{F})(\cdot,\mathbf{t})$ are in $\mathcal{A}^{1,\alpha}(\partial\Delta)$. First, since S^n_ϕ is $\mathcal{C}^3$-smooth and $\xi \mapsto \mathcal{F}(\xi,\mathbf{t})$ is an analytic disc attached to S^n_ϕ, $\mathcal{F}$ is $\mathcal{C}^{2,\alpha}$-smooth in ξ. This gives the $\mathcal{C}^{1,\alpha}$-regularity of $D_\xi\mathcal{F}(\cdot,\mathbf{t})$ on $\partial\Delta$. Next, note that $\mathcal{F}(\xi,\mathbf{t}) = \mathrm{ev}(\xi, F(\mathbf{t},\phi))$, where ev is the map defined in Lemma 3.1. Thus, $D_{\mathbf{t}}\mathcal{F}(\cdot,\mathbf{t})(\mathbf{s}) = D_{\mathbf{t}}F(\mathbf{t},\phi)(\mathbf{s})(\cdot)$. Since $D_{\mathbf{t}}F(\mathbf{t},\phi)$ is a bounded linear transformation from $\mathbb{R}^{n-1}$ to $\mathcal{A}^{1,\alpha}(\partial\Delta;\mathbb{C}^n)$, the entries of $D_{\mathbf{t}}\mathcal{F}(\cdot,\mathbf{t})$ are in $\mathcal{A}^{1,\alpha}(\partial\Delta;\mathbb{C}^n)$, and therefore holomorphic in $\xi \in \Delta$. Thus, the indices of S^n_ϕ along $f_{\mathbf{t}}$ are $2, 0, ..., 0$. □

For the rest of this section, we fix $t \in (0,1)$, $\phi \in N_t \cap \mathcal{C}^{2k+1}(K;\mathbb{C}^n)$ $(k \geq 3)$ and $\mathbf{t} \in D^{n-1}(t)$. We let $\mathcal{M} = M_{\mathrm{TR}}$. Recall that by [7], $f_{\mathbf{t}} : \xi \mapsto \mathcal{F}_\phi(\xi,\mathbf{t})$ is in $\mathcal{A}^{2k,\alpha}(\partial\Delta;\mathbb{C}^n)$ and is a normalized analytic disc attached to S^n_ϕ (see Remark 2). We fix a tubular neighborhood Ω of $f_{\mathbf{t}}(\partial\Delta)$ in $\mathbb{C}^n$ and a map $\rho^\phi : \Omega \to \mathbb{R}^n$ such that

- ▷ $\rho^\phi = (\rho^\phi_1, ..., \rho^\phi_n) \in \mathcal{C}^{2k+1}(\Omega;\mathbb{R}^n)$;
- ▷ $d\rho^\phi_1 \wedge \cdots \wedge d\rho^\phi_n \neq 0$ on Ω;
- ▷ $S^n_\phi \cap \Omega = \{z \in \Omega : \rho^\phi(z) = 0\}$.

Let $X_1(\xi) = \dfrac{\partial f_{\mathbf{t}}}{\partial\theta}(\xi)$. Since $S^n_\phi \cap \Omega$ is $\mathcal{C}^{2k+1}$-smooth and totally real, there exist $\mathcal{C}^{2k}$-smooth maps $X_2, ..., X_n : \partial\Delta \to \mathbb{C}^n$ such that for each $\xi \in \partial\Delta$, the real span of $X_1(\xi), ..., X_n(\xi)$ is the tangent space to S^n_ϕ at $f_{\mathbf{t}}(\xi)$. Given $p = (p_1, ..., p_n) \in \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n)$ and $q = (q_1, ..., q_n) \in \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n)$, let

$$\mathcal{E}(p, q) = \sum_{j=1}^{n} p_j X_j + i \sum_{j=1}^{n} (q_j + iH(q_j))X_j.$$

Note that $\mathcal{E} : \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n) \times \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n) \to \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{C}^n)$ is a linear isomorphism. This is because $X_j(\xi), iX_j(\xi)$, $1 \le j \le n$, form a real basis of $\mathbb{C}^n$ and the standard Hilbert transform $H : \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n) \to \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n)$ is a bounded linear map. There exist neighborhoods $\mathcal{U}_1$ of 0 in $\mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n)$ and $\mathcal{U}_2$ of 0 in $\mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{C}^n)$, and a $\mathcal{C}^k$-smooth map $\mathcal{D} : \mathcal{U}_1 \to \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{C}^n)$ such that

(*i*) for any $f \in \mathcal{U}_2$, $f_{\mathbf{t}} + f$ is attached to S^n_ϕ if and only if $f = \mathcal{D}(p)$ for some $p \in \mathcal{U}_1$; and

(*ii*) there is an $\eta > 0$ such that $\left\|\mathcal{D}(p) - \mathcal{D}(p')\right\|_{k,\alpha} \ge \eta \left\|p - p'\right\|_{k,\alpha}$ for all $p, p' \in \mathcal{U}_1$.

Proof. Let $\mathcal{U}$ be a neighborhood of 0 in $\mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n)$ such that, for all $p, q \in \mathcal{U}$, $f_{\mathbf{t}}(\xi) + \mathcal{E}(p, q)(\xi) \in \Omega$ for all $\xi \in \partial\Delta$. Consider the map

$$\mathcal{R} : (p, q) \mapsto \Big(\xi \mapsto \rho^\phi\big(f_{\mathbf{t}}(\xi) + \mathcal{E}(p, q)(\xi)\big)\Big)$$

on $\mathcal{U} \times \mathcal{U}$. Note that $\mathcal{R}(0, 0) = 0$. By Lemma 5.1 in [12], $\mathcal{R} : \mathcal{U} \times \mathcal{U} \to \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n)$ is a $\mathcal{C}^k$-smooth map. We claim that $(D_q\mathcal{R})(0, 0) : \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n) \to \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n)$ is a linear isomorphism. This is because, for $h = (h_1, ..., h_n) \in \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R}^n)$,

$$\begin{aligned} D_q\mathcal{R}(0, 0)(h) &= \sum_{j=1}^{n} h_j \left\langle \nabla\rho^\phi_j(f_{\mathbf{t}}), iX_k \right\rangle_{\mathbb{R}^{2n}} - \sum_{j=1}^{n} H(h_j) \left\langle \nabla\rho^\phi_j(f_{\mathbf{t}}), X_k \right\rangle_{\mathbb{R}^{2n}} \\ &= \left(\left\langle \nabla\rho^\phi_j(f_{\mathbf{t}}), iX_k \right\rangle_{\mathbb{R}^{2n}} \right) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = C \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \end{aligned}$$

where C is an $n \times n$ matrix with entries in $\mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{R})$. Note that the second equality follows from the fact that $X_j(\xi)$ are tangential to S^n_ϕ at $f_{\mathbf{t}}(\xi)$. It suffices to show the invertibility of C at each $\xi \in \partial\Delta$. If, for some $\xi \in \partial\Delta$, $C(\xi)$ is not invertible, then there exist $a_1, ..., a_n \in \mathbb{R}$ such that $\sum_{j=1}^{n} a_j iX_j(\xi)$ is orthogonal to each $\nabla\rho^\phi_k(f_{\mathbf{t}}(\xi))$

(as vectors in $\mathbb{R}^{2n}$), which contradicts the total reality of S^n_ϕ at $f_{\mathbf{t}}(\xi)$. Thus, by the implicit function theorem applied to $\mathcal{R}$, there exist neighborhoods $\mathcal{U}_1,\mathcal{U}_1' \subseteq \mathcal{U}$, and a $\mathcal{C}^k$-smooth map $\mathcal{Q} : \mathcal{U}_1 \to \mathcal{U}_1'$ such that

$$(p, q) \in \mathcal{U}_1 \times \mathcal{U}_1' \text{ satisfies } \mathcal{R}(p, q) = 0 \iff p \in \mathcal{U}_1 \text{ and } q = \mathcal{Q}(p).$$

Now, setting $\mathcal{D}(p) = \mathcal{E}(p, \mathcal{Q}(p))$, $\mathcal{U}_2 = \mathcal{E}(\mathcal{U}_1 \times \mathcal{U}_1')$, and recalling that $\mathcal{E}$ is a linear isomorphism, we have (i).

