# GENERALIZED BRAUER DIMENSION OF SEMI-GLOBAL FIELDS

by

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## ABSTRACT OF THE DISSERTATION

# Generalized Brauer Dimension Of Semi-global Fields

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Let $F$ be a one variable function field over a complete discretely valued field with residue field $k$. Let $n$ be a positive integer, coprime to the characteristic of $k$. Given a finite subgroup $B$ in the $n$-torsion part of the Brauer group ${}_n\operatorname{Br}(F)$, we define the index of $B$ to be the minimum of the degrees of field extensions which split all elements in $B$. In this thesis, we improve an upper bound for the index of $B$, given by Parimala-Suresh, in terms of arithmetic invariants of $k$ and $k(t)$. As a simple application of our result, given a quadratic form $q/F$, where $F$ is a function field in one variable over an $m$-local field, we provide an upper-bound to the minimum of degrees of field extensions $L/F$ so that the Witt index of $q \otimes L$ becomes the largest possible.

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# Dedication

To the loving memory of my grandparents: Ajji, Ajoba, Mai, Baba.

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# Chapter 1

# Introduction

## 1.1 Two Motivating Problems

The question how arithmetic properties of a field transmit to its finitely generated field extensions has been explored throughout the years. Let $F$ be a field. The $u$-invariant of $F$ is the smallest natural number $u(F)$ such that all degree two homogenous polynomials with coefficients in $F$, and number of variables greater than $u(F)$, have a non-trivial solution. The $u$-invariant is an example of an arithmetic invariant. A natural question arises: how does the $u$-invariant behave under finitely generated field extensions of $F$? If $u(F)$ is finite, then can we say that $u(F(t))$ is finite? These questions are yet to be answered. Even for nicer fields, say when $F$ is a totally imaginary number field, it is not known if $u(F(t))$ is finite.

The finiteness of the $u$-invariant is related to the finiteness of other arithmetic invariants of $F$, coming from the Brauer group. The Brauer group of $F$ can be thought of as the group of Brauer equivalence classes of finite dimensional central simple algebras over $F$ (see Definition 2.1.13) under the operation of tensor product. The identity element of this group is the class of the field $F$. Given a division algebra $D/F$, we can associate two numbers to it, namely the period of $D$, which is simply its order in the Brauer group, and the index of $D$ which is the minimum degree of the field extension that makes $D$ trivial in the Brauer group of that extension. Alternatively the index can also be defined as the square root of the dimension of the division algebra. For example, consider the Hamilton quaternion algebra $(-1,-1)$ over $\mathbb{R}$. One can show that the tensor product $(-1,-1) \otimes_{\mathbb{R}} (-1,-1)$ is Brauer equivalent to the class of the identity element in the Brauer group, i.e., $\mathbb{R}$. This shows that the period of $(-1,-1)$ is two. Since there are no nontrivial division algebras over an algebraically closed field, $(-1,-1)\otimes_{\mathbb{R}}\mathbb{C}$ is trivial in the Brauer group of $\mathbb{C}$. Thus its index is also two.

It turns out that the period divides the index and they share the same prime factors. As a measure of how complicated can the division algebras be, one asks the following question: given any division algebra $D$ over $F$ of a fixed period $\ell$, can we uniformly bound its index in terms of $\ell$? It is not difficult to construct fields where there can be no such uniform bound. However, for nice fields such as finitely generated fields over number fields, finite fields, local fields or algebraically closed fields, there is a folklore conjecture that there is such a uniform bound. This problem is known as the period-index problem. The uniform bound has been referred to as the Brauer $\ell$-dimension in recent times. The question of finiteness of the Brauer $\ell$-dimension for function fields in one variable over number fields remains open. Finiteness of the Brauer 2-dimension in the case of function fields in one variable over totally imaginary number fields, will imply that their $u$-invariant is also finite, in view of a recent result of Suresh (see [Su]).

## 1.2 Generalized Brauer Dimension

In addition to the $u$-invariant and the Brauer dimension, we have another measure of complexity of the arithmetic of the field. Given any finite collection of division algebras $\{D_1,\cdots,D_n\}$ over $F$ of a fixed period $\ell$, can we uniformly bound the degree of field extensions $L/F$ such that $D_i\otimes_F L$ is trivial in the Brauer group of $L$, for every $i=1,\cdots,n$? Again, it is reasonable to ask a similar question only over the nice fields mentioned in the previous paragraph. The existence of such a uniform bound means that division algebras cannot be too "independent". We will briefly expand on this heuristic: it is not too hard to show that there is a uniform bound on the degree of the field extension that kills a finite collection of square classes if and only if the group of square classes is finite. While a uniform bound for division algebras of period two does not mean that the number of division algebras is finite, it does impose some constraints on division algebras of a given period. For example, one can show that there exist central simple algebras, Brauer equivalent to those division algebras, and containing a Galois field extension, with a uniform bound on the cardinality of the Galois group. The smallest uniform bound on the degree of field extensions that split any finite collection of Brauer classes of period dividing $\ell$ is called the Generalized Brauer $\ell$-dimension (see 5.1.1 for a formal definition). The Generalized Brauer $\ell$-dimension was first introduced before by Parimala and Suresh in [Pa-Su3], where they refer to it as "uniform $(2, \ell)$ bound". We prefer the terminology "*Generalized Brauer $\ell$-dimension*" due to its similarity with the Brauer $\ell$-dimension. We will denote the Generalized Brauer $\ell$-dimension of a field $F$ by $\mathrm{GBrd}_\ell(F)$. One knows the Generalized Brauer $\ell$-dimension only for "lower dimensional" fields such as global fields and local fields, and for function fields of curves over local fields. Even the finiteness of the Generalized Brauer $\ell$-dimension is not known for function fields of higher dimensional varieties over local fields, finite fields and algebraically closed fields.

The finiteness of Generalized Brauer $\ell$-dimension provides some handle on the complexity of Galois cohomology classes in degree three and higher, which is otherwise difficult to understand. Combined with the norm residue theorem (see [Hmr-Wei, Pg1, Theorem A]) and a theorem of Krashen [Kra] (and independently, Saltman), there is a uniform bound on the number of symbols required to express a Galois cohomology class with $\mu_\ell$ coefficients, provided the Generalized Brauer $\ell$-dimension is finite. This uniform bound will be called the mod-$\ell$ symbol length. The case when $\ell = 2$ is particularly interesting, since the finiteness of mod-2 symbol length, implies that the $u$-invariant is also finite (see Theorem 5.4.16). In general, the higher Galois cohomology classes with $\mu_\ell$ coefficients (with appropriate twisting) are often invariants of principal homogenous spaces under linear algebraic groups, which themselves are in natural bijection with finite dimensional algebraic structures. In particular, the degree three classes are invariants of principal homogenous spaces under simply connected linear algebraic groups. Therefore, we speculate that the finiteness of the Generalized Brauer $\ell$-dimension will give some information about the arithmetic complexity of principal homogenous spaces, just as it does for quadratic forms by informing us that the $u$-invariant is finite.

## 1.3 Semi-global Fields

As mentioned before, just as the $u$-invariant and Brauer $\ell$-dimension, the Generalized Brauer $\ell$-dimension is not known for many fields of arithmetic and geometric interest. The field patching technique introduced by Harbater and Hartmann in [HH], and later developed further by Harbater, Hartmann and Krashen in a series of papers ([HHK09], [HHK15(1)], [HHK15]), offers a systematic way to deal with such arithmetic problems over certain kinds of fields. These fields are function fields in one variable over complete discretely valued fields. They are being called *semi-global fields* in recent times. Good examples to keep in mind are $k((t))(x)$ in the equicharacteristic case and $\mathbb{Q}_p(t)$ in the mixed characteristic case. Harbater, Hartmann and Krashen in [HHK09] compute the $u$-invariant and the Brauer $\ell$-dimension for semi-global fields in terms of fields of lower arithmetic complexity.

In the increasing direction of arithmetic complexity, semi-global fields serve as intermediate cases: we cannot address the arithmetic questions above, say for fields of the form $k(t,x)$ yet, where $k$ is a nice field, but we can for fields of the form $k((t))(x)$, and the hope is that this will be helpful in dealing with these problems over $k(t,x)$.

## 1.4 Main Result

In this thesis, we provide an upper-bound for the Generalized Brauer $\ell$-dimension of semi-global fields in terms of that of fields of lower arithmetic complexity. The first upper-bound in this case was provided by Parimala and Suresh in [Pa-Su3] (see also Theorem 5.1.5):

**Theorem** (Parimala-Suresh)**.** *Let $F$ be a semi-global field with residue field $k$ such that $char(k) \neq \ell$. We have the following upper bound for the Generalized Brauer $\ell$-dimension.*

$$\mathrm{GBrd}_\ell(F) \leq \ell^3 \cdot [\mathrm{GBrd}_\ell(k(t))]! \cdot [\mathrm{GBrd}_\ell(k)]!.$$

The authors in [Pa-Su3] are interested in providing an upper bound in the bad, mixed characteristic case, i.e., when $char(F) = 0$ and $char(k) = \ell$. Their upper bound in this situation is much better than the above bound. This suggests that in the good characteristic case, the above bound is not optimal.

We manage to cut the above bound and obtain the following:

**Theorem.** *Let $F$ be a semi-global field with residue field $k$. Let $\ell$ be a prime not equal to $char(k)$.*

1. *We have the following upper bound for the Generalized Brauer 2-dimension:*

$$\mathrm{GBrd}_2(F) \leq 2^3 \cdot \mathrm{GBrd}_2(k(t)) \cdot \mathrm{GBrd}_2(k).$$

2. *If $\ell \neq 2$, we obtain*

$$\mathrm{GBrd}_\ell(F) \leq \ell^2 \cdot \mathrm{GBrd}_\ell(k(t)) \cdot \mathrm{GBrd}_\ell(k).$$

This allows us to give an upper bound for the Generalized Brauer $\ell$-dimension for function fields in one variables over $m$-local fields. We use this to give a non-trivial answer to a question concerning the so called"splitting index" of quadratic forms raised in the conference on "Deformation theory and Brauer groups" in 2011 ([AimPL]). Our techniques can also be used to obtain a statement similar to Theorem 5.3.3 for finiteness of mod-$\ell$ symbol length of semi-global fields.

Our strategy is to use the recipe of Saltman used in [Salt], to first "clear out" the bad locus of a given collection of Brauer classes on a two dimensional model of the semi-global field by making a field extension of an appropriate degree. After clearing out this bad locus, we can locally specialize these Brauer classes on the special fiber of a regular model. Note that the function fields of the curves in the special fiber are fields of lower arithmetic complexity. We can split these specialized Brauer classes by making another extension, generically on the special fiber. This means that there are only finitely many closed points on the special fiber where the Brauer classes are non-zero. The most technical part is then to split them on the remaining closed points in a controlled manner. Finally, field patching, which provides us with a local-global principle with respect to over-fields coming from closed points and open sets of the special fiber, enables us to conclude that all the Brauer classes are killed by this field extension.

## 1.5 Open Problems And Future Directions

1. As we noted above, Parimala-Suresh in [Pa-Su3] obtain a much better bound for $\mathrm{GBrd}_\ell(F)$ for a semi-global field $F$ in the mixed characteristic case. This indicates that one can improve the upper bound in the Main Theorem further. One approach to improve the bound can be getting rid of the factor $\mathrm{GBrd}_\ell(k)$ or replacing it with a smaller factor, possibly at the cost of increasing the first factor $\ell^2$.

2. A uniform bound for the splitting dimension of a field $F$ also provides a uniform bound for the index of maximal orthogonal Grassmannians of quadratic forms. Another related question arises: is there a small enough, uniform bound for the indices of other orthogonal Grassmannians of quadratic forms? It looks like such a uniform bound can be provided. One can also obtain upper bounds for the indices of Galois Cohomology classes which are a lot smaller than the one in the Main Theorem. This can be potentially used to obtain better bounds for splitting indices of quadratic forms which lie in a larger power of the fundamental ideal.

3. In another direction, we prove in Theorem 5.4.16 (see Chapter 5) that if a field $F$ has finite 2-cohomological dimension and if $\mathrm{GBrd}_2(F)$ is finite, then $u(L)$ is finite for every finite degree field extension $L/F$. Parimala-Suresh raise the question if the converse of this Theorem is true, and also remark that it is plausible that it is not true. Another related question is that if the Brauer 2-dimension of $F$ is finite, is it true that the Generalized Brauer 2-dimension is finite?

# Chapter 2

# Preliminaries

## 2.1 Central Simple Algebras

**Definition 2.1.1.** An $F$-algebra $A$ is called a *central simple algebra* (c.s.a) over $F$ if $A$ is a finite dimensional $F$-vector space, the only two sided ideals of $A$ are $(0)$ and $A$, and the center of $A$ is $F$.

**Example 2.1.2.**

1. *$A = M_n(F)$ is a central simple algebra over $F$. We will call such central simple algebras* split*.*

2. *If $D$ is a division algebra over $F$ with center $F$, then $D$ is a central simple algebra over $F$. So is $M_n(D)$ for $n \geq 1$. As we will see in Theorem 2.1.5, these are all the examples of central simple algebras.*

3. *When $F = \mathbb{R}$ and $Q$ is the $\mathbb{R}$-algebra with basis $\{1, i, j, ij\}$ given by the relations:*

$$i^2 = -1, j^2 = -1, ij = -ji,$$

*one can check that $Q$ is a central simple algebra over $\mathbb{R}$.*

4. *Let $a$, $b$ be elements in $F^\times$. The $F$-algebra with basis $\{i, i, j, ij\}$ and given by the relations:*

$$i^2 = a, j^2 = b, ij = -ji,$$

*is a central simple algebra over $F$. This algebra goes by the name* generalized quaternion algebra *and will be denoted by $(a, b)_2$.*

5. *Let $L/F$ be a cyclic Galois extension of degree $n$. Let $\sigma$ be a generator of its Galois group. Let $b$ be an element of $F^\times$. Consider the $F$-algebra which, as an $L$-vector*

*space is given by the following equality and subject to the following relations:*

$$A = \bigoplus_{i=0}^{n-1} Lu^i; \qquad u^n = b, \qquad uxu^{-1} = \sigma(x) \ \forall \ x \in L.$$

*For the isomorphism* $\chi : \mathrm{Gal}(L|F) \longrightarrow \mathbb{Z}/n\mathbb{Z}$ *where* $\sigma \mapsto 1$, $A$ *is also denoted by* $(\chi, b)$. *Such algebras are called* cyclic algebras.

*If* $L = F(\sqrt[n]{a})$ *for some* $a$ *in* $F$ *(This happens always when* $F$ *contains* $n^{th}$ *roots of unity), then* $A$ *is also denoted by* $(a, b)_n$

Cyclic algebras form an important class of central simple algebras. A deep theorem of Merkurjev and Suslin says that up to a certain equivalence, every central simple algebra over $F$ can be expressed as tensor products of cyclic algebras, provided $F$ contains enough roots of unity. The definition of cyclic algebras seems somewhat ad hoc but it turns out that whenever a central simple algebra $A$ contains a cyclic Galois extension as a maximal subfield, it is isomorphic to a cyclic algebra. One naturally obtains such a description of $A$ in the course of proving this fact, using the Skolem-Noether theorem (see Theorem 2.1.8)

**Proposition 2.1.3.** *If* $A$ *is a central simple algebras over* $F$ *and* $B$ *is a simple algebra over* $F$, *then*

1. $A \otimes_F B$ *is also a simple algebra.*
2. $Z(A \otimes_F B) = Z(A) \otimes_F Z(B)$

*Proof.* See [Pierce, 12.4, Lemma b, c]. □

Therefore, as a corollary:

**Corollary 2.1.4.** *If* $A$ *and* $B$ *are central simple algebras over* $F$ *and* $K/F$ *is a field extension, then*

1. $A \otimes_F B$ *is central simple algebras over* $F$.
2. $K \otimes_F A$ *is a central simple algebra over* $K$.

**Theorem 2.1.5.** *The following statements are equivalent:*

1. *$A$ is a c.s.a over $F$.*

2. *There exists a uniquely determined division algebra $D$ with center $F$ and some $k \geq 1$ such that $A \cong M_k(D)$.*

3. *There exists a finite separable field extension $L/F$ such that $A \otimes_F L \cong M_n(L)$.*

4. *The $F$-algebra map $\rho : A \otimes A^{op} \longrightarrow End_F(A)$ given by $\rho(a \otimes b)(x) = axb$ is an isomorphism.*

*Proof.* (1) $\iff$ (2) is the Artin-Wedderburn theorem. See [Pierce, Theorem 3.5] for a proof. For (1) $\iff$ (3), see [Pierce, Lemma 13.5]. (1) $\iff$ (4) follows from Corollary 2.1.4. □

**Remark 2.1.6.** *By Theorem 2.1.5, dimension of a c.s.a as an $F$ vector space is a square. We define the degree of $A$, denoted by $\deg(A)$, to be $\sqrt{\dim_F(A)}$.*

The characterization (3) of central simple algebras given in Theorem 2.1.5 is central to this thesis.

**Definition 2.1.7.** Let $A/F$ be a central simple algebra. We say that $L/F$ splits $A$ if $A \otimes_F L \cong M_n(L)$ for some $n$.

There is another important theorem named after Skolem and Noether, that we will need later. We will state it here and use it in the next two sections.

**Theorem 2.1.8** (Skolem-Noether)**.** *Let $A/F$ be a central simple algebra and $B \subseteq A$ be a simple subalgebra of $A$. If $\sigma : B \hookrightarrow A$ is any $F$ algebra embedding, then there exists a unit $u$ in $A^\times$ such that $\sigma(b) = ubu^{-1}$ for every $b$ in $B$.*

*Proof.* See [Pierce, Section 12.6] □

**Remark 2.1.9.** *The Skolem-Noether theorem says that every automorphism of a central simple algebra is inner. In particular, $\mathrm{Aut}_F(M_n(F)) \cong PGL_n(F)$.*

We will state a few facts about maximal subfields in central simple algebras. The driving force behind the proofs of these facts is the important Double Centralizer theorem.

Theorem 2.1.12 will be useful in relating the Brauer group with the second Galois cohomology group.

**Definition 2.1.10.** Let $A/F$ be a central simple algebra. A subalgebra $E \subset F$ which is also a field is called *strictly maximal* if $[E : F] = \deg(A)$

**Proposition 2.1.11.** *Let $D/F$ be a division algebra. The following statements are equivalent:*

1. *$E \subset D$ is a maximal subfield.*

2. *The centralizer of $E$ in $D$, $C_D(E)$, equals $E$.*

3. *$[E : F] = \deg(D)$*

*Proof.* See [Pierce, Corollary 13.3] □

**Theorem 2.1.12.** *Let $A/F$ be a central simple algebra. Let $K/F$ be a Galois extension such that $K$ splits $A$. Then there exists a central simple algebra $B$ Brauer equivalent to $A$ (see Definition 2.1.13), and an embedding of $F$ subalgebras $K \hookrightarrow B$ such that $K$ is a strictly maximal subfield in $B$.*

*Proof.* See [Pierce, Theorem 13.3] □

### 2.1.1 Brauer Group

The Brauer group is an extremely important algebraic object which makes its presence felt in a wide variety of places, from class field theory in number theory to obstruction problems for the existence of fine moduli spaces for moduli problems concerning vector bundles. Recall that the Artin-Wedderburn Theorem tells us that it is sufficient to study division algebras in order to study central simple algebras. We will denote the underlying division algebra of a central simple algebra $A$ by $D_A$. We put the following, seemingly naïve, equivalence relation on the set of isomorphism classes of central simple algebras over $F$:

$$A \sim B \iff D_A \cong D_B.$$

This equivalence relation is called *Brauer equivalence*. Traditionally, it is defined as the following:

**Definition 2.1.13.** Let $A$ and $B$ be central simple algebras over $F$. We say that $A$ and $B$ are Brauer equivalent if:

$$A \sim B \iff M_m(A) \cong M_k(B)$$

for some integers $m, k \geq 1$.