To establish (ii), we note that $(D_p\mathcal{D})(0) : \mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{R}^n) \to \mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{C}^n)$ is the map

$$h \mapsto \sum_{k=1}^{n} h_j X_j. \tag{3.14}$$

This computation uses the linearity of $D_q\mathcal{R}(0, 0)$; details can be found in [11, Lemma 6.2]. Due to the nondegeneracy of the matrix $X = [X_1^{\mathrm{T}}, ..., X_n^{\mathrm{T}}]$, there exists an $\eta > 0$ such that, for all $s \in U$ (after shrinking, if necessary), $(D_p\mathcal{D})(s)$ extends to a linear isomorphism $\mathcal{I}_s : \mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{C}^n) \to \mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{C}^n)$ satisfying $\left\|\mathcal{I}_s(\cdot)\right\|_{k,\alpha} \geq \eta\|\cdot\|_{k,\alpha}$ on $\mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{C}^n)$. Assuming $\mathcal{U}_1$ to be convex, we get $\mathcal{D}(p') - \mathcal{D}(p) = \left(\int_0^1 \mathcal{I}_{p+t(p'-p)} dt\right)(p' - p)$, and thus,

$$\left\|\mathcal{D}(p') - \mathcal{D}(p)\right\|_{k,\alpha} \geq \eta\left\|p' - p\right\|_{k,\alpha} \quad p, p' \in \mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{R}^n).$$

□

The neighborhood $\mathcal{U}_1$ obtained above parametrizes all the $\mathcal{C}^{k,\alpha}$-discs close to $f_{\mathbf{t}}$ that are attached to S^n_ϕ. Next, we find those elements of $\mathcal{U}_1$ that parametrize analytic discs attached to S^n_ϕ. We direct the reader to Remark 2 for the definition of a normalized analytic disc.

There exists an open neighborhood U of 0 in $\mathbb{R}^{n-1}$ and a $\mathcal{C}^k$-smooth map $G : U \to \mathcal{A}^{k,\alpha}(\partial\Delta;\mathbb{C}^n)$ such that

(a) $G(0) = 0$;

(b) for each $\boldsymbol{c} \in U$, $f_{\mathbf{t}} + G(\boldsymbol{c})$ extends to a normalized analytic disc attached to S^n_ϕ;

(*c*) for each neighborhood $V \subset U$ of 0 in $\mathbb{R}^{n-1}$, there is a $\tau_V > 0$ so that if $f \in B_{k,\alpha}(0;\tau_V)$ is such that $f_{\mathbf{t}} + f$ is a normalized analytic disc attached to S^n_ϕ, then $f = G(\boldsymbol{c})$ for some $\boldsymbol{c} \in V$;

(*d*) for each $\boldsymbol{c}_1, \boldsymbol{c}_2 \in U$, $G(\boldsymbol{c}_1) \neq G(\boldsymbol{c}_2)$ if $\boldsymbol{c}_1 \neq \boldsymbol{c}_2$.

(*e*) the map $\mathcal{G} : \overline{\Delta} \times U \to \mathbb{C}^n$ given by $(\xi, \boldsymbol{c}) \mapsto f_{\mathbf{t}} + G(\boldsymbol{c})$ is a $\mathcal{C}^k$-smooth embedding.

Proof. In Lemma 3.4, we proved that the indices of S^n_ϕ along $f_{\mathbf{t}}$ are $2, 0, ..., 0$. By Remark 4, there is a map $\Theta = [\Theta_{j\ell}]_{1 \le j,\ell \le n} \in \mathcal{A}^{k,\alpha}(\partial\Delta; \mathrm{GL}(n;\mathbb{C}))$ such that $X = \Theta Y$ on $\partial\Delta$, where

$$Y(\xi) = \begin{pmatrix} \xi & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{I}_{n-1} \end{pmatrix}, \quad \xi \in \partial\Delta.$$

Since $X_1 = \partial f_{\mathbf{t}}/\partial\theta$, the above equation gives $(\partial f_{\mathbf{t}}/\partial\theta)_1(\xi) = \xi\Theta_{11}(\xi)$. On the other hand, $(\partial \mathfrak{g}_{\mathbf{t}}/\partial\theta)_1(\xi) = i\xi\sqrt{1 - \|\mathbf{t}\|^2}$. Thus, shrinking N_t in Theorem 3.10, if necessary, we can make

$$\left\| \Theta_{11} - i\sqrt{1 - \|\mathbf{t}\|^2} \right\|_{\mathcal{C}^\infty(\partial\Delta)} \le \|f_{\mathbf{t}} - \mathfrak{g}_{\mathbf{t}}\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)}$$

small enough so that $\Theta_{11}(0) = \frac{1}{2\pi}\int_0^{2\pi} \Theta_{11}(e^{i\theta})d\theta \neq 0$. We work under this assumption for the rest of this proof.

Now, let $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{D}$ be as in Lemma 3.4. We determine the maps $f = f_{\mathbf{t}} + \mathcal{D}(p)$, $p \in \mathcal{U}_1$, that extend holomorphically to Δ. We have

$$\begin{aligned} \mathcal{D}(p) = \mathcal{E}(p, \mathcal{Q}(p)) &= \sum_{j=1}^n \left(p_j + i(\mathcal{Q}_j(p) + iH\mathcal{Q}_j(p))\right) X_j \\ &= \Theta \left(\sum_{j=1}^n p_j Y_j + i \sum_{j=1}^n (\mathcal{Q}_j(p) + iH\mathcal{Q}_j(p)) Y_j \right). \end{aligned}$$

Note that $f_{\mathbf{t}}$, Y and $\mathcal{Q}(p) + iH\mathcal{Q}(p)$ extend holomorphically to Δ. Moreover, Θ extends holomorphically to Δ with values in $\mathrm{GL}(n;\mathbb{C})$. Thus, $f = f_{\mathbf{t}} + \mathcal{E}(p, \mathcal{Q}(p))$ extends holomorphically to Δ if and only if

$$\xi \mapsto \sum_{j=1}^n p_j(\xi) Y_j(\xi) = (\xi p_1(\xi), p_2(\xi), ..., p_n(\xi)) \tag{3.15}$$

extends holomorphically to Δ. Let us assume that the map in (3.15) extends holomorphically to Δ. Then, since p_j, $j=1,...,n$, are real-valued, we have that $p_j \equiv c_j$ for some real constants $c_2,...,c_n$. Moreover, $p_1(e^{i\theta}) = \sum_{j\in\mathbb{Z}} a_j e^{ij\theta}$ for some $a_j \in \mathbb{C}$ satisfying $a_0 \in \mathbb{R}$ and $a_j = \overline{a_{-j}}$, $j \in \mathbb{N}$. Thus, $\xi p_1(\xi)$ extends to a holomorphic map on Δ if and only if $a_j = 0$ for all $|j| \geq 2$. Now, let $\mathfrak{r} = (p,q,r) \in \mathbb{R}^3$ and $\boldsymbol{c} = (c_2,...,c_n) \in \mathbb{R}^{n-1}$, and $\mathcal{P} : \mathbb{R}^{n+1} \mapsto \mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{R}^n)$ be the bounded linear map

$$(\mathfrak{r},\boldsymbol{c}) = (p,q,r,c_2,...,c_n) \mapsto ((p-iq)\overline{\xi} + r + (p+iq)\xi, c_2,...,c_n),$$

then, based on the above argument,

$(*)$ $f \in \mathcal{U}_2$ extends holormorphically to Δ if and only if $f = \mathcal{D}(\mathcal{P}(\mathfrak{r},\boldsymbol{c}))$ for some $(\mathfrak{r},\boldsymbol{c}) \in \mathcal{P}^{-1}(\mathcal{U}_1)$.

Next, in order to reduce the dimension of the parameter space, we set $\mathfrak{N} = \widetilde{\pi}_{\mathrm{ev}} \circ \mathcal{D} \circ \mathcal{P}$, where the map $\widetilde{\pi}_{\mathrm{ev}} : \mathcal{A}^{k,\alpha}(\partial\Delta;\mathbb{C}^n) \to \mathbb{R}^3$ is given by

$$(f_1,...,f_n) \mapsto (\mathrm{Re}\, f_1(0), \mathrm{Im}\, f_1(0), \mathrm{Im}(f_1')(0)).$$

Then, $\mathfrak{N} : \mathcal{P}^{-1}(\mathcal{U}_1) \subset \mathbb{R}^3 \times \mathbb{R}^{n-1} \to \mathbb{R}^3$ is a $\mathcal{C}^k$-smooth map with $\mathfrak{N}(0,0) = 0$. We claim that $D_{\mathfrak{r}}\mathfrak{N}(0,0)$ is invertible. For this, using (3.14) and the fact that $X_1 = \frac{\partial f_{\mathbf{t}}}{\partial\theta} = i\xi\frac{\partial f_{\mathbf{t}}}{\partial\xi}$, we note that

$$\begin{aligned}
D_{\mathfrak{r}}\mathfrak{N}(0,0)(u,v,w) &= D\widetilde{\pi}_{\mathrm{ev}}(0) \cdot D\mathcal{D}(0)\left((u-iv)\overline{\xi} + w + (u+iv)\xi, 0,...,0\right) \\
&= D\widetilde{\pi}_{\mathrm{ev}}(0)\left((u-iv)\tfrac{X_1(\xi)}{\xi} + wX_1(\xi) + (u+iv)\xi X_1(\xi)\right) \\
&= (au+bv, bu-av, Bu-Av+aw),
\end{aligned}$$

where $a = \mathrm{Re}(f_{t_1})'(0)$, $b = \mathrm{Im}(f_{t_1})'(0)$, $A = \mathrm{Re}(f_{t_1})''(0)$ and $B = \mathrm{Im}(f_{t_1})''(0)$. Here f_{t_1} is the first component of the normalized analytic disc $f_{\mathbf{t}}$. Thus, $\mathrm{Re}\, f_{t_1}'(0) > 0$ and $D_{\mathfrak{r}}\mathfrak{N}(0,0)$ is invertible. We may, thus, apply the implicit function theorem to obtain neighborhoods U of 0 in $\mathbb{R}^{n-1}$, U' of 0 in $\mathbb{R}^3$, and a $\mathcal{C}^k$-smooth map $\mathcal{A} : W \to \mathbb{R}^3$ such that $\mathfrak{N}(\mathfrak{r},\boldsymbol{c}) = 0$ for $(\mathfrak{r},\boldsymbol{c}) \in U' \times U$ if and only if $\mathfrak{r} = \mathcal{A}(\boldsymbol{c})$.

Finally, we let $G : U \to \mathcal{A}^{k,\alpha}(\partial\Delta;\mathbb{C}^n)$ be the map given by

$$G : \boldsymbol{c} \mapsto \mathcal{D}\big(\mathcal{P}(\mathcal{A}(\boldsymbol{c}), \boldsymbol{c})\big).$$

It is clear that G is $\mathcal{C}^k$-smooth and (a) holds. For (b), we note that $(\pi_1 \circ G)(\boldsymbol{c})(0) = 0$ for all $\boldsymbol{c} \in W$. Furthermore, by shrinking U if necessary, we can ensure that $\left|(\pi_1 \circ G)(\boldsymbol{c})'(0)\right| < \left|(\pi_1 \circ f_{\mathbf{t}})'(0)\right|$ for all $\boldsymbol{c} \in U$. Then, since $(\pi_1 \circ f_{\mathbf{t}})'(0) > 0$ and $\operatorname{Im}(\pi_1 \circ G)(\boldsymbol{c})'(0) = 0$, we have that $\operatorname{Re}(\pi_1 \circ G)(\boldsymbol{c})'(0) > 0$. Claim (d) follows from Lemma 3.4 (ii) and the fact that $\mathcal{P}$ is injective. The argument for (e) is similar to the proof of Theorem 3.10. Now, for (c), we let $V \subset U$ be a neighborhood of 0 in $\mathbb{R}^{n-1}$. Since G is injective and continuous, $G(V)$ is open in $G(U)$ (in the subspace topology inherited from $\mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{C}^n)$). Thus, there is an open set $\mathcal{V} \subset \mathcal{U}_2$ in $\mathcal{C}^{k,\alpha}(\partial\Delta;\mathbb{C}^n)$ such that $G(V) = \mathcal{V} \cap G(U)$, and so $G^{-1}(\mathcal{V}) = V$. Thus, combining Lemma 3.4 (i) and $(*)$, we have that, for $f \in \mathcal{V}$, $f_{\mathbf{t}} + f$ is an analytic disc attached to S^n_ϕ with $f_1(0) = 0$ and $\operatorname{Im} f_1'(0) = 0$ if and only if $f = G(c)$ for some $\boldsymbol{c} \in G^{-1}(\mathcal{V}) = V$. To complete the proof of (c), we choose $\tau_V > 0$ so that $B_{k,\alpha}(0;\tau_V) \subset \mathcal{V}$. □

Remark 5. We may repeat the proof of Lemma 3.4 in the $\mathcal{C}^{1,\alpha}$-category to conclude that there exists an open neighborhood U^* of 0 in $\mathbb{R}^{n-1}$ and a $\mathcal{C}^1$-smooth injective map $G^* : U^* \to \mathcal{A}^{1,\alpha}(\partial\Delta;\mathbb{C}^n)$ with $G^*(0) = 0$ such that for each $\boldsymbol{c} \in U^*$, $f_{\mathbf{t}} + G^*(\boldsymbol{c})$ extends to a normalized analytic disc attached to S^n_ϕ. Moreover,

(†) for each neighborhood $V \subset U^*$ of 0, there is a $\tau^*_V > 0$ so that if $f \in B_{1,\alpha}(0;\tau^*_V)$ and $f_{\mathbf{t}} + f$ is a normalized analytic disc attached to S^n_ϕ, then $f = G^*(\boldsymbol{c})$ for some $\boldsymbol{c} \in V$,

and $\mathcal{G}^* : \overline{\Delta} \times U^* \to \mathbb{C}^n$ given by $(\xi, \boldsymbol{c}) \mapsto f_{\mathbf{t}} + G(\boldsymbol{c})$ is a $\mathcal{C}^1$-smooth embedding.

Proof of Theorem 3.11. Recall that $\mathcal{M} = M_{\mathrm{TR}}$ is the manifold constructed in Theorem 3.10. We let $\mathcal{M}_k = \mathcal{G}(\overline{\Delta} \times U)$ and $\mathcal{M}_1 = \mathcal{G}^*(\overline{\Delta} \times U^*)$, where $\mathcal{G}$ and $\mathcal{G}^*$ are the maps defined in Lemma 3.4 (e) and Remark 5, respectively. Note that $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_k$ each contain the disc $f_{\mathbf{t}}(\overline{\Delta})$. To show that near $f_{\mathbf{t}}(\overline{\Delta})$, these three manifolds coincide, we will use the following proposition from [11].

Proposition 3.13 ([11, Prop. 8.1])**.** *Let X be a Banach space, $\omega \subset \mathbb{R}^n$ a neighborhood of 0 and let K, $L : \omega \to X$ be $\mathcal{C}^1$-smooth maps such that $K(0) = L(0)$ and $(DK)(0)$, $(DL)(0)$ both have rank n. Suppose that for every neighborhood of $V \subset \omega$ of 0, there is a neighborhood $V_1 \subset \omega$ of 0 such that $K(V_1) \subset L(V)$. Then, there are neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 such that $K(\mathcal{V}_1) = L(\mathcal{V}_2)$.*

We first show that $\mathcal{M}$ and $\mathcal{M}_1$ coincide near $f_{\mathbf{t}}(\overline{\Delta})$. Shrinking U^* if necessary, we may assume that $\mathbf{t} + U^* \subset D^{n-1}(t)$. We set $\omega = U^* \subset \mathbb{R}^{n-1}$. For $\boldsymbol{c} \in \omega$, we let $K(\boldsymbol{c}) = F(\mathbf{t}+\boldsymbol{c})$ and $L(\boldsymbol{c}) = f_{\mathbf{t}}+G^*(\boldsymbol{c})$, where F and G^* are the maps in Lemma 3.2 and Remark 5, respectively. Note that $K(0) = L(0) = f_{\mathbf{t}}$ and $DK(0)$ and $DL(0)$ both have rank $n-1$. Now, let $V \subset \omega$ be a neighborhood of 0. We set $V_1 = K^{-1}(B_{1,\alpha}(f_{\mathbf{t}}; \tau))$, where $\tau < \tau_V^*$ is sufficiently small so that $V_1 \subset \omega$. Then, for any $\boldsymbol{c} \in V_1$, $K(\boldsymbol{c})$ is a normalized analytic disc attached to S_ϕ^n with property that $\left\|K(\boldsymbol{c}) - f_{\mathbf{t}}\right\|_{1,\alpha} < \tau < \tau_V^*$. Thus, by (†) in Remark 5, $K(\boldsymbol{c}) = f_{\mathbf{t}} + L(\mathfrak{d})$ for some $\mathfrak{d} \in V$. Thus, $K(V_1) \subset L(V)$. By the above proposition, there exist neighborhoods $\mathcal{V}_1$, $\mathcal{V}_2 \subset \omega$ of 0 such that $K(\mathcal{V}_1) = L(V_2)$. This shows that $\mathcal{M}$ and $\mathcal{M}_1$ coincide near $f_{\mathbf{t}}(\overline{\Delta})$.