**Remark 2.1.14.** *1. Note that $F$ is Brauer equivalent to $M_n(F)$.*

*2. By the Artin-Wedderburn Theorem, one can see that the two equivalence relations defined above coincide.*

Notice that by the Artin-Wedderburn Theorem, this equivalence relation is compatible with tensor products, i.e., if $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 \otimes_F B_1 \sim A_2 \otimes_F B_2$. Further, by Theorem 2.1.5, $A \otimes_F A^{op} \sim F$.

Together with the associative property of tensor products, this shows that the operation of tensor product equips a well-defined Abelian group structure on the set of isomorphism classes of central simple algebras over $F$. This group is called the *Brauer group* of $F$ and is denoted by $\mathrm{Br}(F)$.

Since central simple algebras are split by a finite separable field extension (see 2.1.5), there are no nontrivial central simple algebras over a separably closed field, i.e., $\mathrm{Br}(F) = 0$ when $F$ is separably closed. Of course, we hope to compute Brauer groups of some nontrivial fields. In the decreasing direction of triviality, as a first step, let us compute $\mathrm{Br}(\mathbb{R})$.

**Example 2.1.15.** $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$

Let $D/\mathbb{R}$ be a central division algebra. Let $L$ be a maximal subfield in $D$. Note that either $L = \mathbb{R}$ or $L = \mathbb{C}$. By proposition 2.1.11, $[L : \mathbb{R}] = \sqrt{[D : \mathbb{R}]}$. Therefore, either

$D = \mathbb{R}$ or $[D:\mathbb{R}] = 4$. We will show below that up to isomorphism, there is a unique division algebra with $[D:\mathbb{R}] = 4$, namely the quaternions.

Let us fix a maximal subfield $L$ and make the identification: $L = \mathbb{C}$ since $[L:\mathbb{R}] = 2$. Let $\sigma$ be the element in $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$ acting by conjugation. By Theorem 2.1.8, there exists $j$ in $D^\times$ such that $j(a+bi)j^{-1} = a - bi$ and therefore $jij^{-1} = -i$. Since conjugation is an element of order two, $j^2$ commutes with all the elements in $\mathbb{C}$, i.e., $j^2$ lies in the centralizer $C_D(\mathbb{C})$. Since $\mathbb{C}$ is a maximal subfield, $C_D(\mathbb{C}) = \mathbb{C}$. Thus, $j^2$ lies in $\mathbb{C}$. Now observe that $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$ fixes $j^2$ since it is fixed by conjugation by $j$. So in fact, $j^2$ lies in $\mathbb{R}$. Because $D$ is a division algebra $j^2$ cannot be positive, otherwise $D$ would have zero divisors. Therefore, $j^2$ is negative. Normalizing $j$, we may assume that $j^2 = -1$.

Thus, we have found two elements $i$ and $j$ inside $D$ satisfying $i^2 = -1$, $j^2 = -1$ and $ij = -ji$. Now we only need to show that $\{1, i, j, ij\}$ forms a basis for $D$. We just need to check that this set is linearly independent. If we have the following linear relation $a + bi + cj + dij = 0$, multiplying it by $a - bi - cj - dij$, we get that $a^2 + b^2 + c^2 + d^2 = 0$ for $a, b, c, d$ in $\mathbb{R}$, and therefore $a = b = c = d = 0$. Thus, $D$ is isomorphic to the quaternions $\mathbb{H}$.

Let $\alpha$ be the class of $\mathbb{H}$ in $Br(\mathbb{R})$. Since, every non-trivial division algebra is isomorphic to $\mathbb{H}$, $2\alpha = 0$. Therefore, $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.

Note that the proof above can be easily generalized to show that any degree two central simple algebra over a field of characteristic not equal to two is isomorphic to a generalized quaternion algebra. We therefore obtain the following proposition:

**Proposition 2.1.16.** *Let $F$ be a field with $char(F) \neq 2$. Let $A/F$ be a central simple algebra of degree two. Then there exist elements $a, b$ in $A^\times$ such that $A \cong (a, b)$*

## 2.2 Galois Descent

Note that if $K/F$ is a Galois extension of fields, we have that, $K^{\mathrm{Gal}(K|F)} = F$. Thus, the "algebraic structure" $K/F$ descends to the algebraic structure $F$ viewed as an algebraic structure over itself. This naturally leads us to ask the question:

If $V$ is an algebraic structure over $K$, when does it "descend" to an algebraic structure over $F$? We need to make a couple of things precise here. First, what do we mean by an algebraic structure? And what does it mean for it to descend? While we would like to think that all "arithmetic information" in an algebraic structure should descend to the ground field under the action of a Galois group, it does not happen. And we are thankful for that, since such phenomena gives rise to interesting structures such as central simple algebras.

Let $(V, \Phi)$ be a finite dimensional vector space over $F$ equipped with a tensor $\Phi$ of type $(m, n)$, i.e., $\Phi$ is some element in $V^{\otimes m} \otimes (V^*)^{\otimes n}$, possibly satisfying some more properties. For our purpose, we will take this to mean an algebraic structure, as it covers many of the examples of algebraic structures we have in mind. For example, if $(V, q)$ is a quadratic form, we may think of $q$ as a tensor of type $(0, 2)$. If $A$ is a central simple algebra with multiplication map $m$, $(A, m)$ can be thought of as a tensor of type $(1, 2)$ satisfying some more properties.

Theorem 2.1.5 says that every degree $n$ central simple algebra becomes isomorphic to $M_n(F)$ when one extends scalars to $F^{\text{sep}}$. Or, in other words, $M_n(F^{\text{sep}})$ descends to $A$. Regular quadratic forms of dimension $n$ become isometric to the $n$ dimensional form $\langle 1, \cdots, 1\rangle$ when one extends scalars to $F^{\text{sep}}$. Many such algebraic structures become isomorphic to a distinguished algebraic structure when one passes to $F^{\text{sep}}$.

**Definition 2.2.1.** Let $(V, \Phi)$ be a finite dimensional vector space equipped with a tensor $\Phi$ over $F$. We say that a tensor $(W, \Psi)$ over $F$ is a twisted form of $(V, \Phi)$ if there exists an isomorphism $f : (V, \Phi) \otimes F^{\text{sep}} \longrightarrow (W, \Psi) \otimes F^{\text{sep}}$.

Thus, degree $n$ central simple algebras are twisted forms of $M_n(F)$.

The main theorem of Galois descent helps us in classifying twisted forms of a distinguished algebraic structure. We state it below and classify twisted forms of a tensor in terms of non-Abelian Galois cohomology in the next section. We refer the reader to [JJ] for the proof.

**Theorem 2.2.2.** *1. Let $(V, \Phi)$ be a vector space with a tensor defined over $F^{\text{sep}}$.*

*Suppose that for every $\sigma$ in* $\mathrm{Gal}(F^{\mathrm{sep}}|F)$*, we have maps* $T_\sigma : (V, \Phi) \longrightarrow (V, \Phi)$ *which are $\sigma$-linear map (i.e., $T_\sigma(\lambda v) = \sigma(\lambda)v$ for every $\lambda$ in $F^{\mathrm{sep}}$). Then there exists a tensor of the same type $(W, \Psi)$ over $F$ such that there exists an isomorphism:*

$$\phi : (W, \Psi) \otimes F^{\mathrm{sep}} \xrightarrow{\sim} (V, \Phi).$$

2. *Suppose that $(V_1, \Phi_1)$ and $(V_2, \Phi_2)$ are tensors over $F^{\mathrm{sep}}$ and $\overline{\theta}$, a homomorphism $\overline{\theta} : (V_1, \Phi_1) \longrightarrow (V_2, \Phi_2)$ such that the following diagram commutes*

$$\begin{array}{ccc} (V_1, \Phi_1) & \xrightarrow{\overline{\theta}} & (V_2, \Phi_2) \\ \downarrow{\scriptstyle T_\sigma} & & \downarrow{\scriptstyle T_\sigma} \\ (V_1, \Phi_1) & \xrightarrow{\overline{\theta}} & (V_2, \Phi_2). \end{array}$$

*Then there exist tensors $(W_1, \Psi_1)$ and $(W_2, \Psi_2)$ which are forms of $(V_1, \Phi_1)$ and $(V_2, \Phi_2)$ respectively and a homomorphism $\theta : (W_1, \Phi_1) \longrightarrow (W_2, \Phi_2)$ such that $\theta \otimes F^{\mathrm{sep}} = \overline{\theta}$.*

*Proof.* See [JJ, Theorem 2.2] □

## 2.3 Galois Cohomology

One thinks of central simple algebras as those algebraic structures over $F$ which "come from" matrix algebras over $F^{\mathrm{sep}}$. Theorem 2.1.5 demolishes the naïve belief that only $M_n(F)$ gives rise to $M_n(F^{\mathrm{sep}})$. If one takes the elements in $M_n(F^{\mathrm{sep}})$ that are fixed under the action of $\mathrm{Gal}(F^{\mathrm{sep}}|F)$, depending upon how we act the Galois group, we would not only get $M_n(F)$. Thus the Brauer group can be thought of as the group of "things" which come from matrices over $F^{\mathrm{sep}}$. Cohomology in algebra encodes the information lost when an algebraic structure goes through some process. In our context, if this process is taking Galois invariants, then (with some hindsight) we are led to conclude that there should be some relationship between Cohomology arising from taking Galois invariants, and the Brauer group. We will follow the treatment in [G-S, Chapter 3] in this section.

### 2.3.1 A quick introduction to group cohomology

Let $G$ be a finite group and $M$ be a $G$ module. We define the $G$ invariants of $M$ as

$$M^G := \{m \in M \mid gm = m \ \forall g \in G\}.$$

Observe that the functor $(\_\_)^G$ which takes $G$ invariants of objects in the Abelian category of $G$ modules is not right exact. A simple example comes from Galois theory. Consider the exact sequence of $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$ modules:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times \xrightarrow{\times 2} \mathbb{C}^\times \to 0.$$

If one takes $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$ invariants, one sees that the right-most map is not surjective since negative numbers cannot be squares.

Observe that $M^G \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ for every $G$ module $M$, and the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$. One can further show that the following functors are isomorphic: $(\_\_)^G \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \_\_)$. The defect of right exactness is captured by the right derived functors. Thus we are led to the following definition:

We define the group cohomology of $G$ with coefficients in $M$ as:

$$\mathrm{H}^i(G, M) := \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Thus, in order to compute this group, one would take a projective resolution of the trivial $G$ module $\mathbb{Z}$, apply the functor $(\_\_)$ and take cohomology of this new cochain complex. There is a "computational friendly" free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$ module. This resolution is called the bar resolution.

Let $F_j$ be the free abelian group on the set $G^j$, for $j > 0$, let $F_0 = \mathbb{Z}[G]$, and $F_{-1} := \mathbb{Z}$, where $\mathbb{Z}$ is given the trivial action of $G$. Thus to give a map from $F_j$ to any $G$ module $M$ for $j > 0$, amounts to giving a set map $G^j \to M$. Consider the free resolution:

$$\cdots \to F^j \xrightarrow{d_j} F_{j-1} \xrightarrow{d_{j-1}} F_{j-2} \to \cdots \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

Here $d_j$ is the map defined on the basis of $F_j$ as:

$$d_j(g_1,\cdots,g_j) = g_1(g_2,\cdots,g_j) + \sum_{i=1}^{j-1}(-1)^i(g_1,\cdots,g_ig_{i+1},\cdots g_j) + (-1)^j(g_1,\cdots,g_{j-1})$$

and $\epsilon$ is the augmentation map sending every $g$ in $G$ to 1 in $\mathbb{Z}$.

It will be useful to explicitly see what elements in $\mathrm{H}^1(G,M)$ and $\mathrm{H}^2(G,M)$ look like. Note that elements of the cohomology groups $\mathrm{H}^j(G,M)$ can be thought of as the group of maps $G^j \to M$, up to an equivalence. Using the bar resolution, elements in $\mathrm{H}^1(G,M)$ can be though of as the group of crossed homomorphisms up to an equivalence.

Let $Z^1(G,M)$ be the group of crossed homomorphisms, i.e., $Z^1(G,M)$ is the group of maps $a : G \to M$ satisfying $a_{\sigma\tau} = a_\sigma {}^\sigma a_\tau$.

Let $B^1(G,M)$ be the subgroup of the crossed homomorphisms which are of the form, $\sigma \mapsto {}^\sigma m - m$.

One then gets that

$$\mathrm{H}^1(G,M) = Z^1(G,M)/B^1(G,M).$$

Let $Z^2(G,M)$ be the group of maps $G \times G \to M$ satisfying the identity

$${}^\sigma a_{\tau,\rho} - a_{\sigma\tau,\rho} + a_{\sigma,\tau\rho} - a_{\sigma,\tau} = 0.$$

Let $B^2(G,M)$ be the subgroup of $Z^2(G,M)$ consisting of maps $(\sigma,\tau) \mapsto {}^\sigma b_\tau - b_{\sigma\tau} + b_\sigma$ for maps $b : G \to M$.

Again, one can check that

$$\mathrm{H}^2(G,M) = Z^2(G,M)/B^2(G,M).$$

Let $H$ be a subgroup of $G$ and $N$ be an $H$-module. Note that $M_H^G(N) := \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G],N)$ is a $G$-module with the action of $G$ defined as follows: for $\phi$ in $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G],N)$ and $g'$ in $G$, the map $g'\phi$ sends $g$ to $\phi(gg')$. Such modules are called *coinduced*.

**Lemma 2.3.1** (Shapiro). *For every $i \geq 0$, we have the following canonical isomorphism:*

$$\mathrm{H}^i(G, M_H^G(N)) \xrightarrow{\sim} \mathrm{H}^i(H,N).$$

*Proof.* The isomorphism follows from the canonical isomorphism

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M, M_H^G(N)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}[H]}(M, N)$$

for every $G$ module $M$. Given $\theta$ in $\mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], N))$, we send it to the morphism which is defined as $m \mapsto (\theta(m))(1)$. This lies in $\mathrm{Hom}_{\mathbb{Z}[H]}(M, N)$. For the inverse, given $\phi$ in $\mathrm{Hom}_{\mathbb{Z}[H]}(M, N)$, we obtain the morphism $m \mapsto \phi_m$, where $\phi_m$ is defined as $\phi_m(g) = \phi(gm)$. □

Shapiro's lemma helps us in constructing the restriction maps. For every $G$ module $M$, we have the isomorphism: $M \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], M)$. Viewing a $G$ module homomorphism as an $H$ module homomorphism, we get the map:

$$M \to \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$$

Taking cohomology we obtain the *restriction* map:

$$\mathrm{H}^i(G, M) \xrightarrow{\mathrm{Res}} \mathrm{H}^i(G, \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)) \cong \mathrm{H}^i(H, M)$$

Concretely, if we think of a cocycle as a function $G^i \to M$, the restriction map sends this function to the composition $H^i \hookrightarrow G^i \hookrightarrow M$.

We also have another canonical morphism $\mathrm{Cor} : \mathrm{H}^i(H, M) \longrightarrow \mathrm{H}^i(G, M)$ called *corestriction.*

Let $H$ be a subgroup of $G$ with $[G : H] = n$. To define the corestriction map, we first need to define the map $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], M)$.

Let $\{g_1, \cdots, g_n\}$ be a system of coset representatives of $G/H$. Suppose $\phi$ is an element in $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$, we define the map $\phi_H^G$ which sends $x$ to $\sum_{i=1}^n g_i \phi({g_i}^{-1} x)$. It is standard to check that this map does not depend upon the choice of the coset representatives, and that it is an $H$-module homomorphism. Again, passing to cohomology, one obtains the corestriction homomorphism

$$\mathrm{H}^i(H, M) \xrightarrow{\mathrm{Cor}} \mathrm{H}^i(G, M).$$

The restriction map is similar to the pullback map in Algebraic Geometry and the corestriction map is similar to the pushforward. We also have the following form of the projection formula involving restriction and corestriction.

**Theorem 2.3.2.** *If $H$ is a subgroup of $G$ of index $n$, for every $i \geq 0$ the composition*

$$\mathrm{Cor} \circ \mathrm{Res} : \mathrm{H}^i(G, M) \longrightarrow \mathrm{H}^i(G, M)$$

*coincides with multiplication by $n$.*

*Proof.* It is enough to check this for $i = 0$. If $\phi$ is a morphism in $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], M)$, then $\mathrm{Res}(\phi)$ is the same map viewed as an $H$-module homomorphism. Let $\{g_1, \cdots, g_n\}$ be a set of coset representatives of $G/H$. Now for $x$ in $\mathbb{Z}[G]$, $\mathrm{Cor}(\mathrm{Res}(\phi))(x) = \sum_{i=1}^{n} g_i \phi({g_i}^{-1} x) = nx$, since $\phi$ is a $G$-module homomorphism. □

Finally, we define a map, called the *inflation* map, which relates the cohomology of a quotient of a group and the cohomology of the group. Let $N$ be a normal subgroup of $G$. Then we have the map

$$\mathrm{Inf} : \mathrm{H}^i(G/N, M^N) \to \mathrm{H}^i(G, M).$$

We will describe this map concretely. Thinking of a cocycle as a function $f$ from $(G/N)^i$ to $M^H$ satisfying an identity, we compose it with the standard projection $G \to G/N$ to get a function from $G^i$ to $M$.

For our application, $G$ will be the Galois group of a field. Although Galois groups are not finite, they are profinite, i.e.,

$$\mathrm{Gal}(F^{\mathrm{sep}}|F) = \varprojlim_{K/F} \mathrm{Gal}(K|F),$$

where $K/F$ ranges over finite Galois extensions.

If $M$ is a $\mathrm{Gal}(F^{\mathrm{sep}}|F)$-module which is given the discrete topology, then we define the *Galois cohomology of $F$ with coefficients in $M$* to be:

$$\mathrm{H}^i(F, M) \coloneqq \varinjlim_{K/F} \mathrm{H}^i(\mathrm{Gal}(K|F), M^{\mathrm{Gal}(F^{\mathrm{sep}}|K)}),$$

where $K/F$ ranges over finite Galois extensions.

Following [Serre2], one can also define Galois cohomology as the cohomology of a certain cochain complex. We will denote the Galois group $\mathrm{Gal}(F^{\mathrm{sep}}|F)$ by $\Gamma_F$ in what follows. Let $C_c^{i-1}(\Gamma_F, M)$ be the group of continuous maps $G^{i-1} \xrightarrow{a} M$. Consider the cochain complex:

$$\cdots \to C_c^{i-1}(\Gamma_F, M) \xrightarrow{d^{i-1}} C_c^{i}(\Gamma_F, M) \xrightarrow{d^{i}} C_c^{i+1}(\Gamma_F, M) \to \cdots$$

where the differential $d^{i-1}$ is defined by sending the map $a : G^{i-1} \to M$ to the following map in $C_c^i(G, M)$:

$$(\sigma_1, \cdots, \sigma_i) \mapsto {}^{\sigma_1}a_{\sigma_2,\cdots,\sigma_i} + \sum_{j=1}^{i}(-1)^j a_{\sigma_1,\cdots,\sigma_j\sigma_{j+1},\cdots\sigma_i} + (-1)^{j+1}a_{\sigma_1,\cdots,\sigma_{i-1}}.$$

We have the following important theorem due to Hilbert. Originally stated for cyclic Galois extensions, this version of the theorem will help us in computing various Galois cohomology groups. In the section on Nonabelian Galois cohomology, we will see a more general form of this theorem. We succumb to the temptation of giving a proof which involves a delightful trick!