Next, we use the same approach to show that $\mathcal{M}_1$ and $\mathcal{M}_k$ coincide near $f_{\mathbf{t}}(\overline{\Delta})$. In this case, we set $K(\boldsymbol{c}) = \iota \circ G(\boldsymbol{c})$ and $L(\boldsymbol{c}) = G^*(\boldsymbol{c})$, where G^* and G are the maps in Remark 5 and Lemma 3.4, respectively, and $\iota : \mathcal{C}^{k,\alpha}(\partial\Delta; \mathbb{C}^n) \to \mathcal{C}^{1,\alpha}(\partial\Delta; \mathbb{C}^n)$ is the inclusion map. Now, let $V \subset \omega$ be a neighborhood of 0. We set $V_1 = G^{-1}(B_{k,\alpha}(0; \tau))$, where $\tau < \min\{\tau_\omega, \tau_V^*\}$ is sufficiently small so that $V_1 \subset \omega$. Then, for any $\boldsymbol{c} \in V_1$, $f_{\mathbf{t}} + K(\boldsymbol{c})$ is a normalized analytic disc attached to S_ϕ^n with property that $\left\|K(\boldsymbol{c})\right\|_{1,\alpha} < \left\|G(\boldsymbol{c})\right\|_{k,\alpha} < \tau < \tau_V^*$. Thus, by (†) in Remark 5, $f_{\mathbf{t}} + K(\boldsymbol{c}) = f_{\mathbf{t}} + L(\mathfrak{d})$ for some $\mathfrak{d} \in V$. Thus, $K(V_1) \subset L(V)$. Once again, by the above proposition, there exist neighborhoods $\mathcal{V}_1$, $\mathcal{V}_2 \subset \omega$ of 0 such that $K(\mathcal{V}_1) = L(V_2)$. This shows that $\mathcal{M}_k$ and $\mathcal{M}_1$, and therefore $\mathcal{M}_k$ and $\mathcal{M}$, coincide near $f_{\mathbf{t}}(\overline{\Delta})$. This completes the proof of Theorem 3.11. □

Chapter 4

Proof of Parts 1 to 5 in Theorem 1.1

So far, we have constructed that portion of the manifold $\widetilde{S^n_\phi}$ whose leaves stay bounded away from $\mathrm{Sing}(S^n_\phi)$. We summarize the results from the previous sections as Theorem 4.1 below. Note that we will use the following notation throughout this section. For $t \in (0,1)$,

$$\begin{aligned} S^n_{\lesseqgtr t} &= \{(z_1, x') \in S^n : \|x'\| \lesseqgtr t\}, \\ \mathbf{B}^{n+1}_{\lesseqgtr t} &= \{(z_1, x') \in \mathbf{B}^{n+1} : \|x'\| \lesseqgtr t\}. \end{aligned}$$

We also refer the reader to Section 1.3 for the relationship between ψ, Ψ, $\phi = \mathrm{Inv}(\psi)$ and Φ, and recall that $S^n_\phi = \Psi(S^n)$. Further, we recall that in light of Lemma 2.1, if ψ is sufficiently small, we may assume that $\Psi(\mathrm{Sing}(S^n)) = \mathrm{Sing}(S^n_\phi)$ and $\|\Psi - \mathbf{I}\|_{\mathcal{C}^2} \approx 0$. For the sake of convenience, we denote the extension of $\Psi \circ \iota$ to the tubular neighborhood $K \supset S^n$ by $\widetilde{\Psi}$.

Theorem 4.1. *Let $k \geq 1$. Given δ small enough, there is a $t \in (0,1)$ and an $\varepsilon_t > 0$ such that for all $\psi \in \mathcal{C}^{2k+1}(S^n; \mathbb{C})$ with $\|\psi\|_{\mathcal{C}^3(S^n;\mathbb{C}^n)} < \varepsilon_t$, there is a $\mathcal{C}^k$-diffeomorphism $\varphi : \mathbf{B}^{n+1}_{<t} \to \mathbb{C}^n$ such that*

- *(i)* $\varphi(S^n_{<t}) \subset S^n_\phi$, *and for each* $\mathbf{t} \in D^{n-1}(t)$, $\Delta_{\mathbf{t}} := \varphi\big(\{(z_1, x') \in \mathbf{B}^{n+1} : x' = \mathbf{t}\}\big)$ *is an analytic disc attached to* S^n_ϕ.
- *(ii)* $\|\varphi - \mathbf{I}\|_{\mathcal{C}^1(\mathbf{B}^{n+1}_{<t})} < \delta^2$.
- *(iii) There exist* $0 < t_1 < t < t_2 < 1$ *such that* $\Psi(S^n_{<t_1}) \Subset \varphi(S^n_{<t}) \Subset \Psi(S^n_{<t_2})$.
- *(iv) There is a* $t_3 < t$ *such that for* $\|\mathbf{t}\| \in (t_3, t)$, $\mathrm{diam}(\Delta_{\mathbf{t}}) < 7\delta$ *and* $\sup_{z \in \Delta_{\mathbf{t}}} \mathrm{dist}(z, \mathit{Sing}(S^n_\phi)) < 7\delta$.

Moreover, φ has the same regularity as ψ, when ψ is either $\mathcal{C}^\infty$-smooth or real-analytic on S^n.

Proof. Let $\delta \in (0,1)$ and $t = \sqrt{1-\delta^2}$. Let $\varepsilon_\eta > 0$ be as in Lemma 2.1 for $\eta = \delta^2$. Let $N_t \subset \mathcal{C}^3(K;\mathbb{C}^n)$ be as in Theorem 3.10 (and Theorem 3.11). We choose $\varepsilon(t) > 0$ so that $\|\psi\|_{\mathcal{C}^3(S^n;\mathbb{C}^n)} < \varepsilon(t)$ implies that $\phi = \mathrm{Inv}(\psi) \in N_t$. Finally, we set $\varepsilon_t = \min\{\varepsilon_\eta, \varepsilon(t), \delta^2\}$. Then, (i) and (ii) follow from the construction in the previous section.

For (iii), we let $t_1 = \sqrt{1-4\delta^2}$. Note that $\varphi(S^n_{<t_1}) \Subset \varphi(S^n_{<t})$ are connected open sets in S^n_ϕ, and if $z \in \partial S^n_{<t_1}$ and $w \in \partial S^n_{<t}$,

$$\begin{aligned}\big\|\varphi(z)-\varphi(w)\big\| &\geq \|z-w\| - \big\|\varphi(z)-z\big\| - \big\|\varphi(w)-w\big\| \\ &> \frac{\delta}{2} - \delta^2 - \delta^2 > 2\delta^2,\end{aligned}$$

for sufficiently small δ. Thus, the $(2\delta^2)$-neighborhood of $\varphi(S^n_{<t_1})$ in S^n_ϕ is compactly contained in $\varphi(S^n_{<t})$. But this neighborhood contains $\Psi(S^n_{<t_1})$ since $\|\varphi - \Psi\| < 2\delta^2$. Thus, we have half of (iii). For the second half of (iii), we set $t_2 = \sqrt{1-\delta^2/4}$ and repeat a similar argument.

For (iv), we note that since $\|\psi\|_{\mathcal{C}^3} < \varepsilon_\eta$, we have that $\|\Psi - \mathbf{I}\|_{\mathcal{C}^2(S^n)} < \delta^2$ (see Lemma 2.1). Hence, for $\|\mathbf{t}\| \in \left(\sqrt{1-8\delta^2}, \sqrt{1-\delta^2}\right)$, we have that for any $p, q \in \Delta_{\mathbf{t}}$,

$$\begin{aligned}\|p-q\| &\leq \Big\|p-\varphi^{-1}(p)\Big\| + \Big\|\varphi^{-1}(p)-\varphi^{-1}(q)\Big\| + \Big\|\varphi^{-1}(q)-q\Big\| \\ &\leq \delta^2 + 4\sqrt{2}\delta + \delta^2 < 7\delta,\end{aligned}$$

for sufficiently small δ. A similar argument also gives the second part of (iv). □

To construct M near $\mathrm{Sing}(S^n_\phi)$, we will rely on the deep work of Kenig-Webster and Huang (see [15] and [20], respectively), where the local hull of holomorphy of an n-dimensional submanifold in $\mathbb{C}^n$ at a nondengenerate elliptic CR singularity is completely described. Although their results are local, the proofs in [15] and [20] yield the following version of their result. Once again, we are using the compactness of $\mathrm{Sing}(S^n_\phi)$.