**Theorem 2.3.3.** $\mathrm{H}^1(F, (F^{\mathrm{sep}})^\times) = 1$.

*Proof.* We will show that $\mathrm{H}^1(\mathrm{Gal}(K|F), K^\times) = 1$ for a finite Galois extension $K/F$. Let $G$ be the Galois group $\mathrm{Gal}(K|F)$. Let $a$ be a cocycle in $Z^1(G, K^\times)$. Thus $a$ satisfies the identity

$$a_{\sigma\tau} = a_\sigma {}^\sigma a_\tau.$$

Recall that Dedekind's lemma says that the the elements of $G$ are linearly independent. Thus there exists $b$ in $K^\times$ so that the element

$$c := \sum_{\tau \in G} a_\tau \tau(b)$$

is non-zero. For $\sigma$ in $G$, we have

$$\begin{aligned}\sigma(c) &= \sum_{\tau\in G} {}^{\sigma}a_\tau \sigma(\tau(b)) \\ &= \sum_{\tau\in G} \frac{a_\sigma {}^{\sigma}a_\tau \sigma(\tau(b))}{a_\sigma} \\ &= \frac{1}{a_\sigma} \sum_{\tau\in G} a_{\sigma\tau} \sigma\tau(b) \\ &= \frac{c}{a_\sigma}.\end{aligned}$$

Thus, $a_\sigma = \frac{c}{\sigma(c)}$ and hence $a_\sigma$ is a coboundary in $B^1(G, K^\times)$. Therefore, $\mathrm{H}^1(G, K^\times) = 1$. □

When $K/F$ is a cyclic extension, Hilbert theorem 90 says that elements of norm 1 are of the form $c/\sigma(c)$, for $\sigma$ a generator of $\mathrm{Gal}(K|F)$. One obtains this as a corollary of the above theorem. With this trickery out of the way, let us compute a Galois cohomology group.

**Example 2.3.4.** *Let $F$ be a field and let $n > 1$ be coprime to char$(F)$. Then,*

$$\mathrm{H}^1(F, \mu_n) \cong F^\times/(F^\times)^n.$$

Consider the short exact sequence:

$$1 \to \mu_n \to F^{sep\times} \xrightarrow{\times n} F^{sep\times} \to 1.$$

Passing to the long exact sequence, and using Theorem 2.3.3, we obtain the following exact sequence:

$$1 \to \mu_n \cap F \to F^\times \xrightarrow{\times n} F^\times \to \mathrm{H}^1(F, \mu_n) \to 1.$$

Therefore $F^\times/(F^\times)^n \cong \mathrm{H}^1(F, \mu_n)$, where the isomorphism maps $\lambda$ in $F^\times/(F^\times)^n$ to the crossed homomorphism $\sigma \mapsto \frac{\sigma(\sqrt[n]{\lambda})}{\sqrt[n]{\lambda}}$. We will end by computing the group cohomology of cyclic groups. This will be useful later to compute the relative Brauer groups of cyclic extensions.

**Example 2.3.5.** *Let $G$ be a finite cyclic group of order $n$. Let $M$ be a $G$-module. We will compute all the cohomology groups $\mathrm{H}^i(G, M)$.*

*To do this, consider the following projective resolution of $\mathbb{Z}$:*

$$\cdots \to \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

*where $\epsilon$ is the map sending a generator $g$ of $G$ to $1$ in $\mathbb{Z}$. The map $\sigma - 1$ sends $g$ to $\sigma g - g$, and $N$ sends $g$ to $\sum_{i=0}^{n-1} \sigma^i g$.*

*Note that since $0 = \sigma^n - 1 = (\sigma - 1)N$, we see that $\mathrm{Im}(N) \subseteq \mathrm{Ker}(\sigma - 1)$. Further if $\sum_{i=0}^{n-1} \lambda_i \sigma^i$ lies in $\mathrm{Ker}(N)$, one can check that $\lambda_0 = \lambda_1 = \cdots = \lambda_{n-1}$. Therefore, $\sum_{i=0}^{n-1} \lambda_i \sigma^i$ lies in $\mathrm{Im}(N)$, and hence $\mathrm{Ker}(\sigma - 1) \subseteq \mathrm{Im}(N)$. This shows that the resolution is exact.*

*Applying the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \_\_)$ to this, we obtain the following cochain complex:*

$$0 \to M \xrightarrow{\sigma-1} M \xrightarrow{N} M \xrightarrow{\sigma-1} \cdots$$

*Notice that the kernel of $\sigma - 1$ is the invariant module $M^G$. We therefore obtain:*

$$\mathrm{H}^{2i+1}(G, M) = \mathrm{Ker}(N)/(\sigma - 1)M, \quad \mathrm{H}^{2i+2}(G, M) = M^G/\mathrm{Im}(N), \text{ for } i \geq 0.$$

### 2.3.2 Brauer group and $\mathrm{H}^2$

Since every central simple algebra can be split by a finite separable extension, we can as well assume that this splitting field is Galois by going to the Galois closure. If $A/F$ is split by a Galois extension $K/F$ with Galois group $G$, by Theorem 2.1.12, $A$ is Brauer equivalent to a central simple algebra $B$, with $K$ sitting inside $B$ as a strictly maximal subfield. Skolem-Noether (see Theorem 2.1.8) tells us that automorphisms of $K$ as an $F$ algebra are inner. Therefore, for every $\sigma \in G$ and $x \in K$, there exist elements $u_\sigma \in A^\times$ such that

$$\sigma(x) = u_\sigma x {u_\sigma}^{-1}. \tag{2.1}$$

Since $K$ is strictly maximal, $[K : F] = \deg(B)$. As a result, $B$ has dimension $[K : F] = |G|$ as a $K$-vector space. We have $|G|$ elements in $B^\times$ available to us, namely $\{u_\sigma\}_{\sigma \in G}$. If they form a $K$ linearly independent set, then we obtain a generating set for $B$ as a $K$ algebra and at least one relation.

It does turn out that the set $\{u_\sigma\}_{\sigma \in G}$ is linearly independent. We will not prove this fact here. See [Pierce, Lemma 14.1] for the argument. It is a standard argument as arguments for showing linear independence go. One assumes that there is a minimal relation, and using the relation 2.1, one obtains a contradiction to the minimality.

Therefore as a $K$-vector space, we have that

$$B = \bigoplus_{\sigma \in G} K u_\sigma.$$

We still do not know how to multiply $u_\sigma u_\tau$. We expect this product to be some linear combination involving the elements $u_\sigma$ with coefficients in $K$. We make a guess: perhaps $u_\sigma u_\tau = a_{\sigma,\tau} u_{\sigma\tau}$ for some $a_{\sigma,\tau}$ in $K^\times$. Using (2.1), it is not hard to show that the element $u_\sigma u_\tau u_{\sigma\tau}^{-1}$ commutes with every element in $K$. Since $K$, being strictly maximal is self centralizing (see Proposition 2.1.11), we obtain that the elements $a_{\sigma,\tau}$ lie in $K^\times$ for every $\sigma$, $\tau$ in $G$. Therefore, we have

$$u_\sigma u_\tau = a_{\sigma,\tau} u_{\sigma\tau}. \tag{2.2}$$

Note that, since we have taken $u_1 = 1$, we get that $a_{1,\sigma} = a_{\sigma,1} = 1$.

We have not yet exploited the fact that $B$ is an associative algebra. This fundamental property provides us with relations among the $a_{\sigma,\tau}$ and that relation turns out to be a 2-cocycle! Computig the product $u_\sigma u_\tau u_\rho$ in two different ways, one obtains:

$${}^{\sigma}a_{\tau,\rho} a_{\sigma,\tau\rho} = a_{\sigma,\rho} a_{\sigma\tau,\rho}. \tag{2.3}$$

To summarize: we started with a central simple algebra $A$ which was split by a Galois extension $K/F$ and therefore was Brauer equivalent to the algebra $B$ which contained $K$ as a strictly maximal subfield. Using the Skolem Noether Theorem and Proposition 2.1.11, we saw that the algebra $B$ has a very peculiar structure. From relations among its generators, we obtained a 2-cocycle condition. Therefore we can associate an element in $\mathrm{H}^2(G, K^\times)$ to every element in the subgroup of $\mathrm{Br}(F)$ consisting of elements split by $K$. Central simple algebras with structure similar to that of $B/F$, are called *crossed product algebras*.

We define the *relative Brauer group* $\mathrm{Br}(K/F)$ to be the subgroup of elements in $\mathrm{Br}(F)$ which are split by $K$. We have thus obtained a map:

$$\mathrm{Br}(K/F) \xrightarrow{\theta_K} \mathrm{H}^2(G, K^\times). \tag{2.4}$$

If $L/K$ is a field extension, the restriction of a Brauer class $\alpha$ in $\mathrm{Br}(K)$ to $L$ will also be denoted by $\alpha \otimes L$.

The map $\theta_K$ is in fact surjective. Given a cocycle $a_{\sigma,\tau}$ in $\mathrm{H}^2(G, K^\times)$, one can define a crossed product algebra which is split by $K$, by just reverse engineering the way we described the structure of $B$.

We define the $K$-vector space $B$ of dimension $|G|$. We choose a basis $\{u_\sigma\}$, indexed by elements $\sigma$ in $G$, and subject it to the following relations:

$$B = \bigoplus_{\sigma \in G} K u_\sigma \qquad u_\sigma x {u_\sigma}^{-1} = \sigma(x)\ \forall\ x \in K \qquad u_\sigma u_\tau = a_{\sigma,\tau} u_{\sigma\tau}.$$

It turns out that $\theta_K$ is a group isomorphism. Above, we have roughly sketched that the map $\theta_K$ is a bijection. The isomorphisms are compatible with the inflation and restriction maps in Group cohomology. We collect these facts in the following theorem:

**Theorem 2.3.6.** *1. The map $\theta_K$ in (2.4) is a group isomorphism.*

*2. Further, if $L/F$ is a Galois extension with Galois group $H$ containing $K$ as a subfield, then the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Br}(K/F) \arrow[r, "\theta_K"] \arrow[d, "\iota_{L/K}"] & \mathrm{H}^2(G, K^\times) \arrow[d, "inf"] \\ \mathrm{Br}(L/F) \arrow[r, "\theta_L"] & \mathrm{H}^2(H, L^\times). \end{tikzcd} \tag{2.5}$$

*3. If $L/F$ is a subextension of $K/F$, with Galois group $H$. Then the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Br}(K/F) \arrow[r, "\theta_K"] \arrow[d, "\otimes L"] & \mathrm{H}^2(G, K^\times) \arrow[d, "res"] \\ \mathrm{Br}(K/L) \arrow[r, "\theta_L"] & \mathrm{H}^2(H, L^\times). \end{tikzcd} \tag{2.6}$$

*Proof.* For a proof, see [Pierce, Proposition 14.7(a)]. □

The compatibility of the isomorphisms as in (2.6) therefore establishes the following important isomorphism:

**Corollary 2.3.7.** *The map $\theta$ is an isomorphism:*

$$\mathrm{Br}(F) \xrightarrow{\theta} \mathrm{H}^2(F, F^{sep\times}). \tag{2.7}$$

Since every central simple algebra is split by a Galois extension, for any field $F$,

$$\mathrm{Br}(F) = \bigcup_{K/F} \mathrm{Br}(K/F),$$

where $K/F$ ranges over Galois extensions.

The isomorphism (2.4), together with Example 2.3.5, helps in computing the relative Brauer groups of cyclic extensions.

**Example 2.3.8.** *Let $K/F$ be a cyclic Galois extension, with Galois group $G$. Let $\sigma$ be a generator of $G$. Let $N_{K/F} : K^\times \to F^\times$ be the norm map. Recall that since $K/F$ is Galois, $(K^\times)^G = F^\times$ and $N_{K/F}(x) = \prod_{i=0}^{n-1} \sigma^i(x)$. Then, we have the following isomorphism:*

$$F^\times / N_{K/F}(K^\times) \xrightarrow{\sim} \mathrm{Br}(K/F).$$

*Since the map $\theta_K$ in (2.4) is an isomorphism by Theorem 2.3.6, we may identify $\mathrm{Br}(K/F)$ and $\mathrm{H}^2(G, K^\times)$. By Example 2.3.5, $\mathrm{H}^2(G, K^\times) \cong F^\times / N_{K/F}(K^\times)$.*

*There is an explicit description of the above isomorphism (see [G-S, Corollary 4.7.4]): for $b$ in $F^\times$, the element $bN_{K/F}(K^\times)$ in $F^\times / N_{K/F}(K^\times)$ is sent to $[(\sigma, b)]$ in $\mathrm{Br}(K/F)$, where $[(\sigma, b)]$ is the class of the cyclic algebra as in Example 2.1.2(5.).*

Notice that, this example also shows that $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$. Since, every central simple algebra over $\mathbb{R}$ is split by $\mathbb{C}$, and $\mathrm{Gal}(\mathbb{C}|\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, we have

$$\mathrm{Br}(\mathbb{R}) = \mathrm{Br}(\mathbb{C}/\mathbb{R}) \cong \mathbb{R}^\times / N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times).$$

Since $N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$ is isomorphic to the multiplicative group of positive real numbers $\mathbb{R}^+$, one sees that $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.

**Example 2.3.9.** *Using Example 2.3.8 and Theorem 2.3.3 (Hilbert's original version), one can also shows that for any finite field $\mathbb{F}_q$:*

$$\mathrm{Br}(\mathbb{F}_q) = 0.$$

*Recall that every field extension of $\mathbb{F}_q$ is cyclic with Galois group generated by the $q^{th}$ power map: $\phi_q(x) = x^q$. To establish that $\mathrm{Br}(\mathbb{F}_q) = 0$, it suffices to show that*

$\mathrm{Br}(\mathbb{F}_{q^n}/\mathbb{F}_q) = 0$. *Now consider the sequence:*

$$1 \to H \to \mathbb{F}_{q^n}^{\times} \xrightarrow{N_{\mathbb{F}_{q^n}/\mathbb{F}_q}} \mathbb{F}_q^{\times},$$

*where $H$ denotes the kernel of the norm map. We will show that the norm map is surjective.*

*By Theorem 2.3.3, the elements in $H$ are of the form $\phi_q(y)/y$ for $y$ in $\mathbb{F}_{q^n}^{\times}$. But note that $\phi_q(y)/y = y^{q-1}$. Therefore, we have that $|H| = (q^n-1)/(q-1)$. This implies that the image of the norm map has cardinality equal to $q-1$. Therefore, $N_{\mathbb{F}_{q^n}/\mathbb{F}_q}$ is surjective. By Example 2.3.8, we conclude that* $\mathrm{Br}(\mathbb{F}_{q^n}/\mathbb{F}_q) = 0$.

Finally, observe that the isomorphism in 2.3.8 also provides us with the following useful identities. Let us assume that $F$ contains the group of $n^{th}$ roots of unity. We may therefore denote cyclic algebras as $(a,b)_n$. We have the following identities in $\mathrm{Br}(F)$:

$$[(a_1a_2,b)] = [(a_1,b)] + [(a_2,b)] \tag{2.8}$$

$$[(a,b_1b_2)] = [(a,b_1)] + [(a,b_2)]. \tag{2.9}$$

### 2.3.3 Nonabelian Galois cohomology

The purpose of this section is to define the pointed Galois cohomology set $\mathrm{H}^1(F,G)$, for an algebraic group $G$. For our applications, $G$ will almost always be a linear algebraic group over $F$. Nonabelian Galois cohomology sets classify many important algebraic structures over $F$. For example, $\mathrm{H}^1(F,PGL_n)$ classifies central simple algebras of degree $n$ over $F$ and $\mathrm{H}^1(F,O_n)$ classifies non-degenerate $n$-dimensional quadratic forms over $F$ (at least when $char(F) \neq 2$). Such a classification puts classical local-global principles such as the Albert-Brauer-Hasse-Noether theorem for central simple algebras and the Hasse-Minkowski theorem for quadratic forms on an equal philosophical footing.

Let $G/F$ be an algebraic group. For any extension, $L/F$, $G(L)$ will denote the $L$-rational points of $G$.

**Definition 2.3.10.** Equip the set $G(F^{\mathrm{sep}})$ with the discrete topology and $\mathrm{Gal}(F^{\mathrm{sep}}|F)$ with the profinite topology. We define the set of cocyles $Z^1(F,G)$ to be the set of continuous maps $a : \mathrm{Gal}(F^{\mathrm{sep}}|F) \to G(F^{\mathrm{sep}})$ satisfying $a_{\sigma\tau} = a_\sigma({}^\sigma a_\tau)$. We define the following equivalence relation on $Z^1(F,G)$: $(a_\sigma) \sim (b_\sigma)$ if and only if there exists an element $c$ in $G(F^{\mathrm{sep}})$ such that $a_\sigma = c^{-1} b_\sigma({}^\sigma c)$ for every $\sigma$ in $\mathrm{Gal}(F^{\mathrm{sep}}|F)$. The nonabelian Galois cohomology set $\mathrm{H}^1(F,G)$ is the set of equivalence classes of $Z^1(F,G)$.

Note that $\mathrm{H}^1(F,G)$ is a pointed set with base point given by the equivalence class of the cocycle $a : \mathrm{Gal}(F^{\mathrm{sep}}|F) \to G(F^{\mathrm{sep}})$ which maps every $\sigma$ in $\mathrm{Gal}(F^{\mathrm{sep}}|F)$ to the identity element 1 in $G(F^{\mathrm{sep}})$.

We will prove the statement claimed above, that $\mathrm{H}^1(F, PGL_n)$ classifies central simple algebras of degree $n$. Let $A$ be a degree $n$ central simple algebra over $F$. Fix an isomorphism

$$M_n(F^{\mathrm{sep}}) \overset{f}{\to} A \otimes_F F^{\mathrm{sep}}.$$

Note that $\mathrm{Gal}(F^{\mathrm{sep}}|F)$ acts on $A \otimes F^{\mathrm{sep}}$ as $\sigma(x \otimes \lambda) = x \otimes \sigma(\lambda)$ for $x$ in $A$ and $\lambda$ in $F^{\mathrm{sep}}$. Let $T$ be a matrix in $M_n(F^{\mathrm{sep}})$, and $v$ a vector in $(F^{\mathrm{sep}})^n$; $\mathrm{Gal}(F^{\mathrm{sep}}|F)$ acts on $T$ as ${}^\sigma T(v) = \sigma \circ T \circ \sigma^{-1}(v)$. Consider the diagram:

$$\begin{array}{ccc} M_n(F^{\mathrm{sep}}) & \xrightarrow{f} & A \otimes_F F^{\mathrm{sep}} \\ \big\downarrow{\scriptstyle\sigma} & & \big\downarrow{\scriptstyle\sigma} \\ M_n(F^{\mathrm{sep}}) & \xrightarrow{f} & A \otimes_F F^{\mathrm{sep}}. \end{array} \tag{2.10}$$

This diagram need not be commutative. The following element of $G(F^{\mathrm{sep}})$ is the obstruction for this diagram to be commutative:

$$a_\sigma := f^{-1} \circ \sigma \circ f \circ \sigma^{-1}. \tag{2.11}$$

We will check that $a_\sigma$ is in fact an element in $Z^1(F, PGL_n)$.

$$\begin{aligned} a_\sigma {}^\sigma a_\tau &= f^{-1} \circ \sigma \circ f \circ \sigma^{-1} \circ {}^\sigma(f^{-1} \circ \tau \circ f \circ \tau^{-1}) \\ &= f^{-1} \circ \sigma \circ f \circ \sigma^{-1} \circ \sigma \circ f^{-1} \circ \tau \circ f \circ \tau^{-1} \circ \sigma^{-1} \\ &= f^{-1} \circ \sigma\tau \circ f \circ (\sigma\tau)^{-1} \\ &= a_{\sigma\tau}. \end{aligned}$$

Now, let us rewrite the equation $a_\sigma = f^{-1} \circ \sigma \circ f \circ \sigma^{-1}$ as: $f \circ a_\sigma \circ \sigma = \sigma \circ f$. If we redefine the action of $\mathrm{Gal}(F^{\mathrm{sep}}|F)$ on $M_n(F^{\mathrm{sep}})$ as $\sigma * T = a_\sigma \circ \sigma(T)$, the diagram (2.10) commutes. (Note that $a_\sigma$ acts on $M_n(F^{\mathrm{sep}})$ by conjugation). All we need to check now is that this defines an action:

$$(a_\sigma \circ \sigma)(a_\tau \circ \tau) = a_\sigma \circ (\sigma \circ a_\tau) \circ \sigma^{-1} \circ \sigma \circ \tau = a_{\sigma\tau} \circ \sigma\tau.$$

We will call this the *twisted action* of $\mathrm{Gal}(F^{\mathrm{sep}}|F)$ on $M_n(F^{\mathrm{sep}})$.

We are now in a position to state our theorem.

**Theorem 2.3.11.** *Let $CSA_n(F)$ denote the pointed set of isomorphism classes of central simple algebras of degree $n$ over $F$, with distinguished element, the class of $M_n(F)$. Then there is a one to one correspondence between the following pointed sets, which is natural in $F$:*

$$CSA_n(F) \longleftrightarrow \mathrm{H}^1(F, PGL_n).$$

*Proof.* Let $A$ be a degree $n$ central simple algebra. Fix an isomorphism $M_n(F^{\mathrm{sep}}) \xrightarrow{f} A \otimes_F F^{\mathrm{sep}}$. We get a cocycle $a_\sigma$ defined in Equation (2.11). Suppose $g$ is another such isomorphism with corresponding cocycle $b_\sigma$. Consider the element $c := g^{-1}f$ in $PGL_n(F^{\mathrm{sep}})$. It is standard to check that $a_\sigma = c^{-1}b_\sigma({}^\sigma c)$. Thus, the map is well-defined. For surjectivity, given a cocycle $a_\sigma$, we redefine the action of $\mathrm{Gal}(F^{\mathrm{sep}}|F)$ as $\sigma * T = a_\sigma \circ \sigma(T)$ for $T$ in $M_n(F^{\mathrm{sep}})$. Using Theorem 2.2.2, one sees that there exists a central simple algebra $A$ of degree $n$ which gives rise to the cocycle $a_\sigma$. This also shows injectivity. For if $A$ and $B$ are two central simple algebras of degree $n$ giving rise to the same cocycle $a_\sigma$, consider the two diagrams as in (2.10) involving $A$ and $B$. With the twisted action on $M_n(F^{\mathrm{sep}})$ defined using the cocycle $a_\sigma$, these diagrams are commutative. Again using Theorem 2.2.2, we see that $A$ and $B$ are isomorphic. □

We saw in the course of proving Theorem 2.3.11 that giving a cocycle is the same as giving a "descent datum" for the corresponding twisted form. Taking the limit of the pointed sets $\mathrm{H}^1(F, PGL_n)$ as $n$ varies gives the set of isomorphism classes of central simple algebras over $F$. The same argument as in the proof of Theorem 2.3.11 shows:

**Theorem 2.3.12.** *Let $(V, \Phi)$ be a finite dimensional vector space equipped with a tensor $\Phi$, and let $G = \mathrm{Aut}(V, \Phi)$. Let $\mathrm{TW}(V, \Phi)$ be the pointed set of isomorphism classes* *of twisted forms of* $(V,\Phi)$ *with distinguished element* $(V,\Phi)$. *There is a one to one correspondence between the following pointed sets, which is natural in* $F$*:*

$$\mathrm{TW}(V,\Phi) \longleftrightarrow \mathrm{H}^1(F,G).$$

If $V$ is a finite dimensional vector space over $F^{\mathrm{sep}}$, one would expect $V$ to descend down to a vector space over $F$, with no additional structure. Since there is exactly one vector space of a given dimension up to isomorphism, it is reasonable to guess that $\mathrm{H}^1(F,GL_n) = 1$, and that is indeed true. The following theorem is a generalization of Theorem 2.3.3 and also goes by the name Hilbert Theorem 90. The proof is a modification of the proof of Theorem 2.3.3.

**Theorem 2.3.13.** $\mathrm{H}^1(F,GL_n) = 1$.

*Proof.* See [Serre1, Chapter X, Proposition 3]. □

For any algebraic group $G$ over $F$, $\mathrm{H}^1(F,G)$ classifies the so called principal homogenous spaces under $G$.

**Definition 2.3.14.** Let $P$ be a non-empty left $\mathrm{Gal}(F^{\mathrm{sep}}|F)$ set, equipped with a compatible right action of $G$. We will assume that this action is continuous. We say that $P$ is a *principal homogenous space* under $G$ if $G(F^{\mathrm{sep}})$ acts simply transitively on $P$.

**Theorem 2.3.15.** *Let* $\mathcal{P}_G$ *denote the pointed set of isomorphism classes of principal homogenous space under* $G$ *with distinguished element* $G$. *Then there is a one to one correspondence between the following pointed sets:*

$$\mathcal{P}_G \longleftrightarrow \mathrm{H}^1(F,G).$$

*Proof.* Let $P$ be a principal homogenous space under $G$. Let $p$ be an element in $P$, and $\sigma$ be in $\mathrm{Gal}(F^{\mathrm{sep}}|F)$. Since $G$ acts simply transitively on $P$, there exists a unique element $a_\sigma$ in $G(F^{sep})$ such that ${}^{\sigma}p = pa_\sigma$. One may check that $a_\sigma$ is a cocycle. For surjectivity, if $a_\sigma$ is a cocycle, define a new action of $\mathrm{Gal}(F^{\mathrm{sep}}|F)$ on $G(F^{\mathrm{sep}})$: for $\sigma$ in

$\mathrm{Gal}(F^{\mathrm{sep}}|F)$ and $g$ in $G(F^{\mathrm{sep}})$, $\sigma * g := a_\sigma({}^\sigma g)$. It is not hard to check that this is a principal homogenous space under $G$. □

Note that principal homogenous spaces under a linear algebraic group $G$ may be identified with the $F^{\mathrm{sep}}$-rational points of a variety. To see this, first note that the category of continuous $\mathrm{Gal}(F^{sep}|F)$ sets is equivalent to the category of sheaves of sets on the small étale site $\mathrm{Spec}(k)_{\text{ét}}$. Thus a principal homogenous space under $G$ can be identified with a sheaf of sets on $\mathrm{Spec}(k)_{\text{ét}}$. Such sheaves are also called as *$G$-torsors*. It is shown in [Milne] on Pg 120 that $G$-torsors are representable by varieties.

**Theorem 2.3.16.** *Suppose that $Z$ is a closed subgroup of $G$. We have the following long exact sequence of pointed sets:*

$$1 \to Z(F) \to G(F) \to (G/Z)(F) \to \mathrm{H}^1(F,Z) \to \mathrm{H}^1(F,G).$$

*Further, if $Z$ is contained in the center of $G$, and $H$ denotes the quotient group $G/Z$, then one may extend it to the following longer exact sequence:*

$$1 \to Z(F) \to G(F) \to H(F) \to \mathrm{H}^1(F,Z) \to \mathrm{H}^1(F,G) \to \mathrm{H}^1(F,H) \to \mathrm{H}^2(F,Z).$$

*Proof.* See [Serre2, Proposition 43]. □

Consider the exact sequence

$$1 \to \mathbb{G}_m(F^{\mathrm{sep}}) \to GL_n(F^{\mathrm{sep}}) \to PGL_n(F^{\mathrm{sep}}) \to 1.$$

Taking Galois invariants, and using Theorem 2.3.3, one obtains the following exact sequence

$$1 \to F^\times \to GL_n(F) \to PGL_n(F) \to 1.$$

Extending this long exact sequence, one obtains the injective (in the category of pointed sets) map:

$$1 \to \mathrm{H}^1(F, PGL_n) \to \mathrm{H}^2(F,\mathbb{G}_m) = \mathrm{Br}(F).$$

This map turns out to be as we might expect, sending the isomorphism class of a central simple algebra $A$ to its Brauer class $[A]$.

## 2.4 Brauer Dimension

### 2.4.1 Period-Index problem

**Definition 2.4.1.** Let $\alpha$ be an element of $\mathrm{Br}(F)$. The *period* of $\alpha$, denoted by $per(\alpha)$, is the smallest positive integer $n$ such that $n\alpha = 0$.

**Definition 2.4.2.** Let $\alpha$ be an element of $\mathrm{Br}(F)$, with underlying division algebra $D$. The *index* of $\alpha$, denoted by $ind(\alpha)$, is defined as the degree of $D$.

**Theorem 2.4.3.** *For any $\alpha$ in $\mathrm{Br}(F)$, the index of $\alpha$, denoted by $ind(\alpha)$ is the gcd (and the minimum) of degrees of field extensions $L/F$ such that $\mathrm{Res}_{L/F}(\alpha) = 0$ in $\mathrm{Br}(L)$.*

*Proof.* Let $D$ be the underlying division algebra of $\alpha$. If $L$ is a maximal subfield in $D$, by Theorem 2.1.12, $D \otimes_F L$ is isomorphic to a matrix algebra over $L$. Note that $L$ is also strictly maximal (see Definition 2.1.10). Therefore $[L : F] = \deg(D)$.

Let $K/F$ be any splitting field of $D$. Then $D$ is Brauer equivalent to an algebra $B \cong M_n(D)$ containing $K$ as a strictly maximal subfield in $B$. Therefore $[K : F] = \deg(B) = n \deg(D)$, and $\deg(D)$ divides the degree of every splitting field. As we saw, $\deg(D)$ also equals the degree of a splitting field. This establishes that $ind(\alpha)$ is the gcd and the minimum of degrees of splitting fields of $\alpha$, and is equal to the degree of a maximal subfield in its underlying division algebra. $\square$

**Theorem 2.4.4.**

1. *$\mathrm{Br}(F)$ is a torsion group.*
2. *For every element $\alpha$ in $\mathrm{Br}(F)$, the period of $\alpha$ divides the index of $\alpha$.*
3. *Moreover, $per(\alpha)$ and $ind(\alpha)$ share the same prime factors.*

*Proof.* Let $L/F$ be a maximal subfield of the underlying division algebra. By Theorem 2.4.3, $\mathrm{Res}_{L/F}(\alpha) = 0$ and $ind(\alpha) = [L : F]$. Note that, $[L : F]\alpha = \mathrm{Cor} \circ \mathrm{Res}(\alpha) = 0$. The period of $\alpha$, being the order of $\alpha$, therefore divides $ind(\alpha)$. This also shows that $\mathrm{Br}(F)$ is torsion.

We now establish the last statement. Let $\alpha$ be a non-trivial element of $\mathrm{Br}(F)$, and $p$ be a prime factor of $ind(\alpha)$. Let $L/F$ be a separable splitting field of $\alpha$ with degree equal to $ind(\alpha)$. Denote the Galois closure of $L/F$ by $\widetilde{L}/F$. Let $H_p$ be the Sylow $p$ subgroup of the Galois group of $\widetilde{L}/F$. Set $K \coloneqq \widetilde{L}^{H_p}$. Note that $per(\alpha \otimes_F K)$ divides $per(\alpha)$. Since $\widetilde{L}/K$ splits $\alpha \otimes K$, by Theorem 2.4.3, $ind(\alpha \otimes K)$ divides $[\widetilde{L} : K]$. Thus, $ind(\alpha \otimes K)$ is a power of $p$. By the second statement of the theorem proved in the previous paragraph, $per(\alpha \otimes K)$ divides $ind(\alpha \otimes K)$. As a result, $per(\alpha \otimes K)$ is a power of $p$. Therefore, $p$ divides $per(\alpha)$. □

**Remark 2.4.5.** *Theorem 2.4.4 shows that every cyclic algebra of prime degree $\ell$ has period $\ell$. In particular, every non-trivial quaternion algebra has period 2.*

By Theorem 2.4.4, for every $\alpha$ in $\mathrm{Br}(F)$, $ind(\alpha)$ divides some power of $per(\alpha)$, i.e.,

$$ind(\alpha) | [per(\alpha)]^{N(\alpha)},$$

where $N(\alpha)$ is the smallest positive integer satisfying the divisibility condition above for $\alpha$. Note that $ind(\alpha)$ measures how large the underlying division algebra can be, or equivalently how hard it is to split the division algebra. It is natural to wonder whether there is a uniform bound for this measure. In other words, can we uniformly bound the integer $N(\alpha)$? Of course, the question whether or not there is a bound, should depend upon the field.

**Definition 2.4.6.** Let $F$ be a field and $\ell$ be a prime. The *Brauer $\ell$-dimension* of $F$, denoted as $\mathrm{Brd}_\ell(F)$ is the supremum of $N(\alpha)$ as $\alpha$ varies in the $\ell$-primary torsion subgroup of $\mathrm{Br}(L)$, and $L/F$ varies over finite degree field extensions.
The *Brauer dimension* of $F$, denoted by $\mathrm{Brd}(F)$, is the supremum of $\mathrm{Brd}_\ell(F)$ as $\ell$ varies.

When $F = \mathbb{R}$, there is a uniform bound, since the only non-trivial division algebra over $\mathbb{R}$ is the quaternion algebra $(-1, -1)$. Therefore $\mathrm{Brd}(\mathbb{R}) = 1$. However, determining this uniform bound is not an easy problem in general, since finding the index of a given division algebra is not easy. Even to determine whether a given central simple algebra is division or not is hard enough.

We would like to compute $\mathrm{Brd}(F)$ for nice fields arising in arithmetic and geometry, starting with local fields and global fields. The question that, for number fields whether $\mathrm{Brd}(F) = 1$, was asked by Brauer in a letter to Hasse (see [Roq]). For $C_2$ fields, Artin asks whether $\mathrm{Brd}(F) = 1$? In view of some circumstantial evidence, we record the following folklore problem here. The problem is known as the period-index problem.

**Question 1.** *Let $F$ be the function field of a variety over either an algebraically closed field, finite field, non-archimedean local field or a global field (any reasonably "nice" field). Compute* $\mathrm{Brd}(F)$.

We record some known answers below. The techniques used in obtaining these answers vary considerably. One of the major advances in systematically dealing with the problem is the approach of Lieblich, which uses the theory of moduli spaces of twisted sheaves. The field patching technique of Harbater-Hartmann-Krashen succesfully deals with this problem over function fields of curves over complete discretely valued fields.

**Example 2.4.7.**

1. *If $F$ is a local field or a global field, then* $\mathrm{Brd}(F) = 1$. *This is a classical result going back to the work of Albert, Brauer, Hasse and Noether.*

2. *If $F$ is the function field of a curve over a local field, then* $\mathrm{Brd}_\ell(F) = 2$. *For $\ell$ not equal to the residue characteristic, this was proved by Saltman in [Salt]. Parimala-Suresh in [Pa-Su2] completed this computation in the bad, mixed characteristic case (i.e., when characteristic of the residue field equals $\ell$).*

3. *If $F$ is the function field of a curve over an $m$-local field, then* $\mathrm{Brd}_\ell(F) = m + 1$, *when $\ell$ is not equal to the characteristic of the smallest residue field. This was proved by Lieblich in [Lie1], and independently by Harbater-Hartmann-Krashen in [HHK09].*

4. *If $F$ is the function field of a surface over a finite field, then* $\mathrm{Brd}(F) = 2$. *This was proved by Lieblich in [Lie].*

5. *If $F$ is the function field of a surface over an algebraically closed field, then* $\mathrm{Brd}(F) = 1$. *This is due to deJong (for characteristic* $0$*), proved in [dJ].*

6. *Let $F$ be the function field of a surface over $\mathbb{Q}_p$. Recently, Antieau-Auel-Ingalls-Krashen-Lieblich in [AAIKL] show that* $\mathrm{Brd}_\ell(F) = 3$ *for $\ell$ coprime to* $6p$.

We will end this section with a Lemma which says that to compute $\mathrm{Brd}_\ell(F)$, it is enough to compute the indices of Brauer classes with period $\ell$.

**Lemma 2.4.8.** *Suppose that for every finite extension $L/F$ and every Brauer class $\alpha$ in the $\ell$-torsion part of* $\mathrm{Br}(L)$*, $ind(\alpha)$ divides $\ell^N$. Further, $N$ is the smallest such integer. Then* $\mathrm{Brd}_\ell(F) = N$.

*Proof.* Let $L/F$ be a finite extension. Let $\beta$ be a Brauer class in $\mathrm{Br}(L/F)$ of period $\ell^d$. We will show by induction on $d$ that $ind(\beta)$ divides $\ell^{dN}$.

The base case $d = 1$ follows by the hypothesis in the Lemma.

Now let $\beta' = \ell\beta$. Thus, the period of $\beta'$ is $\ell^{d-1}$. By the induction hypothesis, there exists a field extension $M/L$ of degree dividing $\ell^{(d-1)N}$. Thus, $\beta' \otimes M = 0$, which means that $\beta \otimes M$ has period $\ell$. Therefore, there exists a field extension $N/M$ of degree $\ell^N$ which splits $\beta \otimes M$. Thus, $N/L$ splits $\beta$. Now the degree of $[N : L]$ is $\ell^N \ell^{(d-1)N} = \ell^{dN}$. Or in other words, $\mathrm{Brd}_\ell(F) = N$. □

### 2.4.2 $C_i$ property and cohomological dimension

We will see how arithmetic properties behave when passes to a higher transcendence degree field or to the completion, one of the theme that is explored in this thesis. At this point, we should also collect more examples of division algebras over specific fields, and Brauer $\ell$-dimension of some fields. Let us start with fields sharing some arithmetic properties with finite fields. We will take a detour into the $C_i$ property and cohomological dimension of fields, and apply the Theorems in computing the Brauer group of certain fields (see Example 2.4.13), and also the Brauer $\ell$-dimension for $C_2$ fields for $\ell = 2, 3$ (see Proposition 2.4.20).

**Definition 2.4.9.** Let $F$ be a field. We say that $F$ satisfies the $C_i$ *property* (or simply, $F$ is $C_i$) if every homogenous polynomial of degree $d$ in $n$ variables satisfying $n > d^i$ has a nontrivial solution.

Note that any algebraically closed field is a $C_0$ field, and in fact $C_i$ for every $i \geq 0$. We will see some examples of $C_i$ fields for $i > 0$. The very first example of $C_1$ fields which are not $C_0$ are:

**Theorem 2.4.10** (Chevalley). *Finite fields have the $C_1$ property.*

*Proof.* See [G-S, Theorem 6.2.6] □

**Theorem 2.4.11** (Lang). *Let $F$ be a field satisfying the $C_i$ property. If $L/F$ is an extension of transcendence degree $d$, then $L$ is a $C_{i+d}$ field.*

*Proof.* See [Lang, Theorem 6] □

**Example 2.4.12.** *Let $F$ be a $C_1$ field. We claim that $\mathrm{Br}(F) = 0$. Let $D$ be a division algebra over $F$. We choose a basis for $D$ as an $F$-vector space, and hence view it as an affine space $\mathbb{A}_F^{n^2}$ over $F$. Thus the reduced norm on $D$ (denoted by $Nrd$) can be viewed as a homogenous polynomial of degree $n$ in $n^2$ variables. Since $F$ is a $C_1$ field, the equation $Nrd = 0$ has a non-trivial solution if $n > 1$. As a result, there exists $d$ in $D^\times$ such that $Nrd(d) = 0$. But that is impossible since $d$ is invertible and $Nrd$ is multiplicative.*

**Remark 2.4.13.**

1. *The above example and Theorem 2.4.10 shows that $\mathrm{Br}(\mathbb{F}_q) = 0$.*
2. *If $F$ is the function field of a curve over an algebraically closed field, then $\mathrm{Br}(F) = 0$ by Theorem 2.4.11 and the above Example.*

The $C_i$ property is a measure of how far a field is from being algebraically closed. It however does not seem that natural. There is a more natural notion called cohomological dimension, which in some sense also measures how "easy" is it for varieties to have rational points.

**Definition 2.4.14.** Let $F$ be with $char(F) \neq p$. Then, we say that the $p$-cohomological dimension of $F$, denoted by $\mathrm{cd}_p(F)$ is at most $n$ if for every algebraic extension $K/F$, $\mathrm{H}^{n+1}(K, \mu_p) = 0$.

**Remark 2.4.15.** *This is not the most correct way to define p-cohomological dimension since we had to make the assumption that char*$(F) \neq p$. *The definition in [Serre2] goes as: we say that p-cohomological dimension of* $F$ *(possibly also of characteristic* $p$*) is at most* $n$ *if the* $p$*-primary component of* $\mathrm{H}^{n+1}(F, A)$ *is* $0$ *for every torsion, discrete Galois module* $A$.