Theorem 4.2 (Kenig-Webster [15], Huang [20]). *Let $k >> 8$ and $m_k = \lfloor \frac{k-1}{7} \rfloor$. There exist $\delta_j > 0$, $j = 1, 2, 3$, and $\varepsilon^* > 0$ such that for any $\psi \in \mathcal{C}^k(S^n; \mathbb{C}^n)$ with $\|\psi\|_{\mathcal{C}^3(S^n;\mathbb{C}^n)} < \varepsilon^*$, there is a $\mathcal{C}^{m_k}$-smooth $(n+1)$-dimensional manifold $\widetilde{M}^{\phi}_{\delta_1,\delta_2}$ in $\mathbb{C}^n$ that contains some neighborhood of Sing(S^n_ϕ) in S^n_ϕ as an open subset of its boundary and is such that*

(a) *Any analytic disc $f : \Delta \to \mathbb{C}^n$ that is smooth up to the boundary with $f(\partial\Delta) \subset S^n_\phi$, $\operatorname{diam}(f(\Delta)) < \delta_1$ and $\sup_{z\in f(\Delta)} \operatorname{dist}(z, Sing(S^n_\phi)) < \delta_2$, is a reparametrization of a leaf in $\widetilde{M}^{\phi}_{\delta_1,\delta_2}$.*

(b) *$\Psi\left(\{z \in S^n : \operatorname{dist}(z, Sing(S^n)) < \delta_3\}\right) \Subset \partial\widetilde{M}^{\phi}_{\delta_1,\delta}$. Further, if $p \in \widetilde{M}^{\phi}_{\delta_1,\delta}$ is such that $\operatorname{dist}(\Psi^{-1}(p), Sing(S^n)) < \delta_3$, then there is an embedded disk, $f : \Delta \to \mathbb{C}^n$ (unique up to reparametrization) that is smooth up to the boundary, with $p \in f(\Delta)$, $f(\partial\Delta) \subset S^n_\phi$ and $f(\overline{\Delta}) \subset \widetilde{M}^{\phi}_{\delta_1,\delta_2}$, and the union of all such disks is a smooth $(n+1)$-dimensional submanifold, $\widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3'}$, of $\widetilde{M}^{\phi}_{\delta_1,\delta_2}$.*

(c) *If Π is the projection map $(z_1, x' + iy') \mapsto (y')$ on $\mathbb{C}^n$, then $\left\| \Pi\big|_{\widetilde{M}^{\phi}_{\delta_1,\delta_2}} \right\|_{\mathcal{C}^1} \approx 0$.*

Moreover, $\widetilde{M}^{\phi}_{\delta_1,\delta}$ has the same regularity as ψ, when ψ is either $\mathcal{C}^\infty$-smooth or real-analytic on S^n.

Now, given δ_j, $j = 1, 2, 3$, and $\varepsilon^* > 0$ as in Theorem 4.2, we let $\delta = \min\{\frac{\delta}{7}, \frac{\delta_2}{7}, \frac{\delta_3}{3}\}$ and $\varepsilon = \min\{\varepsilon_t, \varepsilon^*\}$, where $t > 0$ and $\varepsilon_t > 0$ correspond to δ as in Theorem 4.1 (shrinking δ further, if necessary). Then, for $\psi \in \mathcal{C}^k(S^n; \mathbb{C}^n)$ with $\|\psi\|_{\mathcal{C}^3(S^n;\mathbb{C}^n)} < \varepsilon$, we let

$$M_1 = \varphi(\mathbf{B}^{n+1}_{<t}) \cup \widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}.$$

We now proceed to show that this indeed leads to the desired manifold. First, by Theorem 4.1 (*iii*) and Theorem 4.2 (*b*),

$$M_1 = \Psi\left(S^n_{<\sqrt{1-4\delta^2}}\right) \cup \Psi\left(S^n_{\geq\sqrt{1-4\delta^2}}\right) \subset \partial M_1 \subseteq S^n_\phi.$$

This follows from the fact that $\operatorname{dist}(z, \operatorname{Sing}(S^n)) \lesssim 2\delta < \delta_3$, when $z \in S^n_{\geq\sqrt{1-4\delta^2}}$.

Next, for the foliated structure and the regularity of M, we need only focus on $\varphi(\mathbf{B}^{n+1}_{<t}) \cap \widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}$. Let $p \in \varphi(\mathbf{B}^{n+1}_{<t}) \cap \widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}$. Then, $p = \varphi(z_1, \mathbf{t})$ for some $(z_1, \mathbf{t}) \in \mathbf{B}^{n+1}_{<t}$, where recall that $t = \sqrt{1-\delta^2}$. We first assume that $\|\mathbf{t}\| > t_3 = \sqrt{1-8\delta^2}$. Then, by the choice of δ and Theorem 4.1 (iv), $\operatorname{diam}(\Delta_{\mathbf{t}}) < \delta_1$, $\sup_{z\in\Delta_{\mathbf{t}}} \operatorname{dist}(z, \operatorname{Sing}(S^n_\phi)) < \delta_2$ and $\operatorname{dist}(p, \operatorname{Sing}(S^n_\phi)) < \delta_3$. Thus, by Theorem 4.2 (b), $\overline{\Delta}_{\mathbf{t}} \subset \varphi(\mathbf{B}^{n+1}_{<t}) \cap \widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}$. By this argument, we see that the smooth $(n+1)$-dimensional manifold

$$B_{t_3,t} := \bigcup_{t_3<\|\mathbf{s}\|<t} \overline{\Delta}_{\mathbf{t}}$$

lies in $\varphi(\mathbf{B}^{n+1}_{<t}) \cap \widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}$. Thus, M is a smooth manifold in a neighborhood of p.

Next, suppose $p = \varphi(z_1, \mathbf{t}) \in \varphi(\mathbf{B}^{n+1}_{<t}) \cap \widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}$ is such that $\|\mathbf{t}\| \le \sqrt{1-8\delta^2}$. We observe that the complement of $\partial B_{t_3,t}$ in $S^n_\phi \cap \partial \widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}$ consists of two disjoint submanifolds of S^n_ϕ — one, say S_{I}, containing $\operatorname{Sing}(S^n_\phi)$ and contained in a (2δ)-neighborhood of $\operatorname{Sing}(S^n_\phi)$, and another, say S_{II}, with the property that $\operatorname{dist}(S_{\mathrm{II}}, \operatorname{Sing}(S^n_\phi)) = 2\sqrt{2}\delta + O(\delta^2) > 2\delta$. Since $p \in \widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}$, it lies on some analytic disc $f(\Delta)$ attached to $S^n_\phi \cap \partial M^{\phi}_{\delta_1,\delta_2,\delta_3}$. By the uniqueness of these discs, $f(\partial\Delta)$ cannot intersect $\partial B_{t_3,t}$ because any disc whose boundary intersects $\partial B_{t_3,t}$ lies completely in $B_{t_3,t}$ (as seen above), and $p \in f(\Delta)$ does not. Thus, either $f(\partial\Delta) \subset S_{\mathrm{I}}$ or $f(\partial\Delta) \subset S_{\mathrm{II}}$ (as the two are disjoint). But since S_{I} lies in the tubular (2δ)-neighborhood of $\operatorname{Sing}(S^n_\phi)$, which is a polynomially convex set, we must have that if $f(\partial\Delta) \subset S_{\mathrm{I}}$, then $\operatorname{dist}(p, \operatorname{Sing}(S^n_\phi)) < 2\delta$. This contradicts the fact that $p = \varphi(z_1, \mathbf{t})$ with $\|\mathbf{t}\| \le \sqrt{1-8\delta^2}$. Thus, $f(\partial\Delta) \subset S_{\mathrm{II}}$. This, and the fact that

$$\Psi\left(S^n_{\ge\sqrt{1-4\delta^2}}\right) \subset S_{\mathrm{I}} \cup \partial B_{t_3,t}$$

shows that if we shrink $\widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}$ by removing S_{II} and the discs attached to it, then

$$M = \varphi(\mathbf{B}^{n+1}_{<t}) \cup \widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}$$

is an $(n+1)$-dimensional manifold, as smooth as $\widetilde{M}^{\phi}_{\delta_1,\delta_2,\delta_3}$, and is foliated by analytic discs attached to its boundary S^n_ϕ. Moreover, M is a $\mathcal{C}^1$-small perturbation of $\mathbf{B}^{n+1}$ in $\mathbb{C}^n$.