*One can show that if* $F$ *has characteristic equal to* $p$, *then* $\mathrm{cd}_p(F) \leq 1$ *(see [Serre2, Chapter 2, Section 2.2]).*

**Example 2.4.16.** *Let* $F$ *be a* $C_1$ *field. Then* $\mathrm{cd}_p(F) \leq 1$. *By the above Remark, we may assume that char*$(F)$ *is not equal to* $p$. *Thus, we need to show that* $\mathrm{H}^2(F, \mu_p) = 0$. *By the long exact sequence in Galois cohomology of the short exact sequence*

$$1 \to \mu_p \to \mathbb{G}_m \xrightarrow{\times p} \mathbb{G}_m \to 1,$$

*and the Hilbert Theorem 90 (see 2.3.3), one sees that* ${}_p\mathrm{Br}(F) \cong \mathrm{H}^2(F, \mu_p)$. *Since* $F$ *is* $C_1$, *by Remark 2.4.13,* $\mathrm{Br}(F) = 0$. *Therefore,* ${}_p\mathrm{Br}(F) = 0$. *Thus,* $\mathrm{H}^2(F, \mu_p) = 0$.

Serre in [Serre2] asks the question: If $F$ is $C_i$, then does it imply that $\mathrm{cd}_p(F) \leq i$? He shows that this is indeed true when $p = 2$ using Milnor's conjecture. For other primes $p$, it is not known whether this is true.

There is a transition theorem analogous to Theorem 2.4.11 for cohomological dimension.

**Theorem 2.4.17.** *Let* $K/F$ *be a field of transcendence degree at most* $d$, *then*

$$\mathrm{cd}_p(K) \leq \mathrm{cd}_p(F) + d.$$

*Proof.* Clearly, when $d = 0$, the inequality is satisfied, following from the definition of $p$-cohomological dimension. It is enough to show the statement holds when the transcendence degree of $K/F$ is one. The general case follows inductively. If $K$ has transcendence degree one, then $K$ is a finite extension of $F(t)$. Thus, we just need to prove the statement for $K = F(t)$. Consider the extension $L := F^{\mathrm{sep}}(t)$, and denote the separable closure of $F(t)$ by $M$. Note that $\mathrm{Gal}(F^{\mathrm{sep}}(t)|F(t))$ can be identified with $\mathrm{Gal}(F^{\mathrm{sep}}|F)$. Since $L$ is a $C_1$ field, $\mathrm{cd}_p(L) \leq 1$. Finally using [Serre2, Chapter 1 Proposition 15], one sees that $\mathrm{cd}_p(F(t)) \leq \mathrm{cd}_p(F) + 1$. ◻

**Theorem 2.4.18.** *Let $F$ be a complete discretely valued field with residue field $k$. Then*

$$\mathrm{cd}_p(F) \le \mathrm{cd}_p(k) + 1.$$

*Proof.* Let $F^{ur}$ be the maximal unramified extension of $F$. We can identify $\mathrm{Gal}(F^{ur}|F)$ with $\mathrm{Gal}(k^{\mathrm{sep}}|k)$. It is shown in [Lang], that $K^{ur}$ satisfies the $C_1$ property. Thus $\mathrm{cd}_p(K^{ur}) \le 1$. Again, using [Serre2, Chapter 1 Proposition 15], the inequality follows. □

**Remark 2.4.19.** *Note that the above theorem has no analogue for the $C_i$ property. It is known that $\mathbb{Q}_2$ does not satisfy the $C_2$ property (see references in [Serre2]).*

We will end this section by computing the Brauer 2-dimension and Brauer 3-dimension of fields satisfying the $C_2$ property.

**Proposition 2.4.20** (Artin)**.** *Let $F$ be a field satisfying the $C_2$ property. We will assume that char$(F) \neq 2, 3$. Then* $\mathrm{Brd}_2(F) \le 1$ *and* $\mathrm{Brd}_3(F) \le 1$.

*Proof.* First, by Lemma 2.4.8, it is enough to show this for period 2 and period 3 algebras. In view of Theorem 2.4.11, it is also enough to assume that these algebras are over $F$.

We will only prove that $\mathrm{Brd}_3(F) \le 1$. The statement that $\mathrm{Brd}_2(F) \le 1$ follows from a similar reasoning. To show that $\mathrm{Brd}_3(F) \le 1$, we will first prove that any two degree 3 division algebras, $A$ and $B$, can be simultaneously split by a field extension of degree 3. The proof for the degree 2 case is similar. If $F(\alpha)/F$ is a degree 3 splitting field of $A$, $F(\alpha)$ sits inside $A$.

Let $\{e_i\}$ be a basis for the $F$-vector space $A$. Consider the element $\sum_{i=1}^{9} x_i e_i$ in $A \otimes_F F(x_1, \cdots, x_9)$. The reduced characteristic polynomial of this generic element is called the generic reduced characteristic polynomial. If one extends scalars to $F^{\mathrm{sep}}$, this polynomial agrees with the characteristic polynomial of the generic $3 \times 3$ matrix $[x_{ij}]$. Thus the generic reduced characteristic polynomial is a degree 3 polynomial: $t^3 + a_1 t^2 + a_2 t + a_3$, where each $a_i$ is a homogenous polynomial of degree $i$ with coefficients in $F$.

The reduced characteristic polynomial of a generic element in $A$ is the same as the minimal polynomial of a generic degree 3 subfield in $A$. Therefore a field extension $F(\alpha)$ sits inside $A$ and $B$ if and only if there is a solution in $F$ to the equations given by setting the coefficients of the generic reduced characteristic polynomials of $A$ and $B$ equal.

Let $a_1$, $a_2$ and $a_3$ be the coefficients of the generic reduced characteristic polynomial of $A$, and let $b_1$, $b_2$ and $b_3$ be the coefficients of the generic reduced characteristic polynomial of $B$. Consider the system of homogenous polynomials: $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

This is a system of polynomials on $\mathbb{A}(A) \times \mathbb{A}(B)$ involving 18 variables. Note that $18 > 1^2 + 2^2 + 3^2$. Since $F$ satisfies the $C_2$ property, this system has a non-trivial solution. Therefore there exists a common degree 3 field extension in both $A$ and $B$. Since, this is a strictly maximal subfield of both $A$ and $B$, it splits both the division algebras $A$ and $B$.

Without loss of generality, we can assume that $F$ contains the cube roots of unity. By the Merkurjev-Suslin theorem (see [Me-Sus]), any period 3 algebra $A$ is Brauer equivalent to a tensor product of cyclic algebras. Let $A = D_1 \otimes \cdots \otimes D_n$, where each $D_i$ is a cyclic algebra. We will show that $ind(A) = 3$, by induction on $n$. If $n = 1$, there is nothing to prove. Consider the algebra $A' = D_1 \otimes \cdots \otimes D_{n-1}$. By the induction hypothesis, $ind(A') = 3$. Replace $A'$ with its underlying division algebra $D'$. Since $D$ and $D'$ have a common degree three splitting field extension, it follows that $ind(A) = 3$. □

### 2.4.3 Witt exact sequence

The Brauer group of a complete field can be described in terms of the Brauer group of the residue field and the group of characters of the Galois group of the residue field. The Witt exact sequence (see 2.4.24) makes this statement precise. The ramification map, one of the homomorphisms appearing in the sequence, helps in determining whether a given Brauer class over a given field is non-trivial, or equivalently whether a division algebra is split. Usually, one passes to the completion at a suitable place of that field, and shows that the Brauer class is non-trivial there by inspecting its image under the ramification map.

**Proposition 2.4.21.** *Let $K$ be a complete discretely valued field with residue field $k$. Assume that char$(k) = p$. Let $\alpha$ be in $\mathrm{Br}(K)$ with period prime to $p$. Then $\alpha \otimes_F F^{nr} = 0$.*

*Proof.* We will sketch the proof here and refer the reader to [Serre1]. It is sufficient to show that if $\alpha$ is in $\mathrm{Br}(K^{nr})$ with period prime to $p$, then $\alpha = 0$. The main idea is to pass to the maximal prime-to-$p$ extension $L/F^{ur}$ and observe that $\alpha$ should split $L$, since $\alpha$ has period $p$ and hence should have index a power of $p$, and therefore cannot be split by a prime to $p$ extension. The advantage in passing to $L$ is that the Galois group $\mathrm{Gal}(L/F^{nr})$ is isomorphic to $\widehat{\mathbb{Z}}' = \varprojlim_{(n,p)=1} \mathbb{Z}/n\mathbb{Z}$. This is because $L/F^{nr}$ is a tamely ramified extension (see [Serre2, Chapter II, Section 4.3]). One then shows that $\mathrm{Br}(L/K^{nr}) \cong \mathrm{H}^2(\widehat{\mathbb{Z}}', L^\times) \cong \mathrm{H}^2(\widehat{\mathbb{Z}}', \mu)$, where $\mu$ is the group of roots of unity. The group on the right hand side then vanishes for cohomological dimension reasons. □

We will make the same assumption as in the statement of Proposition 2.4.21. Note that the Galois group $\mathrm{Gal}(K^{nr}|K)$ can be identified with $\mathrm{Gal}(k^{\mathrm{sep}}|k)$. Let $U_{nr}$ be the kernel of the valuation maps. We therefore have the split exact sequence of $\mathrm{Gal}(k^{\mathrm{sep}|k})$-modules:

$$0 \to U_{nr} \to K^\times \xrightarrow{v} \mathbb{Z} \to 0. \tag{2.12}$$

The map $s : \mathbb{Z} \to K^\times$ which sends 1 to $\pi$, where $\pi$ is a parameter, defines a splitting of the sequence. The long exact sequence of $\mathrm{Gal}(k^{\mathrm{sep}}|k)$-modules therefore splits into the following split short exact sequences, for every $i > 0$:

$$0 \to \mathrm{H}^i(k, U_{nr}) \to \mathrm{H}^i(k, K^\times) \to \mathrm{H}^i(k, \mathbb{Z}) \to 0.$$

Now consider the exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

We thus have the following long exact sequence in Galois cohomology:

$$\cdots \to \mathrm{H}^{i-1}(k, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^i(k, \mathbb{Z}) \to \mathrm{H}^i(k, \mathbb{Q}) \to \mathrm{H}^i(k, \mathbb{Q}/\mathbb{Z}) \to \cdots.$$

Now notice that $\mathrm{H}^i(G, \mathbb{Q})$ is a $\mathbb{Q}$ vector space and also torsion for a finite group $G$. Therefore, $\mathrm{H}^i(k, \mathbb{Q}) = 0$ since $\mathrm{Gal}(k^{\mathrm{sep}}|k)$ is profinite. Therefore, we have:

$$\mathrm{H}^i(k, \mathbb{Z}) \cong \mathrm{H}^{i-1}(k, \mathbb{Q}/\mathbb{Z}),$$

for $i > 1$. We record the following technical lemma:

**Lemma 2.4.22.** *For $i > 1$, we have the following isomorphism:*

$$\mathrm{H}^i(k, U_{nr}) \cong \mathrm{H}^i(k, k^{\mathrm{sep}\times}).$$

*Proof.* We will show this in the special case when $K = k((t))$ and $char(k) = 0$, referring the reader to [Serre1] for the proof in the general case. Note that $K^{nr} = k^{\mathrm{sep}}((t))$ and $U_{nr} = k^{\mathrm{sep}}[[t]]^\times$. Observe that there exists a homomorphism $k^{\mathrm{sep}}[[t]]^\times \to (k^{\mathrm{sep}})^\times$ obtained by sending the formal power series $f(t)$ to $f(0)$. Set $U^1 := 1 + tk^{\mathrm{sep}}[[t]]$. Now, consider the short exact sequence:

$$0 \to U^1 \to k^{\mathrm{sep}}[[t]]^\times \to k^{\mathrm{sep}\times} \to 0.$$

To establish the result, it is enough to show that $\mathrm{H}^i(k, U^1) = 0$. Note that by Hensel's lemma, it is easy to show that $U^1$ is a uniquely divisible $\mathrm{Gal}(k^{\mathrm{sep}}|k)$-module. Therefore, $U^1$ has the structure of a $\mathbb{Q}$-vector space. This establishes the claim. □

We will denote the prime-to-$p$ part of the Brauer group of a field $K$ by $\mathrm{Br}(K)'$, and $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(k^{\mathrm{sep}}|k), \mathbb{Q}/\mathbb{Z})'$ will denote the prime-to-$p$ part of the group of continuous characters of the Galois group $\mathrm{Gal}(k^{\mathrm{sep}}|k)$. Putting together the above discussion, we obtain:

**Theorem 2.4.23** (Witt)**.** *Let $K$ be a complete discretely valued field with residue field $k$. Suppose that $char(k) = p$. Then, we have the following split exact sequence:*

$$0 \to \mathrm{Br}(k)' \to \mathrm{Br}(K)' \xrightarrow{\partial_v} \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(k^{\mathrm{sep}}|k)\mathbb{Q}/\mathbb{Z})' \to 0. \tag{2.13}$$

*Proof.* Note that that $\mathrm{H}^2(\Gamma_k, K^\times) \cong \mathrm{Br}(K^{nr}|K)$. By Proposition 2.4.21, it follows that $\mathrm{Br}(K)' \cong \mathrm{H}^2(k, K^\times)'$. Finally since $\mathbb{Q}/\mathbb{Z}$ is a trivial $\mathrm{Gal}(k^{\mathrm{sep}}|k)$-module, $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ can be identified with $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(k^{\mathrm{sep}}|k), \mathbb{Q}/\mathbb{Z})$. □

In fact, we also have the following generalization of (2.13). We will however not prove it here. The main ideas used in the proof, to a great extent, are contained in obtaining Theorem 2.4.23.

**Theorem 2.4.24.** *Let $n > 1$ be coprime to the char(k). For $i, j > 0$, we have the following split exact sequence:*

$$0 \to \mathrm{H}^i(k, {\mu_n}^{\otimes j}) \to \mathrm{H}^i(K, {\mu_n}^{\otimes j}) \xrightarrow{\partial_v^i} \mathrm{H}^{i-1}(k, {\mu_n}^{\otimes (j-1)}) \to 0. \tag{2.14}$$

*Proof.* See [G-S, Corollary 6.8.8]. □

The map $\partial_v^i$ is called the *ramification map*. For $i = 2$ and $j = 1$, the map $\partial_v^2 : {}_n\mathrm{Br}(K) \to \mathrm{H}^1(k, \mathbb{Z}/n\mathbb{Z})$ can be described explicitly on Brauer classes of cyclic algebras as $\partial_v^2([(\chi, \pi)]) = \chi$. For $i = j = 1$, the ramification map can be identified with valuation mod $n$: $K^\times/K^{\times n} \xrightarrow{v} \mathbb{Z}/n\mathbb{Z}$. Thus the ramification map can be thought of as a higher cohomological analogue of the valuation map. This also justifies the term "ramification" since for $a$ in $K^\times/K^{\times n}$, the fact that mod $n$ valuation $v(a)$ equals 0 implies that the extension $K(\sqrt[n]{a})/K$ is unramified. Analogous to this, if the ramification of a Brauer class is trivial at a discrete valuation, then by Theorem 2.4.23, the class can be specialized uniquely to the residue field. In this sense, the ramification map determines whether a Brauer class can be locally specialized.

Let $F$ be the function field of a smooth projective variety $X$, and $\alpha$ be an element in $\mathrm{Br}(F)$. The codimension one points of $X$ provide discrete valuations of $F$. The ramification maps at these discrete valuations, help in determining whether $\alpha$ can be specialized at the residue fields of these codimension one points. The subgroup of unramified Brauer classes of $X$ is a very useful birational invariant. It has been used to produce counterexamples of unirational varieties that are not rational (see [CT]).

**Proposition 2.4.25.** *Let $L/K$ be an extension of complete discretely valued fields with ramification index $e$. Let $l/k$ be the corresponding extension of residue fields. We assume as before that char$(k)$ is coprime to $n$. Then the following diagram is commutative:*

$$\begin{tikzcd} \mathrm{H}^2(K, \mu_n) \arrow[r, "\partial_v"] \arrow[d, "res_{L/K}"] & \mathrm{H}^1(k, \mathbb{Z}/n\mathbb{Z}) \arrow[d, "e\ res_{l/k}"] \\ \mathrm{H}^2(L, \mu_n) \arrow[r, "\partial_v"] & \mathrm{H}^1(l, \mathbb{Z}/n\mathbb{Z}). \end{tikzcd} \tag{2.15}$$

*Proof.* See [CT, Proposition 3.3.1]. □

One can compute the Brauer group of local fields using the Witt exact sequence. Although we have established it away from the characteristic of the residue field, by modifying the proof suitably, one obtains the following computation of Brauer groups of nonarchimedean local fields:

**Example 2.4.26.** *1. Let $K$ be the completion of a number field at a nonarchimedean place (so a finite extension of $\mathbb{Q}_p$). Then the ramification map gives the isomorphism:*

$$\mathrm{Br}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

*2. Let $K$ be the completion of a function field of a smooth curve over a finite field (so isomorphic to $\mathbb{F}_q((t))$). Then the ramification map gives the isomorphism:*

$$\mathrm{Br}(\mathbb{F}_q((t))) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

The isomorphisms follow from the Witt exact sequence and the fact that the residue fields in both cases is a finite field, and thus has trivial Brauer group. Recall that the absolute Galois group of a finite field is $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$. We claim that $\mathrm{Hom}_{\mathrm{cont}}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$. Let $\theta$ be the canonical, topological generator of $\widehat{\mathbb{Z}}$, i.e., the closure of the subgroup of $\widehat{\mathbb{Z}}$ generated by $\theta$ equals $\widehat{\mathbb{Z}}$. Let $f$ be in $\mathrm{Hom}_{\mathrm{cont}}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$. Define the map $ev$ : $\mathrm{Hom}_{\mathrm{cont}}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ such that $ev(f) = f(\theta)$. This map is injective since if for some $f$, $f(\theta) = 0$ then $f$ vanishes on the subgroup generated by $\theta$. Since this subgroup is dense in $\widehat{\mathbb{Z}}$ and $f$ is continuous, it follows that $f = 0$. Thus the map defined above is injective. For surjectivity: if $\mu$ is an element in $\mathbb{Q}/\mathbb{Z}$, we obtain a map $f : \widehat{\mathbb{Z}} \to \mathbb{Q}/\mathbb{Z}$ such that $f(\theta) = \mu$.

**Theorem 2.4.27.** *Let $F$ be a nonarchimedean local field. For $\alpha$ in $\mathrm{Br}(F)$, $per(\alpha) = ind(\alpha)$. In other words, $\mathrm{Brd}(F) = 1$.*

*Proof.* By the primary decomposition theorem, we may assume that $per(\alpha)$ is $\ell$ primary for a prime $\ell$. Using Lemma 2.4.8, we may further assume that $per(\alpha) = \ell$. Let $k$ be the residue field of $F$. Note that $k$ is a finite field. Let $\chi$ be the ramification of $\alpha$ with the corresponding cyclic extension $l/k$. Let $L/K$ be the unramified lift of $l/k$. Since

$\mathrm{Ker}(\chi) = \mathrm{Gal}(k^{\mathrm{sep}}|l)$, $\chi|\Gamma_l = 0$ and therefore by the commutativity of diagram (2.15), $\alpha \otimes L = 0$. From this it follows that, $ind(\alpha)$ divides $\ell$. Since $\alpha$ is non-trivial and $\ell$ is a prime, $ind(\alpha) = \ell$. ☐

The following proposition is an exercise in [G-S]. It can be seen as a generalization of the above theorem. It will be used in Proposition 2.4.29 in this section, and a couple of times later as well.

**Proposition 2.4.28.** *Let $\alpha = \alpha_0 + (\chi, \pi)$ be a class in $\mathrm{Br}(K)$ with period $\ell$ coprime to $p$, where $\alpha_0$ is a class in $\mathrm{Br}(k)$ and $\chi$ is a character of $\Gamma_k$. Let $m/k$ be the cyclic extension of degree $\ell$ corresponding to $\chi$ Then,*

$$ind(\alpha) = \ell \cdot ind(\alpha_0 \otimes m).$$

*Proof.* Let $M/K$ be the unramified lift of $m/k$. Observe that $M/K$ splits $\alpha_0$. If $M'/M$ is a splitting field of $\alpha_0 \otimes M$, then the extension $M'/K$ splits $\alpha$. Therefore $\ell \cdot ind(\alpha_0 \otimes m)$ divides $ind(\alpha)$. Thus it suffices to show that $ind(\alpha)$ divides $\ell \cdot ind(\alpha_0 \otimes m)$.
Now let $L/K$ be any splitting field of $\alpha$ with residue field extension $l/k$, and ramification index $e$. Let $K'/K$ be the largest unramified extension in $L/K$. By the commutativity of the diagram (2.15), and the fact that L/K splits $\alpha$, it follows that $0 = \partial(\alpha \otimes L) = e \cdot \chi|\Gamma_l$. If $e$ is coprime to $\ell$, $\Gamma_l \subseteq \mathrm{Ker}(\chi) = \Gamma_m$. This implies that $m \subseteq l$ and hence $M \subseteq K'$. Note that $\alpha \otimes M = \alpha_0 \otimes M$. Since $L/M$ splits $\alpha \otimes M$, it splits $\alpha_0 \otimes M$. Thus $ind(\alpha_0 \otimes m)$ divides $[L : M]$. Therefore, $\ell \cdot ind(\alpha_0 \otimes m)$ divides $[L : M][M : K] = [L : K]$.

If $\ell$ divides $e$, then note that $\ell$ divides $[L : K']$. We claim that $K'$ splits $\alpha_0 \otimes M$. Since $K'/K$ is the largest unramified extension in $L$, the residue fields of $K'$ and $L$ are the same. Therefore, the reduction of $\alpha_0 \otimes K'$ to the residue field is trivial since $L$ splits $\alpha_0 \otimes K'$. Thus, $\alpha_0 \otimes K' = 0$. Thus, $ind(\alpha_0 \otimes m)$ divides $[K' : M]$. From this, it follows that $\ell \cdot ind(\alpha_0 \otimes m)$ divides $[L : K]$. Since $\ell \cdot ind(\alpha_0 \otimes m)$ divides the degree of any splitting field, $\ell \cdot ind(\alpha_0 \otimes m)$ divides $ind(\alpha)$ by Theorem 2.4.3. ☐

One needs to restrict to "nice" fields to answer Question 1 (see Subsection 2.4.1). As we see below in Proposition 2.4.29, one can construct Brauer classes of arbitrary high index.

**Proposition 2.4.29.** *Let $F$ be a purely transcendental function field in countably many variables over $\mathbb{C}$. Then there exists a family of Brauer classes $\{\alpha_i\}_{i=1}^{\infty}$ such that $ind(\alpha_i) = 2^i$ for every $i \geq 1$.*

*Proof.* Let $F = \mathbb{C}(x_1, y_1, x_2, y_2, \cdots)$. Define $\alpha_i$ to be the sum of $i$ Quaternion algebras:

$$\alpha_i := \sum_{j=1}^{i} (x_j, y_j).$$

We need to show that $ind(\alpha_i) = 2^i$. We will show this by induction on $i$. For $i = 1$, we first need to show that $\alpha_1 = (x_1, y_1)$ is a division algebra. Since it is a quaternion algebra, it suffices to show that it is not split. We will show that $\alpha_1$ is not split by passing to a larger field: $F_1 := \mathbb{C}(x_2, y_2, \cdots)((x_1))((y_1))$. The ramification of $\alpha_1$ over this field is $\partial(x_1, y_1) = (x_1)$, where $(x_1)$ denotes the square class of $x_1$ in the residue field $\mathbb{C}(x_2, y_2, \cdots)((x_1))$. Since the valuation of $x_1$ is 1, the square class $(x_1)$ is nonzero. Thus $(x_1, y_1)$ is not split. In fact, we have shown that $(x_1, y_1)$ is not split on $\mathbb{C}(x_1, y_1)$.

By the induction hypothesis, $ind(\alpha_{d-1}) = 2^{d-1}$. Note that $\alpha_d = \alpha_{d-1} + (x_d, y_d)$. Let

$$F_d := \mathbb{C}(x_1, y_1, \cdots, x_{d-1}, y_{d-1}, x_{d+1}, y_{d+1}, \cdots)((x_d))((y_d))$$

with residue field $K_d$. By Proposition 2.4.28, $ind(\alpha_d \otimes F_d) = 2\ ind(\alpha_{d-1} \otimes K_d(\sqrt{x_d}))$. Note that the residue field of $K_d(\sqrt{x_d})$ is $\mathbb{C}(x_1, y_1, \cdots, x_{d-1}, y_{d-1}, x_{d+1}, y_{d+1}, \cdots)$ and $\alpha_{d-1}$ is unramified on $K_d(\sqrt{x_d})$. Thus, again using Proposition 2.4.28, $ind(\alpha_{d-1} \otimes K_d(\sqrt{x_d})) = ind(\alpha_{d-1}) = 2^{d-1}$. Thus $ind(\alpha_d \otimes F_d) = 2^d$. Note that $ind(\alpha_d)$ divides $2^d$ since $\alpha_d = \alpha_{d-1} + (x_d, y_d)$. Also, $ind(\alpha_d \otimes F_d)$ divides $ind(\alpha_d)$, i.e., $2^d$ divides $ind(\alpha_d)$. Therefore $ind(\alpha_d) = 2^d$. □

Proposition 2.4.28 also helps in constructing Brauer classes of high index over function fields of varieties. The following proposition is a generalization of another exercise in [G-S] suggested by Colliot-Thélène.

**Theorem 2.4.30.** *Let $F$ be a field having $char(F) \neq 2$. Assume that the dimension of the $\mathbb{F}_2$-vector space $F^{\times}/(F^{\times})^2$ is at least $d$. Then there exists a Brauer class $\alpha$ in $\mathrm{Br}(F(x_1, \cdots, x_d))$ with $ind(\alpha) = 2^d$.*

*Proof.* Let $\{a_1, \cdots, a_d\}$ be a linearly independent set in the $\mathbb{F}_2$-vector space $F^\times/(F^\times)^2$. Consider the Brauer class

$$\alpha_d = \sum_{i=1}^{d} (x_i, a_i)$$

We will show by induction on $d$ that $ind(\alpha_d) = 2^d$. For $d = 1$, it suffices to show that the Brauer class $(a_1, x_1)$ is not split. Computing the residue at the valuation on $F((x_1))$: $\partial(a_1, x_1) = (a_1)$, we see that it is non-zero.

Let $F_d := F(x_1, \cdots, x_{d-1})((x_d))$ and $K_{d-1}$ be its residue field. By Proposition 2.4.28, $ind(\alpha_d \otimes F_d) = 2\, ind(\alpha_{d-1} \otimes K_{d-1}(\sqrt{a_d}))$. By the induction hypothesis, $ind(\alpha_{d-1} \otimes K_{d-1}) = 2^{d-1}$. Note that $\{a_1, \cdots, a_{d-1}\}$ is linearly independent over the group of square classes of $L := F(\sqrt{a_d})$, i.e., $L^\times/(L^\times)^2$. Therefore, $ind(\alpha_d) = 2^d$. □

### 2.4.4 Brauer dimension of global fields

Now that the local picture is clear, a natural question arises: can we compute the Brauer group of global fields? Or more generally, can we compute Brauer groups of function fields of curves over any field? We will sketch an answer for $F(t)$ in the form of the Faddeev exact sequence. We will also state the Albert-Brauer-Hasse-Noether (ABNH) theorem which computes the Brauer group of global fields. The ABHN theorem (see Theorem 2.4.35) will also help us in computing the Brauer dimension of number fields.

We observed that the Witt exact sequence is a higher cohomological analogue of the short exact sequence in 2.12.

We will denote the projective line over a *perfect field* $k$ by $\mathbb{P}^1_k$. Since $k$ is perfect, the separable closure of $k$, $k^{\text{sep}}$ equals its algebraic closure $\bar{k}$.

To compute the Brauer group of $k(t)$, we start by mimicking the proof, as a first step. There is a "globalized" form of this short exact sequence of Galois modules.

$$0 \to \bar{k}(t)^\times/\bar{k}^\times \to \mathrm{Div}(\mathbb{P}^1_{\bar{k}}) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Note that this sequence is split, where the splitting map is defined by sending 1 to a rational point. Therefore the long exact sequence breaks into the following split short exact sequences for every $i \geq 1$:

$$0 \to \mathrm{H}^i(k, \bar{k}(t)^\times/\bar{k}^\times) \to \mathrm{H}^i(k, \mathrm{Div}(\mathbb{P}^1_{\bar{k}})) \to \mathrm{H}^i(k, \mathbb{Z}) \to 0.$$

Now consider the following exact sequence of $\mathrm{Gal}(\bar{k}|k)$-modules:

$$0 \to \bar{k}^\times \to \bar{k}(t)^\times \to \bar{k}(t)^\times/\bar{k}^\times \to 0.$$

Therefore, we have the following long exact sequence:

$$\cdots \to \mathrm{H}^i(k, \bar{k}^\times) \to \mathrm{H}^i(k, \bar{k}(t)^\times) \to \mathrm{H}^i(k, \bar{k}(t)^\times/\bar{k}^\times) \to \cdots.$$

The map $\mathrm{H}^i(k, \bar{k}^\times) \to \mathrm{H}^i(k, \bar{k}(t)^\times)$ can be shown to be injective (see [G-S, Corollary 6.4.6]). Thus, one obtains the following exact sequence:

$$0 \to \mathrm{H}^i(k, \bar{k}^\times) \to \mathrm{H}^i(k, \bar{k}(t)^\times) \to \mathrm{H}^i(k, \mathrm{Div}(\mathbb{P}^1_{\bar{k}})) \to \mathrm{H}^i(k, \mathbb{Z}) \to 0.$$

We shall make the last two terms of the sequence more familiar.

Note that we have the following isomorphism of Galois modules

$$\mathrm{Div}(\mathbb{P}^1_{\bar{k}}) \cong \bigoplus_{P \in \mathbb{P}^1_k} (\bigoplus_{Q \mapsto P} \mathbb{Z}).$$

For a closed point $P$ in $\mathbb{P}^1_k$, the notation $Q \mapsto P$ means that $Q$ lies over the point $P$ under the projection map $\mathbb{P}^1_{\bar{k}} \to \mathbb{P}^1_k$. Therefore we have the following identification:

$$\mathrm{H}^i(k, \mathrm{Div}(\mathbb{P}^1_{\bar{k}})) \cong \bigoplus_{P \in \mathbb{P}^1_k} \mathrm{H}^i(k, \bigoplus_{Q \mapsto P} \mathbb{Z}).$$

Let $\{Q \mapsto P\}$ denote the set of closed points $Q$ in $\mathbb{P}^1_{\bar{k}}$ in the preimage of $P$ under the projection map. Note that the cardinality of $\{Q \mapsto P\}$ equals $[k(P) : k]$. Since $k(P)/K$ is a separable field extension (since $k$ is perfect), $[k(P) : k]$ equals the number of distinct coset representatives of $\mathrm{Gal}(\bar{k}|k(P))$ in $\mathrm{Gal}(\bar{k}|k)$. Using this fact, one can show that we have the following isomorphism (see [G-S, Lemma 6.4.1])

$$M^k_{k(P)}(\mathbb{Z}) \cong \bigoplus_{Q \mapsto P} \mathbb{Z},$$

where $M^k_{k(P)}(\mathbb{Z})$ is the coinduced module of the $\mathrm{Gal}(\bar{k}|k(P))$-module $\mathbb{Z}$ (see the discussion before Lemma 2.3.1 for the definition of coinduced modules). Therefore, we have made the following identifications for $i \geq 2$:

$$\mathrm{H}^i(k, \mathrm{Div}(\mathbb{P}^1_{\bar{k}})) \cong \bigoplus_{P \in \mathbb{P}^1_k} \mathrm{H}^i(k(P), \mathbb{Z}) \cong \bigoplus_{P \in \mathbb{P}^1_k} \mathrm{H}^{i-1}(k(P), \mathbb{Q}/\mathbb{Z}).$$

We thus obtain:

**Theorem 2.4.31** (Faddeev)**.** *For $i \geq 1$, we have the following exact sequence:*

$$0 \to \mathrm{H}^i(k, \bar{k}^\times) \to \mathrm{H}^i(k, \bar{k}(t)^\times) \xrightarrow{\oplus \partial_P} \bigoplus_{P \in \mathbb{P}^1} \mathrm{H}^{i-1}(k(P), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sum \mathrm{Cor}} \mathrm{H}^{i-1}(k, \mathbb{Q}/\mathbb{Z}) \to 0.$$

The map in the middle is the direct sum of the ramification maps at places coming from closed points. There are still a couple of things to check here. For example, why can we identify the last two maps with $\oplus \partial_P$ and $\sum \mathrm{Cor}$? We refer the reader to [G-S, Theorem 6.4.4] for this.

We also have a version of the Faddeev exact sequence with finite coefficients:

**Theorem 2.4.32.** *Let $m > 1$ be an integer coprime to the characteristic of $k$. Then for $i, j > 0$, we have the following exact sequence:*

$$0 \to \mathrm{H}^i(k, \mu_m^{\otimes j}) \to \mathrm{H}^i(k(t), \mu_m^{\otimes j}) \to \bigoplus_{P \in \mathbb{A}^1_k} \mathrm{H}^{i-1}(k(P), \mu_m^{\otimes (j-1)}) \to 0.$$

This comes close to the function field (over finite field) analogue of the Albert-Brauer-Hasse-Noether Theorem (although we have only established it for function field of a projective line). In view of the isomorphism 2.4.26 which we obtained using the Witt exact sequence, the middle term can be identified with the Brauer group of local fields of positive characteristic.

**Theorem 2.4.33** (Hasse)**.** *Let $F$ be the function field of a smooth curve $C$ over a finite field. Then the following sequence is exact:*

$$0 \to \mathrm{Br}(K) \to \bigoplus_{P \in C_0} \mathrm{Br}(\widehat{K_P}) \to \mathbb{Q}/\mathbb{Z} \to 0. \tag{2.16}$$

As a Corollary, we can compute the Brauer dimension of function fields of curves over finite fields:

**Corollary 2.4.34.** *Let $F$ be the function field of a smooth curve $C$ over a finite field. Let $\ell$ be a prime not equal to the characteristic of $K$. Then* $\mathrm{Brd}_\ell(F) = 1$*.*

*Proof.* In view of Lemma 2.4.8, it is sufficient to compute the indices of Brauer classes of prime period. Let $K/F$ be a finite field extension. Let $\ell$ be a prime not equal to the characteristic of $F$ and $\alpha$ be a Brauer class in $\mathrm{Br}(K)$ of period $\ell$. Let $\mathrm{Ram}(\alpha)$ be the finite set of closed points $P$ in $C$ such that $\alpha \otimes \widehat{K_P} \neq 0$. By Theorem 2.4.27, the index of $\alpha \otimes \widehat{K_P}$ equals $\ell$ for every $P$ in $\mathrm{Ram}(\alpha)$. Let $\widehat{L_P}/\widehat{K_P}$ be a degree $\ell$ extension splitting $\alpha$ locally. By weak approximation and Krasner's lemma, one can construct a global field extension $L/K$ such that $L \otimes_K \widehat{K_P} \cong \widehat{L_P}$. Note that $L$ is the function field of the normalization of $C$ in $L$. By construction of $L$, $\alpha$ splits over every completion of $L$. Therefore, $\alpha \otimes L = 0$. Since $L/K$ has degree $\ell$, it follows that $\mathrm{Brd}_\ell(F) = 1$. $\square$

Note that the we have only used the Hasse principle part of the exact sequence (2.16) in the above proof, namely the injectivity of the first map. Using the Albert-Brauer-Hasse-Noether Theorem, one can show that the period of every Brauer class equals its index, provided that it is ramified at only archimedean places. This establishes the fact that $\mathrm{Brd}_\ell(F) = 1$ for totally imaginary number fields. However, to complete the computation, we have to deal with archimedean completions also. To get around this, we will state the Grunwald–Wang theorem and use it show, not only that the Brauer dimension of number fields is 1, but also that every central simple algebra over a number field is cyclic. At this point, let us state the Albert-Brauer-Hasse-Noether theorem:

**Theorem 2.4.35** (Albert-Brauer-Hasse-Noether)**.** *Let $K$ be a number field. Let $\Omega_K$ denote the set of places of $K$. Then the following sequence is exact:*

$$0 \to \mathrm{Br}(K) \to \bigoplus_{v \in \Omega_K} \mathrm{Br}(\widehat{K_v}) \to \mathbb{Q}/\mathbb{Z} \to 0.$$

The proof of this theorem is quite deep, using the full force of Class field theory: Hasse Norm Theorem, the Artin reciprocity Law and Chebotarev density theorem. Since we only need the Hasse principle bit in Theorem 2.4.35, and the Grunwald–Wang theorem, we will only sketch proof of the Hasse principle, i.e., the injectivity of the first map in Theorem 2.4.37 below.

Even to show the Hasse principle is quite difficult. One shows it by first establishing an equivalent statement, called the Hasse Norm Theorem. This theorem can be seen as a local global principle for principal homogenous spaces under norm one tori of cyclic extensions.

We will denote the set of valuations of $F$ by $\Omega_F$. We will denote the completion of $F$ at $v$ by $F_v$ (dropping the hat).

**Theorem 2.4.36** (Hasse Norm Theorem)**.** *Let $K/F$ be a cyclic extension of number fields. Let $a$ be an element of $F^\times$. Suppose that for every completion $v$, $a$ lies in $N_{K_v/F_v}({K_v}^\times)$. Then $a$ is an element in $N_{K/F}(K^\times)$.*

In view of Example 2.3.8, we see that this is equivalent to the Hasse principle for Brauer classes split by a cyclic extension.

**Theorem 2.4.37.** *Let $F$ be a number field and $\alpha$ in $\mathrm{Br}(F)$. If $\alpha \otimes F_v = 0$ for every $v$ in $\Omega_F$. Then $\alpha = 0$.*

*Proof.* As we remarked above, if $\alpha$ is split by a cyclic extension, then the statement follows from Theorem 2.4.36. We will briefly expand on this: since $\alpha$ is split by a cyclic extension, $\alpha$ is Brauer equivalent to a cyclic algebra $(\chi, b)$. Since $(\chi, b) \otimes F_v = 0$, we have that $(\chi_v, b) = 0$, where $\chi_v$ denotes the restriction of the character $\chi$ to $F_v$. Since $(\chi_v, b) = 0$, by Example 2.3.8, $b$ is a norm of the cyclic extension corresponding to $\chi_v$. This cyclic extension can be identified with $K_v$. Therefore $b$ is an element in $N_{K_v/F_v}({K_v}^\times)$ for every $v$ in $\Omega_F$. Thus $b$ is an element in $N_{K/F}(K^\times)$. Therefore, $(\chi, b) = 0$.

In general, let $p$ be a prime dividing $ind(\alpha)$. Then there exists a field extension $E/F$ such that $\alpha \otimes E$ is cyclic, $ind(\alpha \otimes E) = p$, and $v_p([E : F]) < v_p(ind(\alpha))$. Since $\alpha \otimes F_v = 0$ for every $v$ in $\Omega_F$. Therefore, $(\alpha \otimes E) \otimes E_w = 0$ for every $w$ in $\Omega_E$. Thus, $\alpha \otimes E = 0$. Therefore, $ind(\alpha)$ divides $[E : F]$. However, since $v_p([E : F]) < v_p(ind(\alpha))$, this is impossible. Thus $\alpha = 0$. □

**Remark 2.4.38.** *With some additional work, using the fact that extensions of number* *fields are ramified at a finite number of places and the product formula for number fields (see [Pierce]), one can show that every Brauer class is split at all but finitely many places. This finally establishes the injectivity of the first map in Theorem 2.4.35.*

We will state the Grunwald-Wang theorem (the version in [Pierce]) and use it later to show that every central simple algebra over a number field is cyclic, and Brauer dimension of number fields equals one.