Chapter 5

Holomorphic/Polynomial Convexity

5.1 Preliminary Results

5.1.1 M as a graph

Let $\Pi : \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{R}^{n-1}$ be the map $(z_1, x' + iy') \mapsto (z_1, x')$. For $\psi \in \mathcal{C}^k(S^n; \mathbb{C}^n)$ as above, we note that since S^n_ϕ is a $\mathcal{C}^3$-small perturbation of S^n, we may write $S^n_\phi = \text{Graph}(h)$ for some $\mathcal{C}^k$-smooth $h : \partial\Omega \to \mathbb{R}^{n-1}$, where Ω is a $\mathcal{C}^k$-smooth strongly convex domain in $\mathbb{C}_{z_1} \times \mathbb{R}^{n-1}_{x'}$, and h and $\partial\Omega$ are $\mathcal{C}^3$-small perturbations of the constant zero map and S^n_ϕ, respectively. We make two observations. Since S^n_ϕ lies in $\overline{\Omega} \times i\mathbb{R}^{n-1}$, which is strongly convex, $M \subset \overline{\Omega} \times i\mathbb{R}^{n-1}$ with $\text{int}\, M \subset \Omega \times i\mathbb{R}^{n-1}$.

Next, since $T_p(M)$ at any $p \in M$ is a small perturbation of $T_{\Pi(p)}(\overline{\Omega})$ (as manifolds with boundary in $\mathbb{C}^n$), $\Pi : M \to \overline{\Omega}$ is a local diffeomorphism that restricts to a diffeomorphism between S^n_ϕ and $\partial\Omega$. Thus, Π extends to a $\mathcal{C}^{m_k}$-smooth diffeomorphism from M to $\overline{\Omega}$, and we may write $M = \text{Graph}\,(h^*)$ for some $\mathcal{C}^1$-small $h^* : \overline{\Omega} \to \mathbb{R}^{n-1}$.

5.1.2 On the analytic extendability of M

In this section, we fix our attention on real-analytic perturbations of S^n. So far, we have: given $\delta > 0$, there is an $\varepsilon > 0$ so that for any any $\psi \in \mathcal{C}^\omega(S^n; \mathbb{C})$ with $\|\psi\|_{\mathcal{C}^3(S^n)} < \varepsilon$, there is a $\mathcal{C}^\omega$-domain $\Omega_\phi \subset \mathbb{C} \times \mathbb{R}^{n-1}$, and a $\mathcal{C}^\omega$-map $H : \overline{\Omega}_\phi \to \mathbb{R}^{n-1}$, such that

- ⋆ $\partial\Omega_\phi$ and $H|_{\partial\Omega_\phi}$ are ε-small perturbations (in $\mathcal{C}^3$-norm) of S^n and the zero map, respectively,
- ⋆ $\text{Graph}_{\Omega_\phi}(H)$ is foliated by an $(n-1)$-parameter family of embedded analytic discs attached to S^n_ϕ, and $\|H\|_{\mathcal{C}^1(\overline{\Omega}_\phi)} < \delta$.

In this subsection, we show that given $\rho > 0$, $\delta > 0$, there is a $\rho' > 0$, $\varepsilon > 0$ such that

$$\text{if the complexified map } \psi_{\mathbb{C}} \text{ extends holomorphically to } \mathcal{N}_\rho S^n_{\mathbb{C}} \text{ with } \sup_{\overline{\mathcal{N}_\rho S^n_{\mathbb{C}}}} \|\psi\| < \varepsilon,$$
$$\text{then } H \text{ extends real-analytically to } (1+\rho')\Omega_\phi, \text{ and } \|H\|_{\mathcal{C}^2((1+\rho')\overline{\Omega}_\phi)} < \delta.$$

Near $\mathrm{Sing}(S^n_\phi)$, this follows from the results in [15] and [20], where uniform analytic extendability of the local hulls of holomorphy past real-analytic nondegenerate elliptic points is established . Away from $\mathrm{Sing}(S^n_\phi)$, we obtain this by complexifying the construction of M_{TR}, and establishing a lower bound on the radius of convergence of its parametrizing map $\mathcal{F}_\phi : \overline{\Delta} \times D^{n-1}(t) \to \mathbb{C}^n$ for every ϕ (or ψ) sufficiently small. We briefly elaborate on this below.

In order to complexify the construction in Section 3.2, we need to expand our collection of function spaces. First, recall that $S^n_{\mathbb{C}} = \{(z, \overline{z}) \in \mathbb{C}^{2n} : z \in S^n\}$ and $\mathcal{N}_r S^n_{\mathbb{C}} = \{\xi \in \mathbb{C}^{2n} : \mathrm{dist}(\xi, S^n_{\mathbb{C}}) < r\}$. For $s \in (0,1)$, we set, $\Delta_s = (1+s)\Delta$ and $\mathrm{Ann}_s = \{z \in \mathbb{C} : 1 - s < |z| < 1 + s\}$. We define $\mathcal{A}^{1,\alpha}(\partial\Delta_s)$ and $\mathcal{A}^{1,\alpha}(\mathrm{Ann}_s)$ in analogy with $\mathcal{A}^{1,\alpha}(\partial\Delta)$; see (3.1). For any open set $U \in \mathbb{C}^n$, we let $A(U)$ be the Banach spaces of continuous functions on $\overline{U}$, whose restrictions to U are holomorphic.

$$\begin{aligned}
X^n(s) &= \mathcal{A}^{1,\alpha}(\partial\Delta_s; \mathbb{C}^n) \times \mathcal{A}^{1,\alpha}(\mathrm{Ann}_s; \mathbb{C}^n), \\
X^n_{\mathbb{R}}(s) &= \{(f, h) \in X^n(s) : h|_{\partial\Delta} = \overline{f}|_{\partial\Delta}\}, \\
Y^n(r) &= A(\mathcal{N}_r S^n_{\mathbb{C}}; \mathbb{C}^n), \\
Y^{2n}_{\mathbb{R}}(r) &= \{(\varphi_1, ..., \varphi_{2n}) \in Y^{2n}(r) : \varphi_2(z, \overline{z}) = \overline{\varphi_1}(z, \overline{z}), \mathrm{Im}\, \phi_j(z, \overline{z}) = 0, j = 3, ..., 2n\}, \\
Z^n(r, s) &= \{(\varphi, \eta, f, h) \in Y^{2n}(r) \times X^n(s) : (f, h)(\mathrm{Ann}_s) \subset \mathcal{N}_r S^n_{\mathbb{C}}\}, \\
Z^n_{\mathbb{R}}(r, s) &= Z^n \cap \left(Y^{2n}_{\mathbb{R}}(r) \times X^n_{\mathbb{R}}(s)\right).
\end{aligned}$$

We need the bounded linear map $K_{r,s} : \mathbb{R} \times Y^n(2r) \times \mathcal{A}^{1,\alpha}(\partial\Delta_{2s}; \mathbb{C}^n) \to \mathbb{C} \times Y^{2n}(r) \times X^n(s)$ given by

$$(x, \phi_1, ..., \phi_n, f) \mapsto (x + i0, \underbrace{\phi_1, \phi_1^*, (\mathrm{Re}\, \phi_2)^*, (\mathrm{Im}\, \phi_2)^* ..., (\mathrm{Re}\, \phi_n)^*, (\mathrm{Im}\, \phi_n)^*}_{=:(\phi, \phi^*)}, f, f^*),$$

where ϕ_1^*, $(\operatorname{Re}\phi_j)^*$, $(\operatorname{Im}\phi_j)^*$ and f^* are obtained by taking the holomorphic extensions of the real analytic functions $\overline{\phi_1}\big|_{S^n_{\mathbb{C}}}$, $(\operatorname{Re}\phi_j)\big|_{S^n_{\mathbb{C}}}$, $(\operatorname{Im}\phi_j)\big|_{S^n_{\mathbb{C}}}$, and $\overline{f}\big|_{\partial\Delta}$, respectively. To keep the exposition short, we will now only discuss this for the case $n = 2$.

Now, fixing $r = \rho/2$ and $s = r/2$, and dropping all inessential references to r and s, we solve the following complexified version of (3.10) on Ann_s: given $\varphi \in Y^2$, find $(f, h) \in X^2$ satisfying

$$(f_1 - \varphi_1(f,h))(h_1 - \varphi_2(f,h)) + \left(\frac{f_2 + h_2}{2} - \varphi_3(f,h)\right)^2 = 1$$
$$f_2 - h_2 = \varphi_4(f,h),$$

so that $(f, h) \in X^2_{\mathbb{R}}$ if $\varphi \in Y^2_{\mathbb{R}}$. For this, we first define the following maps on $\mathbb{C} \times Z^2$.

$$\begin{aligned}
\Sigma^{\mathbb{C}} \quad &: \quad (\eta, \varphi, f, h) \mapsto \big(\eta + H_{\mathbb{C}}(\varphi_4(f,h)) - \varphi_3(f,h)\big)^2, \text{ and} \\
P^{\mathbb{C}} \quad &: \quad (\eta, \varphi, f, h) \mapsto \big(\phi_1(f,h), \varphi_2(f,h), 1 - \Sigma(\varphi, \eta, f, h)\big),
\end{aligned}$$

where $H_{\mathbb{C}} : \mathcal{A}^{1,\alpha}(\mathrm{Ann}_s) \to \mathcal{A}^{1,\alpha}(\mathrm{Ann}_s)$ is the complexified Hilbert transform (see [12]). We let $\Omega^{\mathbb{C}} \subset A(\mathrm{Ann}_r)^2 \times A(\mathrm{Ann}_r; \mathbb{C} \setminus (-\infty, 0))$ be the domain of the operator $E^{\mathbb{C}}$ obtained by complexifying the map E constructed in Lemma 3.2. The range of $E^{\mathbb{C}}$ lies in X^1, and if $(f, h) = E^{\mathbb{C}}(\varphi, \sigma)$, then

- on Ann_s, $(f - \varphi_1)(h - \varphi_2) = \sigma$,
- if $\varphi \in Y^2_{\mathbb{R}}$ and $\sigma|_{\partial\Delta} > 0$, then $(f, h) = (E(\phi, \sqrt{\sigma}), \overline{E(\phi, \sqrt{\sigma})})$ on $\partial\Delta$, i.e., $(f, h) \in X^2_{\mathbb{R}}$,
- for $c \in \mathbb{C} \setminus (-\infty, 0]$, $E^{\mathbb{C}}(0, 0, c) = (\sqrt{c}\xi, \sqrt{c}/\xi)$.

Finally, we set $\mathcal{W}^{\mathbb{C}} = \{\zeta \in \mathbb{C} \times Z^2 : P^{\mathbb{C}}(\zeta) \in \Omega^{\mathbb{C}}\}$, and define the map $R^{\mathbb{C}} : \mathcal{W}^{\mathbb{C}} \to X^2$ as follows

$$\zeta = (\eta, \varphi, f, h) \mapsto (f, g) - \big(E^{\mathbb{C}} \circ P^{\mathbb{C}}(\zeta), \eta + H_{\mathbb{C}}(\varphi_4(f,h)) + i\varphi_4(f,h), \eta + H_{\mathbb{C}}(\varphi_4(f,h)) - i\varphi_4(f,h)\big).$$

All the complexified maps constructed are holomorphic on their respective domains, and therefore, so is $R^{\mathbb{C}}$. Moreover, $(\eta, 0, \mathfrak{g}_\eta) \in \mathcal{W}^{\mathbb{C}}$, $R^{\mathbb{C}}(0, \eta, \mathfrak{g}_\eta) = 0$ and $D_3 R^{\mathbb{C}}(0, \eta, \mathfrak{g}_\eta) = \mathbf{I}$, for $\eta \in Q(1, s) = (-1, 1) \times (-is, is)$, where

$$\mathfrak{g}_\eta(\xi) = (\sqrt{1-\eta^2}\xi, \eta, \sqrt{1-\eta^2}\xi^{-1}, \eta).$$

Thus, by repeating the argument in §3.2, given $t_0 < 1 - s$, $s_0 < s$, there is an $\varepsilon > 0$, such that for each $\|\varphi\|_{Y^4} < \varepsilon$, there is a holomorphic embedding $\mathcal{F}^{\mathbb{C}}_\varphi : \Delta_s \times Q(t_0, s_0) \to \mathbb{C}^4$ whose image is a disjoint union of analytic discs in $\mathbb{C}^4$ with boundaries in $\mathbf{S}^2_{\mathbb{C}}$. Moreover, there is a $C > 0$ (independent of φ) such that $\sup_{\Delta_s \times Q(t_0,s_0)} \left\| \mathcal{F}^{\mathbb{C}}_\varphi \right\| \leq C \sup_{\mathcal{N}_r S^n} \|\varphi\|$. By shrinking s, t_0, s_0 slightly, and using Cauchy estimates, we can ensure that for a given $\delta > 0$,

$$\left\| \mathcal{F}^{\mathbb{C}}_\varphi \right\|_{\mathcal{C}^2\left(\overline{\Delta_s \times Q(t_0,s_0)}\right)} < \delta, \qquad \text{for all } \|\varphi\|_{Y^4} < \varepsilon. \tag{5.1}$$

Now, let $\varphi = (\phi, \phi^*) \in Y^4_{\mathbb{R}}$ with $\|\varphi\|_{Y^4} < \varepsilon$. Setting $\mathcal{F}_\phi = \pi \circ \mathcal{F}^{\mathbb{C}}_\varphi \big|_{\Delta_s \times (-t_0, t_0)}$, where $\pi : \mathbb{C}^4_{z,w} \mapsto \mathbb{C}^2_z$ is the projection map, we have

(*a*) $\mathcal{F}_\phi : \overline{\Delta} \times (-t_0, t_0) \to \mathbb{C}^2$ is an anlytic map with radius of convergence at least s_0,

(*b*) $\mathcal{F}_\phi(\partial\Delta \times \{t\}) \subset \mathbf{S}^2_\phi$ for every $t \in (-t_0, t_0)$. This follows from the fact that $R^{\mathbb{C}}$ complexifies the map $R^{\mathbb{R}} : (\mathbf{t}, \phi, f) \mapsto \pi \circ R^{\mathbb{C}}(\mathbf{t} + i0, K(\phi, f))$, and $R^{\mathbb{R}} = 0$ gives equations (3.10).

(*c*) $M' = \mathcal{F}_\phi(\Delta_s \times (-t_0, t_0)) \subset \mathbb{C}^2$ is an embedded 3-manifold with boundary that is a graph over a domain $\Omega' \subset \mathbb{C} \times \mathbb{R}$. Due to (*a*) above, there is a $\rho' > 0$ (depending only on ρ and δ) such that $(1 + \rho')\Omega_\phi \subset \Omega'$.

5.2 Polynomial Hull of S^n_ϕ

We note that if M is as constructed in the previous section, then due to its foliated structure, M is contained in both the schlicht part of $\widetilde{S^n_\phi}$, and in $\widehat{S^n_\phi}$. In this section, we show that when the perturbations are real-analytic and admit a uniform lower bound on their radii of convergence, then M is in fact polynomially convex. This will complete the proof of Theorem 1.1. Our strategy is to globally ‘flatten’ M, which allows for M to be expressed as the intersection of $n-1$ Levi-flat hypersurfaces, to each of which we can apply Lemma 5.2. We note that when $n=2$, the flattening is unnecessary, and the final claim follows directly from Lemma 5.2 (as seen in Bedford’s paper [3]).

There is a neighborhood $\mathcal{W}$ of $\overline{\Omega}_\phi$ in $\mathbb{C}^n$ and a biholomorphism $G:\mathcal{W}\to\mathbb{C}^n$ such that $M\Subset G(\mathcal{W})$ and $normG-\mathbf{I}_{\mathcal{C}^1}\lesssim\delta$.

Proof. We let $M'=\{(z_1,z')\in V_{\rho'/2}M_\phi:\overline{z}'=\mathfrak{H}(z_1,\overline{z_1},z')\}$, where ρ' and $\mathfrak{H}$ are as in section 5.1.1. since M' is a small perturbation of Graph(0) and is foliated by analytic discs, it admits a tangential $(1,0)$-vector field, $L=\frac{\partial}{\partial z_1}+a_2\frac{\partial}{\partial z_2}+\cdots a_n\frac{\partial}{\partial z_n}$, $a_2,...,a_n\in\mathcal{C}^\omega(M';\mathbb{C})$, such that $[L,\overline{L}]\in\text{span}\{L,\overline{L}\}$ mod $HM'\otimes_{\mathbb{R}}\mathbb{C}$ on M'. The conditions on L give that

(a) $\overline{L}(\boldsymbol{a})\equiv 0$ on M', i.e., $\boldsymbol{a}$ is a CR-map on M', where $\boldsymbol{a}=(a_2,...,a_n)$, and

(b) $\boldsymbol{a}(z_1,z')=\dfrac{\partial\mathfrak{H}}{\partial\overline{z_1}}(z_1,\overline{z_1},z')$ along M', since $L(\overline{z}'-\mathfrak{H}(z_1,\overline{z_1},z'))=0$.