**Theorem 2.4.39.** *Let $F$ be a number field. Let $\{(v_1, n_1), \cdots, (n_k, v_k)\}$ be a set of pairs of natural numbers $n_i$ and valuations $v_i$. We assume that $n_i = 1$ if $v_i$ is a complex place and $n_i \leq 2$ if $v_i$ is a real place. Then for every $n$ divisible by $\operatorname{lcm}\{n_1, \cdots, n_k\}$, there exists a cyclic extension $K/F$ of degree $n$ such that $n_i$ divides the degrees of the local extensions $[K_{v_i} : F_{v_i}]$.*

**Theorem 2.4.40.** *If $F$ is a number field, then every central simple algebra over $F$ is cyclic, and $\operatorname{Brd}_\ell(F) = 1$ for every prime $\ell$.*

*Proof.* Let $D/F$ be a division algebra of degree $n$ with Brauer class $\alpha$. Let $\operatorname{Ram}(\alpha) = \{v_1, \cdots, v_k\}$ be the set of places $v$ where $\alpha \otimes F_v \neq 0$. Note that by Remark 2.4.38, $\operatorname{Ram}(\alpha)$ is a finite set. Define $n_i := ind(\alpha \otimes F_{v_i})$ for each $v_i$ in $\operatorname{Ram}(\alpha)$. Set $m := \operatorname{lcm}\{n_1, \cdots, n_k\}$. Note that $n_i$ divides $\deg(D)$, and therefore $m$ divides $n$. By Theorem 2.4.39, there exists a cyclic field extension $K/F$ of degree $n$ such that $n_i$ divides $[K_{v_i} : F_{v_i}]$ for every $i$ from $1, \cdots, k$. This means that $K/F$ splits $D$. Since $\deg(D) = [K : F] = n$, $K$ can be identified with a maximal subfield of $D$. Therefore, $D/F$ is cyclic.
Again using Theorem 2.4.39, there exists a field extension $L/F$ of degree $m$ such that $n_i$ divides $[L_{v_i} : F_{v_i}]$. Thus, $L/F$ splits $\alpha$, implying that $ind(\alpha)$ divides $m$. By what, we have shown in the previous paragraph, $ind(\alpha) = n = m$. If $k$ is the period of $\alpha$, and since $n_i$ also equals the period of $\alpha \otimes F_{v_i}$ (see Theorem 2.4.27), each $n_i$ divides $k$. Therefore, $m$ divides $k$. As a result, $ind(\alpha)$ divides $per(\alpha)$, and we get that $\operatorname{Brd}_\ell(F) = 1$. □

**Remark 2.4.41.** *Note that in the course of proving the above theorem, we obtained a formula for $ind(\alpha)$ in terms of local indices:*

$$ind(\alpha) = \operatorname{lcm}_{v \in \Omega_F}(ind(\alpha \otimes F_v)). \tag{2.17}$$

*We will later obtain such a formula for function fields of curves over complete discretely valued fields (and therefore, also a local-global principle).*

## 2.5 Some Quadratic Form Theory

We will state a few basic facts about quadratic forms which we will need later. Besides the fact that the algebraic theory of quadratic forms is extremely rich, the Milnor conjectures inform us that there is a deep relationship between quadratic forms over a field and the Galois cohomology of the field, and therefore also the arithmetic of the field.

**Definition 2.5.1.** Let $F$ be a field and $V$ be a finite dimensional vector space on $F$. A quadratic form is a pair $(V, q)$ where $q$ is a map $q : V \to F$ satisfying the following properties:

1. $q(\lambda v) = \lambda^2 v$ for every $\lambda$ in $F$ and $v$ in $V$.
2. The pairing $b_q : V \times V \to F$ given by $b_q(v, w) = q(v+w) - q(v) - q(w)$ is a symmetric bilinear pairing.

We say that $q/F$ is non-degenerate if the pairing $b_q$ is non-degenerate. Note also that if $char(F) \neq 2$ (which will most often be the case for us), there is a one to one correspondence between quadratic forms and symmetric bilinear forms: given a symmetric bilinear form $b$, one can obtain a quadratic form $q(v) := b(v, v)/2$. One can check that the corresponding bilinear form of this quadratic form is $b$ itself. Thus, we will not distinguish between quadratic forms and symmetric bilinear forms.

Let us choose a basis for $V$, say $\{e_1, \cdots, e_n\}$. Let $G_b := [b(e_i, e_j)]$ be the Gram matrix associated to the symmetric bilinear form $b$. If $X$ and $Y$ are the column matrices associated to the vectors $x$ and $y$, then one sees that $b(x, y) = X G_b Y^T$. Thus after choosing a basis for $V$, we can associate to a quadratic form, a degree two homogenous polynomial with coefficients in the field $F$. Let $\{e'_1, \cdots, e'_n\}$ be another basis. Then the new homogenous polynomial is related to the old homogenous polynomial by a change of variable given by the change of basis matrix. Thus we can think of quadratic forms as degree two homogenous polynomials up to an equivalence.

Since we are assuming that $char(F) \neq 2$, a standard Gram-Schmidt process allows us to choose an orthogonal basis. Therefore, the Gram matrix is a diagonal matrix. After choosing, such a basis, we use the following convenient notation for quadratic forms: $q = \langle a_1, \cdots, a_n \rangle$ where the $a_i$ are the diagonal entries of the Gram matrix with respect to the orthogonal basis. In this case, the corresponding homogenous polynomial is given by $q(x_1, \cdots, x_n) = a_1 x_1^2 + \cdots + a_n x_n^2$.

**Definition 2.5.2.** Let $q/F$ be a quadratic form with underlying vector space $V$. We say that $q$ is *isotropic* if there exists a non-zero vector $v$ in $V$ such that $q(v) = 0$. Otherwise, we say that $q$ is *anisotropic*.

Note that a quadratic form $q$ defines a degree two hypersurface in $\mathbb{P}(V)$. We see this quite easily: if we choose a basis for $V$ (i.e., coordinates on $\mathbb{P}(V)$), then the vanishing locus of the degree two homogenous polynomial is the hypersurface. To say that $q$ is isotropic is the same as saying that the corresponding hypersurface has a rational point.

If $(V, q)$ and $(W, p)$ are two quadratic forms, then we define their orthogonal sum to be a quadratic form written as $q \perp p$, with underlying vector space $V \oplus W$ satisfying $(q \perp p)(v, w) = q(v) + p(w)$. If $q = \langle a_1, \cdots, a_n \rangle$ and $p = \langle b_1, \cdots, b_m \rangle$, then $q \perp p$ is given by $q \perp p = \langle a_1, \cdots, a_n, b_1, \cdots, b_m \rangle$.

There is a distinguished two dimensional quadratic form, called the *hyperbolic plane*. We may define it as the nondegenerate two dimensional form $\mathbb{H} = \langle 1, -1 \rangle$ (this is not the most satisfactory definition). The reason why hyperbolic planes are special is that they are subforms of isotropic quadratic forms.

**Theorem 2.5.3.** *If $q/F$ is a non-degenerate quadratic form, then*

$$q \cong q_{an} \perp n\mathbb{H},$$

*where $q_{an}$ is an anisotropic form and $n \geq 0$ is some non-negative integer. Here $n\mathbb{H}$ denotes the orthogonal sum of $n$ copies of $\mathbb{H}$.*

We will call $q_{an}$, the anisotropic kernel of $q$. The integer $n$ also has a name:

**Definition 2.5.4.** We define the *Witt index* of a quadratic form $q/F$ to be the largest integer $n$ such that $n\mathbb{H}$ is a subform of $q$. We will denote the Witt index of $q$ by $i_W(q)$.

The Witt index is a measure of the extent that a given quadratic form is isotropic. It turns out that the Witt index $i_W(q)$ is the dimension of the largest isotropic subspace in the underlying vector space of $q$ (see [Lam]).

We can define an arithmetic invariant of a field coming from the isotropy properties of quadratic forms. It is yet another measure of how hard it is for polynomial equations to have solutions.

**Definition 2.5.5.** Let $F$ be a field. The $u$-invariant of $F$, denoted by $u(F)$ is the supremum of the dimensions of non-degenerate anisotropic forms over $F$.

Thus any form with dimension greater than $u(F)$ is isotropic. Notice that if $F$ satisfies the $C_i$ property, then $u(F) \leq 2^i$. Therefore, if $F$ is the function field of a $d$-dimensional variety over a separably closed field, then $u(F) \leq 2^d$. Since finite fields are $C_1$ (see Theorem 2.4.10), if $F$ is the function field of a $d$-dimensional variety over a finite field (of characteristic not 2), $u(F) \leq 2^{d+1}$. A natural question in analogy with Question 1 (see subsection 2.4.1) arises: when $F$ is the function field of a variety over a "nice" field, can we compute the $u$-invariant? Is it always a power of 2 for these nice fields? This is an open problem. It has been conjectured that the $u$-invariant of function fields of curves over totally imaginary number fields, such as $\mathbb{Q}(i)(x)$, is 8.

There has been much progress on the $u$-invariants of function fields of varieties over non-archimedian local fields. The first significant step in this direction is a result of Parimala-Suresh (see [Pa-Su1]) where they compute the $u$-invariant of function fields of curves over $p$-adic fields to be 8. Independently, Harbater-Hartmann-Krashen, using their field patching technique compute the $u$-invariant of function fields of curves over complete discretely valued fields (see [HHK09]). In particular, they compute that the $u$-invariant of function fields of curves over $m$-local fields is $2^{m+2}$. We will prove this fact using patching, but by a slightly different method in a later chapter (see Corollary 4.3.3). The $u$-invariant for function fields of higher dimensional varieties over $p$-adic fields was computed by Leep in [Leep] using a theorem of Heath-Brown on systems of quadratic forms over $p$-adic fields. These results however have been shown when the characteristic of the residue field is not equal to 2. In the bad, mixed characteristic setting, Parimala-Suresh in another paper show that the $u$-invariant of curves over dyadic fields is also 8.

It is not even known, whether the $u$-invariant of function fields of curves over totally imaginary number fields is finite. Using Proposition 5.4.16 and a recent result of Suresh (see [Su]), one can show that the finiteness of $u$-invariant of function fields in one variable over totally imaginary number fields is equivalent to the finiteness of their Brauer 2-dimension.

One can compute $u$-invariants of complete discretely valued fields using Hensel's lemma. If $F$ is a complete discretely valued field with parameter $\pi$, and $q/F$ is a quadratic form, then we may write $q \cong q_1 \perp \pi q_2$, where $q_1$ and $q_2$ has units as entries in the ring of integers $R$. We will need the following weak analogue of the Witt exact sequence for that.

**Proposition 2.5.6.** *Let $F$ be a complete discretely valued field with parameter $\pi$ and residue field $k$. Let $q/F$ be a non-degenerate quadratic form with $q = q_1 \perp \pi q_2$, where $q_1$ and $q_2$ have units as entries in the ring of integers of $F$. The form $q/F$ is isotropic if and only if either of the reductions, $\overline{q_1}/k$ or $\overline{q_2}/k$ is isotropic.*

*Proof.* Suppose that $q/F$ is isotropic, and both $\overline{q_1}/k$ and $\overline{q_2}/k$ are isotropic. Then there exist vectors $v$ and $w$ such that $q_1(v) + \pi q_2(w) = 0$. Since, $q_1$ and $q_2$ are homogenous, at least one of $v$ and $w$ are primitive. Reducing modulo $\pi$, we get that $\overline{q_1}(\overline{v}) = 0$. Since, $\overline{q_1}/k$ is anisotropic, $\overline{v} = 0$. This means that there exists a vector $u$ such that $v = \pi u$. Thus, we get that $\pi^2 q_1(u) + \pi q_2(w) = 0$. Therefore, $\pi q_1(u) + q_2(w) = 0$. Again reducing modulo $\pi$, we get that $\overline{q_2}(\overline{w}) = 0$. Since $\overline{q_2}/k$ is anisotropic, $\overline{w} = 0$. But this implies that $w$ is not primitive, a contradiction.

Now suppose that $\overline{q_1}/k$ is isotropic. We will show that $q_1/F$ is isotropic. Let $\overline{v}$ be a non-zero isotropic vector for $\overline{q_1}/k$. Let $v$ be a lift of $\overline{v}$ in $F$. Choose a basis $\{v_1, \cdots, v_n\}$ for

$q_1$. With respect to this basis, we may regard $q_1$ as a degree two homogenous polynomial in variables $x_1, \cdots, x_n$. Let $b_{q_1}$ be the bilinear form corresponding to $q_1$. Note that $\partial\overline{q_1}/\partial x_i(\overline{v}) = b_{q_1}(\overline{v}, \overline{v_i})$. Since $b_{q_1}$ is non-degenerate, there exists $i$ such that $b_{q_1}(\overline{v}, \overline{v_i}) \neq 0$. By Hensel's lemma, this implies that $q_1(v) = 0$. □

The above proposition shows that $u(F) = 4$ when $F$ is a nonarchimedean local field. As a consequence, using the Hasse-Minkowski theorem, one can compute the $u$-invariant of global fields with no real embeddings.

**Theorem 2.5.7.** *If $F$ is a totally imaginary number field, or function field of a curve over a finite field, then $u(F) = 4$.*

*Proof.* Let $q/F$ be a non-degenerate quadratic form with $\dim(q) \geq 5$. By the above proposition, $q \otimes F_v$ is isotropic for every completion $F_v$. Therefore by the Hasse-Minkowski theorem, $q/F$ is isotropic. This shows that $u(F) \leq 4$. Note that every global field admits a non-trivial quaternion algebra over $F$. The norm form of a non-trivial quaternion algebra over $F$ is a four dimensional anisotropic quadratic form. This shows that $u(F) = 4$. □

We will now briefly introduce the Witt ring and some facts about Pfister forms which we will need later. The following Theorem is proved in [Lam].

**Theorem 2.5.8** (Witt cancellation). *Let $q, p, p'$ be non-degenerate quadratic forms over $F$. Suppose that $q \perp p \cong q \perp p'$. Then $p \cong p'$.*

The theorem says that the monoid $M(F)$ of isometry classes of quadratic forms under orthogonal sum is a cancellative monoid. Recall that one can form the Grothendieck group of $M(F)$ defined by an equivalence relation on $M(F) \times M(F)$ as: $(p_1, q_1) \sim (p_2, q_2)$ if and only if $p_1 \perp q_2 \cong p_2 \perp q_1$. This group is called the *Grothendieck-Witt group* and is denoted by $\mathrm{GW}(F)$. Note that under tensor product, $\mathrm{GW}(F)$ also has a commutative ring structure. Henceforth, we will refer to $\mathrm{GW}(F)$ as the *Grothendieck-Witt ring.* Note that the group generated by $\mathbb{H}$ is an ideal in $\mathrm{GW}(F)$. In view of Theorem 2.5.3, we may as well replace quadratic forms by their anisotropic parts. This is reflected by quotienting $\mathrm{GW}(F)$ by its ideal $\mathbb{Z}[\mathbb{H}]$. The ring that we obtain is denoted by $W(F)$.

$$W(F) := \mathrm{GW}(F)/\mathbb{Z}[\mathbb{H}].$$

This ring is called the *Witt-ring* of $F$. Observe that, $\langle -a\rangle = -\langle a\rangle$ in $W(F)$. By Theorem 2.5.8, anisotropic forms are in one to one correspondence with elements in $W(F)$. Observe that if a quadratic form is a sum of hyperbolic planes, then its class is the 0 element in $W(F)$. We will call such quadratic forms *split*.

The Witt ring has an important filtration by powers of a special ideal $I(F)$, called the *fundamental ideal*. Note that we have a ring homomorphism $W(F) \to \mathbb{Z}/2\mathbb{Z}$, taking a class of $q$ to $\dim(q)$ mod 2. By Theorem 2.5.3, this is a well defined homomorphism. The fundamental ideal $I(F)$ is the kernel of this homomorphism, and consists of classes of even dimensional quadratic forms. Notice that in $W(F)$, any class of a binary form $\langle a, b\rangle$ can be written as $\langle a, b\rangle = \langle 1, a\rangle - \langle 1, -b\rangle$. Thus $I(F)$ is additively generated by binary forms: $\langle 1, -a\rangle$. The form $\langle 1, -a\rangle$ determines the square class $(a)$ in $F^\times/(F^\times)^2$. And conversely, a square class $(a)$ gives rise to a quadratic field extension $F(\sqrt{a})/F$, and the norm form of $F(\sqrt{a})/F$ is the quadratic form $\langle 1, -a\rangle$. In fact, we have the following isomorphism due to Pfister (see [Lam] for a proof):

$$\frac{I(F)}{I^2(F)} \xrightarrow{\sim} F^\times/(F^\times)^2.$$

This maps the class of an even dimensional form to its signed discriminant.

Observe that $I^2(F)$ is additively generated by forms of the shape $\langle 1, -a\rangle \otimes \langle 1, -b\rangle$. Such forms are norm forms of quaternion algebras $(a, b)$. This suggests that there must be a relationship between $I^2(F)$ and ${}_2\operatorname{Br}(F)$. In fact, the suggestion is to look at the following filtration:

$$W(F) \supseteq I(F) \supseteq I^2(F) \supseteq I^3(F) \supseteq \cdots$$

and the associated graded ring since there could be a relationship between the Galois cohomology ring with $\mu_2$ coefficients and this graded ring. It was conjectured by Milnor that these graded rings are isomorphic. Milnor's conjecture was finally proved by Orlov-Vishik-Voevodsky in [OVV]. The following deep theorem of Merkurjev shows that the degree two terms of the graded rings are isomorphic.

**Theorem 2.5.9** (Merkurjev)**.** *Let $F$ be a field with $char(F) \neq 2$. We have the following group isomorphism:*

$$\frac{I^2(F)}{I^3(F)} \xrightarrow{\sim} {}_2\operatorname{Br}(F),$$

*which sends the generator $\langle 1, -a\rangle \otimes \langle 1, -b\rangle$ to the Brauer class $[(a, b)]$.*

This theorem tells us in particular that ${}_2\operatorname{Br}(F)$ is generated by classes of quaternion algebras. In fact, the surjectivity of this map is the most difficult part of the proof. The above theorem and Pfister's theorem suggests that the additive generators of $I^n(F)$ should correspond to the cup products of elements in $\mathrm{H}^1(F, \mu_2)$. These quadratic forms are called *Pfister forms*:

**Definition 2.5.10.** An *$n$-fold Pfister form* $q$ over $F$ is a quadratic form of the following shape:

$$q = \langle 1, -a_1\rangle \otimes \langle 1, -a_2\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle.$$

We will denote the set of non-zero values of a quadratic form $q$ over a field $F$ by $D_F(q)$. If $q/F$ is isotropic, then $\langle 1, -1\rangle$ is a subform by Theorem 2.5.3. For any $a$ in $F^\times$, observe that $\langle 1, -1\rangle \cong \langle a, -a\rangle$. Therefore, $a$ is a value of $q$. To summarize: if $q$ is isotropic, then $D_F(q) = F^\times$.

Note that if $q/F$ is a Pfister form, 1 lies in $D_F(q)$. The orthogonal complement of the one dimensional form $\langle 1\rangle$ is called the pure subform of $q$ and is denoted by $q'$.

**Theorem 2.5.11.** *Let $q/F$ be an $n$-fold Pfister form. Suppose that $q/F$ is isotropic, then $q/F$ is split, i.e., a sum of hyperbolic planes.*

*Proof.* Since $q/F$ is isotropic, $\langle 1, -1\rangle$ is a subform of $q$. Therefore by Theorem 2.5.8, $-1$ is a value of the pure subform $q'$. By the Pure Subform Theorem (see [Lam, Chapter X, Theorem 1.5]), $q \cong \mathbb{H} \otimes p$, for some $(n-1)$-fold Pfister form. Therefore $q$ is a sum of hyperbolic forms. $\square$

# Chapter 3

# Field Patching

The field patching technique was introduced by Harbater and Hartmann in [HH] to deal with patching problems arising in algebra. It has been extremely successful in dealing with problems concerning central simple algebras, quadratic forms and more generally principal homogenous spaces under a linear algebraic group, over certain types of fields, which may be thought of as function fields of thickenings of curves. These fields are being called *semi-global* in recent times (see Definition 3.0.1).