Thus, we get that $\boldsymbol{a}$ extends as a holomorphic map, say $\boldsymbol{A}$, to some neighborhood of M'. Since, $\mathfrak{H}$ (and, therefore $\boldsymbol{a}$) has radius of convergence at least $\rho'/2$ on M', $\boldsymbol{A}$ is holomorphic on $V_{\rho'/2}(M_\phi)$. Further, we have that $\boldsymbol{A}(z_1,z')=\boldsymbol{a}(z_1,\overline{z_1},z',\mathfrak{H}(z_1,\overline{z_1},z'))$ on $V_{\rho'/2}(M_\phi)$, which gives the bound $\|\boldsymbol{A}\|_{\mathcal{C}^1}\lesssim\delta$ on $V_{\rho'/2}(M_\phi)$ (since $\|a\|_{\mathcal{C}^1}<\delta$ on M', from (b)).

We now construct the flattening map. By applying the implicit function theorem to the equation $\overline{z}'=\mathfrak{H}(z_1,\overline{z_1},z')$ on $V_{\rho'/2}(M_\phi)$, we can solve for y' in terms of x_1, y_1 and z' to write $M'=\text{Graph}_{\Omega'}H$, where Ω' is the $(1+\rho'/2)$-tubular neighborhood of Ω_ϕ in $\mathbb{C}\times\mathbb{R}^{n-1}$, and $H:\Omega'\to\mathbb{R}^{n-1}$ is a $\mathcal{C}^\omega$-map with $\|H\|_{\mathcal{C}^1}\lesssim\delta$. Shrinking ε further, we may assume that $\Omega_\phi\subset B\subset\Omega'$, where $B=(1+\rho'/4)\overline{\mathbf{B}^{n+1}}$. Given $(z_1,x')\in B$, we let $w(z_1,x')=x'+iH(x_1,y_1,x')$. Now, on the metric space $\mathcal{F}=\{g\in\mathcal{C}(B;\mathbb{R}^{n-1}):\sup_B\|g-w\|<\rho'/2\}$, endowed with the sup-norm, we consider the map

$$Q:g\mapsto(Qg)(z_1,x')=x'+iH(0,0,x')+\int_0^{z_1}A(\xi,g(\xi,x'))d\xi.$$

To see this, note that for g, g_1, $g_2\in\mathcal{F}$, we have

$$\sup_B \|Qg - w\| \leq \sup_B \big\|H(0,0,x') - H(x_1,y_1,x')\big\| + \sup_{V_{\rho'/2}(M_\phi)} \|A\| \operatorname{diam}(B) \lesssim \delta\left(1+\tfrac{\rho'}{4}\right),$$
$$\sup_B \|Qg_1 - Qg_2\| \leq \sup_{V_{\rho'/2}(M_\phi)} \|DA\| \operatorname{diam}(B) \sup_B \|g_1 - g_2\| \lesssim \delta\left(1+\tfrac{\rho'}{4}\right) \sup_B \|g_1 - g_2\|.$$

Shrinking $\varepsilon > 0$ further, if necessary, we can ensure that $\delta(1+\rho'/4) < \min\{\rho'/2, 1\}$. Thus, $Q(\mathcal{F}) \subset \mathcal{F}$, and Q is a contraction, i.e., $\|Qg_1 - Qg_2\|_{\mathcal{F}} < \|g_1 - g_2\|_{\mathcal{F}}$, for all $g_1, g_2 \in \mathcal{F}$. By the Banach fixed point theorem, there is a unique $g_0 \in \mathcal{F}$ such that $Q(g_0) = g_0$. In other words, $G : (z_1, x') \mapsto (z_1, g_0(z_1, x'))$ is a solution of the flow equation

$$\begin{aligned} \frac{\partial g}{\partial z_1}(z_1, x') &= \left(1, \boldsymbol{A}(z_1, g(z_1, x'))\right), && \text{on } B, \\ g(0, x') &= x' + iH(0,0,x'), && \text{on } B_0 = B \cap \{z_1 = 0\}. \end{aligned}$$

By the local uniqueness and regularity of solutions to quasilinear PDEs with real-analytic Cauchy data, G must be real-analytic in z_1 and x'. Moreover, $\|G - \mathbf{I}\|_{\mathcal{C}^1(B)} \lesssim \delta$. Thus, G extends to a biholomorphism in some neighborhood $\mathcal{W}$ of B. Now, since $G_*(\partial/\partial z_1) = L$ and $G(B_0) \subset M'$, by the uniqueness of integral curves, $G(B) \subset M'$. Finally, if $z \in \partial B$, then $\big\|\Pi \circ G(z) - z\big\| \lesssim \delta$, where $\Pi : \mathbb{C}^n \to \mathbb{C} \times \mathbb{R}^{n-1}$ is the projection map, and δ can be made sufficiently small (by shrinking ε) so that $\Omega_\phi \subset (\Pi \circ G)(B)$, and thus, $M \Subset G(B) \subset G(\mathcal{W})$. This settles our claim. □

Now, to complete the proof of the polynomial (and holomorphic convexity) of M, we need the following lemma.

Let $D' \subset \mathbb{C}^{n-1} \times \mathbb{R}$ be a domain containing the origin, and $F : D' \to \mathbb{R}$ be a smooth function such that $\mathcal{L}' = \mathrm{Graph}_{D'}(F)$ is a Levi-flat hypersurface. Then, for any strongly convex domain $D \Subset D'$ containing the origin, the set $\mathcal{L} = \mathrm{Graph}_{\overline{D}}(F)$ is polynomially convex.

Proof. We fix a $t_0 \in (0,1)$ such that $D_t = (1+t)D \Subset D'$ for all $t \leq t_0$. Now, set $C = 2(t_0 + \sup_{\overline{D}_{t_0}} |F|)$. Since $\overline{D}_{t_0} \times [-iC, iC]$ is polynomially convex in $\mathbb{C}^n$, by a theorem due to Docquier and Grauert (see [?]), it suffices to produce a family of pseudoconvex domains, $\{U_t\}_{0<t}$ in $D_{t_0} \times (-iC, iC)$ such that

$$\overline{U_s} \subset U_t \text{ if } s < t, \quad \bigcap_{s>t} \operatorname{int} U_s = U_t, \quad \bigcup_{s<t} U_s = U_t, \quad \mathcal{L} = \bigcap_{0<t} U_t, \text{ and } D_{t_0} \times (-iC, iC) = \bigcup_{0<t} U_t.$$

We use the notation (z^*, w) to denote a point in $\mathbb{C}^{n-1} \times \mathbb{C}$, with $w = u + iv$. Now, consider the following pseudoconvex domains.

$$U_t = \begin{cases} \{(z^*, w) : (z^*, u) \in D_t, \left|v - F(z^*, u)\right| < t\}, & 0 < t \leq t_0, \\ \{(z^*, w) : (z^*, u) \in D_{t_0}, \max(-C, F(z^*, u) - t) < v < \min(F(z^*, u) + t, C)\}, & t > t_0. \end{cases}$$

The claim now follows. □

Finally, given $j = 2, ..., n$, let Y_j denote the hyperplane $\{z \in \mathbb{C}^n : \operatorname{Im} z_j = 0\}$. We set

$$\mathcal{L}'_j = G\left(\mathcal{W} \cap Y_j\right)$$

Shrinking ε further, if necessary, we have that $\mathcal{L}'_j$ is a graph of some smooth function F^j over some open set $D'_j \subset Y_j \cong \mathbb{C}^{n-1} \times \mathbb{R}$ such that $\Omega_\phi \Subset D'_j \subset \mathcal{W} \cap Y_j$. We now choose a strongly convex domain $\mathcal{E} \subset \mathbb{C}^n$ such that

* $\mathcal{E} \cap Y_j \Subset D'_j$, and
* $\mathcal{E} \cap Y_2 \cap \cdots \cap Y_n = \Omega_\phi$.

This can be obtained, for instance, by letting $\mathcal{E} = \{\tau^2 p(z, x') + \left\|y'\right\|^2 < 0\}$, where p is a smooth strongly convex exhaustion function of Ω_ϕ with $p \geq -1$ (see [?]), and $\tau > 0$ is small enough. Now, we apply Lemma 5.2 to D'_j, F^j and $D_j = \mathcal{E} \cap Y_j$, and conclude that $\mathcal{L}_j = \text{Graph}_{\overline{\mathcal{E}_j}}(F^j)$ is polynomially convex. However,

$$M = \bigcap_{j=2}^{n} \mathcal{L}_j.$$

Thus, M is polynomially convex.

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