The main idea is inspired from cut-and-paste techniques in analysis and topology: we are interested in understanding an algebraic structure over a semi-global field. We define over-fields, called patches, coming from the geometry of the semi-global field. The main theorem of field patching says that algebraic structures on these patches that are compatible on their "overlaps" glue together to give a unique global algebraic structure. One therefore obtains a local-global principle: if two algebraic structures are isomorphic on the patches in a compatible sense, then they are isomorphic over the original field.

The technique has been subsequently improved, with many interesting applications by Harbater, Hartmann and Krashen in a series of papers ([HHK09], [HHK15(1)], [HHK15], [HHK15(2)]). Some of the applications include: a solution to the long-standing $u$-invariant problem concerning quadratic forms, computation of Brauer dimension and the group admissibility problem. Furthermore, the technique provides a conceptual explanation of why local global principles for various algebraic structures hold, or do not hold in certain situations.

We are interested in this technique because we would like to give upper bounds to the

Generalized Brauer dimension of semi-global fields in terms of arithmetic invariants of simpler fields. Our approach in this chapter is heavily inspired from Harbater's notes on field patching (see [H]).

**Definition 3.0.1.** Let $K$ be a complete discretely valued field with residue field $k$. We say that a field $F$ is *semi-global* if it is the function field of a regular projective curve over $K$.

When the characteristic of the residue field is the same as the characteristic of $K$, all semi-global fields look like finite extensions of $k((t))(x)$ (see [Serre1, section II.4]). In the mixed characteristic situation, good examples to keep in mind are finite extensions of $\mathbb{Q}_p(t)$. This class of examples also explains the terminology "*semi-global*". Fields of the form $\mathbb{F}_p(t)$ are called global fields. Note that $\mathbb{Q}_p(t)$ is the function field of $\mathbb{P}^1_{\mathbb{Z}_p}$. We can view $\mathbb{P}^1_{\mathbb{Z}_p}$ as an infinitesimal thickening of $\mathbb{P}^1_{\mathbb{F}_p}$. Thus $\mathbb{Q}_p(t)$ is somehow "epsilon more than global", and indeed many arithmetic properties of function fields of curves over the residue field $k$ carry over to the semi-global field, adding to them an additional layer of complexity.

## 3.1 Patches

Let us start with making the word "thickening" precise. This will also help us in intuitively understanding the ingredients that go in the patching technique a little better.

We follow the maxim: to understand a space, it is enough to understand the set (sheaf) of functions on it. Consider the following example: the two dimensional affine space $\mathbb{A}^2_k$ over a field $k$ is completely determined by the polynomial ring $k[x, y]$. The point at the origin is determined by the values of polynomials at the origin, or equivalently polynomials up to the ideal $(x, y)$, i.e., $k[x, y]/(x, y)$. The topology of a space is the repository of local information around points. Depending on the information we want, the Zariski topology may not be enough. Suppose we want to know a little bit more around the origin, say in the "horizontal direction". Heuristically, we think of this "bit more" information as a fuzz in the horizontal direction. A good candidate for the set of functions which captures this "point and fuzz" information would be those functions that have a first-order vanishing in the horizontal direction, namely the ring $k[x,y]/(x^2,y) \cong k[x]/(x^2)$. If we want a little more fuzz, we should look at the ring $k[x,y]/(x^3,y) \cong k[x]/(x^3)$, and so on. Compatibly putting these rings together, we obtain the ring that captures extremely local information in the horizontal direction, viz: the formal power series ring in one variable $\varprojlim k[x]/\langle x^n \rangle = k[[x]]$. This is what we mean by "thickening of the point in the horizontal direction".

With this in mind, we refer to the projective line over $k[[t]]$, $\mathbb{P}^1_{k[[t]]}$, as a thickening of the projective line over $k$. Note that the function field of $\mathbb{P}^1_{k[[t]]}$ is the semi-global field $F := k((t))(x)$. We want to "break" this field into simpler overfields, so that compatible algebraic structures defined on these overfields glue together. If these fields are to be simpler, they should somehow come from another simple field, where we have some information about these algebraic structures. Assuming we know something about the residue field $k$ and function field in over variable over $k$, it would be useful to define them by utilizing the special fiber of the projective line $\mathbb{P}^1_{k[[t]]} \to \mathrm{Spec}(k[[t]])$ over the closed point, namely $\mathbb{P}^1_k$. Divide $\mathbb{P}^1_k$ into the standard open affine set $U := \mathbb{A}^1_k$ and the point at infinity $P$.

Note that the ring characterizing $U$ inside $\mathbb{P}^1_{k[[t]]}$ may be defined as the set of meromorphic functions in $F$, that are regular on $U$, i.e., functions with reduced expression $g/h$, where $g$ and $h$ are in $k[[t]][x]$ ($k[[t]][x]$ is a UFD) such that $h$ mod $t$ lies in $k^\times$. We will denote this ring by $R_U$. To thicken $R_U$ in the horizontal direction, we complete it $t$-adically, and denote this completed ring by $\widehat{R_U}$. Since we are interested in fields, we take the fraction field of $\widehat{R_U}$ and denote it by $F_U$.

The ring characterizing the point, may be taken to be the stalk of the structure sheaf at $P$ (or equivalently rational functions regular at $P$), i.e., $R_P := k[[t]][x]_{(t,x)}$. We could thicken this point in the horizontal direction. However we also want the fraction field of thickening of $R_P$ to have some overlap with $F_U$. Thus we take a "two dimensional thickening" of $R_P$, namely the completion of $R_P$ at its maximal ideal. As one can check, this completion turns out to be $\widehat{R_P} := k[[t,x]]$. Finally, we take its fraction field

$F_P = k((t,x))$.

There is an "overlap" between the thickened fields $F_U$ and $F_P$, occurring along the branch $\wp$ of the special fiber $\mathbb{P}^1_k$ at $P$. This branch is given by the ideal $(t)$ in the ring $k[[t,x]]$. Thus a way to capture this overlap is by taking the localization of $\widehat{R_P}$ at the ideal $(t)$. We will denote this localization by $R_\wp$. Note that $R_\wp$ is a discrete valuation ring with residue field $k((x))$. We will be more careful and complete $R_\wp$ at the ideal $(t)$ to get the complete discrete valuation ring $\widehat{R_\wp} := k((x))[[t]]$. We will denote its fraction field by $F_\wp$.

We will setup some notation and formally define these over fields $F_U$, $F_P$ and $F_\wp$ for open sets $U$, closed points $P$ and branches $\wp$ at these closed points lying on the special fiber. We will define them quite generally, i.e., for any regular projective curve over a complete discretely valued field. Note that models of these curves over the complete ring need not be smooth. One can find regular models, and we could work with regular models, however it will be extremely useful to work with just normal models at the expense of complicating matters somewhat. For now, let us fix some notation, which we will stick to throughout unless stated otherwise:

**Notation 1.** *$R$ will denote a complete discretely valued ring with parameter $t$, residue field $k$ and fraction field $K$. Let $X/K$ be a regular projective curve. We will denote its function field by $F$. We will denote normal models of $F$ over $R$ by $\pi : \mathscr{X} \to \operatorname{Spec} R$ or simply by $\mathscr{X}$.*

*The scheme $\mathscr{X}$ is projective and flat over* $\operatorname{Spec}(R)$*, with generic fiber isomorphic to the curve $X/K$. The* special fiber*, or the closed fiber is the fiber of the structure morphism $\pi$ over the closed point $(t)$. It is isomorphic to $\mathscr{X} \times_{\operatorname{Spec} R} \operatorname{Spec} k$, and will be denoted by $\mathscr{X}_k$.*

Note that $\mathscr{X}_k$ need not be reduced. Since we only use the topology of $\mathscr{X}_k$, we will most often give it the reduced induced subscheme structure. Parimala-Suresh often make the assumption that the original curve $X/K$ is geometrically integral. The advantages of making this assumption is that the morphism $\mathcal{O}_{\operatorname{Spec} R} \to \pi_* \mathcal{O}_{\mathscr{X}}$ is an isomorphism.

By Zariski's connectedness principle (see [Liu, Chapter 5, Theorem 3.15]), one may therefore assume that the fiber $\mathscr{X}_k$ is geometrically connected. Further, this also implies that $F$ and $\overline{K}$ are linearly disjoint. We do need connectedness of the closed fiber. It is sufficient to just assume that $X/K$ is geometrically connected.

Let $\mathcal{P}$ be a nonempty finite set of closed points on the special fiber $\mathscr{X}_k$ which include the set of points where distinct irreducible components of $\mathscr{X}_k$ meet. Let $\mathcal{U}$ be the set of irreducible components of the open subscheme $\mathscr{X}_k \smallsetminus \mathcal{P}$. Let $\mathcal{B}$ be the set of branches incident at the points in $\mathcal{P}$ lying on the special fiber. These branches are in one to one correspondence with the set of height one prime ideals of $\widehat{R_P}$ containing the parameter $t$ for every $P$ in $\mathcal{P}$.

For every $U$ in $\mathcal{U}$, define $R_U$ to be the ring of rational functions which are regular on $U$, or equivalently:

$$R_U \coloneqq \bigcap_{Q \in U} \mathcal{O}_{\mathscr{X},Q}.$$

Note that the rational function $t$ vanishes on $U$. Thus the parameter $t$ is regular on $U$, and hence lies in $R_U$. We define $\widehat{R_U}$ to be the $t$-adic completion of $R_U$. Let $F_U$ be the fraction field of $\widehat{R_U}$.

For every $P$ in $\mathcal{P}$, define $\widehat{R_P}$ to be the completion of $\mathcal{O}_{\mathscr{X},P}$ at its maximal ideal. Note that since $\mathcal{O}_{\mathscr{X},P}$ is a normal, excellent domain, $\widehat{R_P}$ is a normal domain (see [Liu, Chapter 8, Proposition 2.41]).

For every branch $\wp$ in $\mathcal{B}$ incident at $P$, let $R_\wp$ be the localization of $\widehat{R_P}$ at the height one prime ideal corresponding to $\wp$. Since $\mathcal{O}_{\mathscr{X},P}$ is excellent and normal, so is $\widehat{R_P}$. Let $\widehat{R_\wp}$ be the completion of $R_\wp$ at this prime ideal. Since $\widehat{R_P}$ is normal, $R_\wp$ is a discrete valuation ring. Let $\widehat{R_\wp}$ be its completion at its maximal ideal. This ring is a complete discretely valued ring. Abusively, we will denote its unique maximal ideal by $\wp$. Let $F_\wp$ be the fraction field of $\widehat{R_\wp}$. Note that $F_\wp$ is a complete discretely valued field.

Let $P$ be a point in $\mathcal{P}$ such that $P \in \overline{U}$ for some $U$ in $\mathcal{U}$. Here the closure is taken in $\mathscr{X}_k$. Since the closure of $U$ is irreducible, $U$ is irreducible. Thus the ideal defining the reduced subscheme $U$ of $\operatorname{Spec}(\widehat{R_U})$ is prime. We will denote this prime ideal by $\eta$. Set

$\mathfrak{p} := \wp \cap \mathcal{O}_{\mathscr{X},P}$. The universal property of localization shows that there is a canonical isomorphism between the localizations $R_{U,\eta}$ and $R_{P,\mathfrak{p}}$. Therefore we have the canonical inclusion $R_U \hookrightarrow \widehat{R_\wp}$. Notice that the contraction of the prime ideal $\wp$ on $R_U$ gives $\eta$. Note also that $\eta$ is the radical ideal of the ideal $(t)$. Thus the $t$-adic completion of $R_U$ canonically sits inside $\widehat{R_\wp}$. Therefore for every $P$ in $\overline{U}$, we have the canonical embedding $F_U \subset F_\wp$. The fields of the form $F_U$ and $F_P$ will be called *patches*, and fields of the form $F_\wp$ will be called *branch fields.*

If $\wp$ is a branch incident at $P$, then clearly $F_P \subset F_\wp$. Summarizing the above discussion, for every $P$ in $\overline{U}$ and for every branch $\wp$ incident at $P$ lying on $U$, we have the following diamond of canonical field embeddings:

$$\begin{tikzcd} & F_\wp & \\ F_U \arrow[ur, hook] & & F_P \arrow[ul, hook'] \\ & F \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd} \tag{3.1}$$

We have seen an example of what these fields look like when we were getting an intuitive feel for them. We will recall that example here, and point out some common mistakes one might make if they are not cautious. We will stick to simple examples which serve to give us a feeling about these fields.

**Example 3.1.1.** *1. Let $\mathscr{X} = \mathbb{P}^1_{k[[t]]}$. Let $\mathcal{P} = \{\infty_k\}$. Therefore, $\mathcal{U} = \{\mathbb{A}^1_k\}$. Observe that $\widehat{R_U} = k[x][[t]]$, and therefore $F_U = Frac(k[x][[t]])$. Note that $k[x][[t]] \neq k[[t]][x]$. The latter is a polynomial ring with power series coefficients. This is quite easy to see: $\sum_{i\geq 0} x^i t^i$ is contained in $k[x][[t]]$, but not in $k[[t]][x]$. Also, $Frac(k[x][[t]]) \neq k(x)((t))$, unlike in the case of polynomial rings. (See the discussion after this example)*

*As we have seen before, $F_P = k((t,x))$. This is the fraction field of $k[[t,x]]$. Note that $k[[t,x]] \subset k[[t]][[x]] \subset Frac(k[[t]][[x]]) \subsetneq k((t))((x))$. The latter containment is strict (See the discussion after this example). Therefore, $k((t,x)) \subsetneq k((t))((x))$. Finally, $F_\wp = k((x))((t))$.*

*2. Let $\mathscr{X} = \mathbb{P}^1_{k[[t]]}$ with $\mathcal{P} = \{P_1 = 0, P_2 = \infty\}$. Let $\wp_1$ be the branch incident at*

*$P_1$ and $\wp_2$ be the branch incident at $P_2$. Therefore, $\mathcal{U} = \{\mathbb{A}^1_k \smallsetminus \{0\}\}$. Observe that $\widehat{R_U} = k[x, x^{-1}][[t]]$. Further, $F_{P_1} = k((t, x))$ and $F_{P_2} = k((t, x^{-1}))$, whereas $F_{\wp_1} = k((x))((t))$ and $F_{\wp_2} = k((x^{-1}))((t))$.*

If $R$ is a domain with field of fractions $F$, then $K := \mathrm{Frac}(R[[t]]) \subsetneq F((t))$ in general. A typical element in $K$ can be written as $\lambda := t^j(\frac{\sum_i f_i t^i}{\sum g_i t^i})$, where $f_0$ and $g_0$ are not equal to 0. Consider the element:

$$\begin{aligned}\frac{1}{\sum_i g_i t^i} &= \frac{1}{g_0 - h(t)} \\ &= \frac{g_0^{-1}}{1 - h(t)/g_0} \\ &= g_0^{-1} \sum_i \frac{h(t)^i}{g_0^i}\end{aligned}$$

Thus $\lambda$ has the form $\sum_{i \geq j} h_i t^i$, where all $h_i$ lie in the ring $R[1/g_0]$. Clearly this element lies in $F((t))$. However not every element in $F((t))$ can be expected to have this form in general. For example, consider $R = k[x]$. Let $\{p_1, p_2, \cdots\}$ be a set of primes in $k[x]$. Note that the element $\sum_{i \geq 0} \frac{1}{\prod_{j=1}^i p_j} t^i$ lies in $k(x)((t))$, but not in $\mathrm{Frac}(k[x][[t]])$, by the above discussion.

Even when $R = k[[x]]$, it turns out that $K := \mathrm{Frac}(R[[t]]) \subsetneq k((x))((t))$. Let us look at a typical element in $K$:

$$\begin{aligned}\lambda :&= t^j(\frac{\sum_i f_i t^i}{\sum_i g_i t^i}) \\ &= (\frac{f_0}{g_0})t^j + (\frac{f_1 g_0 - f_0 g_1}{g_0{}^2})t^{j+1} + (\frac{f_2 g_0{}^2 - f_1 g_1 g_0 + f_0 g_1{}^2 - f_0 g_0 g_2}{g_0{}^3})t^{j+2} + \cdots\end{aligned}$$

Write $\lambda$ as $\sum_i h_i t^{j+i}$. Denote the valuation of $h_i$ with respect to $x$ by $n_i$ and the valuation of $g_0$ by $m$. We will assume that $m > 0$. Since the numerators are elements of $R$, notice that $n_i \geq -(i+1)m$ for every $i$. (Some of the numerators can be zero, but we are ignoring this point here since we are interested in exhibiting just one element in k((x))((t)) which does not lie in $K$). Now consider the element $\mu := \sum_i \frac{t^i}{x^{i^2}}$. This element certainly lies in $k((x))((t))$. The valuation of the $i^{th}$ coefficient of $\mu$ with respect to $x$ is $-i^2$. Note however that $-i^2 < -(i+1)m$ for $i >> 0$ for any positive constant $m$. Thus, $\mu$ lies in $k((x))((t))$ but not in $K$. We learnt about this example from an answer of Tony Scholl on MathOverflow to a question asked by Pete Clark.

We will end this section with two facts about complete local rings that will be useful later. We will only sketch their proofs and refer the reader to [Milne] for complete proofs. They should be seen as examples of the meta-principle: "unramiffed algebraic structures over complete rings uniquely come from the residue field". The Witt exact sequence (see Theorem 2.4.24) is another instance of this principle.

**Lemma 3.1.2.** *Let $(A, \mathfrak{m})$ be a complete local ring. Let $B$ be a finite étale $A$-algebra. Let $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the structure morphism. If there exists a point $\mathfrak{n}$ in $\mathrm{Spec}(B)$ such that $k(\mathfrak{n}) = k(\mathfrak{m})$, then there exists a section of $f$ mapping $\mathfrak{m}$ to $\mathfrak{n}$.*

*Proof.* Since finite étale algebras over complete local rings are isomorphic to product of local rings étale over $A$ (See [Milne, Chapter 1, Theorem 4.2]), it is sufficient to show this when $B$ is a local $A$-algebra. Note that since $B$ is finitely generated, flat over $A$ and $A$ is local, $B$ is a free $A$-module. Recall by hypothesis, $k(y) = k(x)$. Thus $k(y) = B \otimes_A k(y) \cong B \otimes_A k(x)$. Thus $B \otimes_A k(x)$ is a one dimensional vector space over the residue field $k(x)$ of $A$. Thus by Nakayama lemma, $B$ is a free rank one $A$-module. Therefore, $B \cong A$ as an $A$-algebra. This isomorphism gives the section. □

The above lemma says that sections of étale morphisms over closed points can be lifted.

**Theorem 3.1.3.** *Let $A$ be a complete local ring with residue field $k$. There is an equivalence of categories between finite étale $A$-algebras and finite étale $k$-algebras given by the functor $B \mapsto B \otimes_A k$.*

*Proof.* Let $B_1$ and $B_2$ be finite étale $A$-algebras. We will first show that the functor is fully faithful, i.e.,

$$\mathrm{Hom}_A(B_1, B_2) \to \mathrm{Hom}_k(B_1 \otimes k, B_2 \otimes k)$$

is a bijection. Let $\phi_1, \phi_2 : B_1 \to B_2$ be $A$-algebra homomorphisms. If $\phi_1 \otimes k = \phi_2 \otimes k$, then the corresponding maps of affine schemes $\phi_1', \phi_2' : \mathrm{Spec}(B_2) \to \mathrm{Spec}(B_1)$ agree on the fiber over the closed point of $\mathrm{Spec}(A)$. Note that both $\phi_1'$ and $\phi_2'$ are separated morphisms. Since $B_2$ is local, $\mathrm{Spec}(B_2)$ is connected. Since such étale morphisms are uniquely determined by their image on one point (see [Milne, Chapter 1, Corollary 3.13]), it follows that $\phi_1' = \phi_2'$. Therefore $\phi_1 = \phi_2$. This shows injectivity.

For surjectivity, let $\psi : B_1 \otimes k \to B_2 \otimes k$ be a $k$-algebra homomorphism. Composing $\psi$ with the map $B_1 \to B_1 \otimes k$, we obtain the homomorphism: $B_1 \to B_2 \otimes k$. Therefore by the universal property of tensor products, we get the map: $\theta : B_1 \otimes_A B_2 \to B_2 \otimes k$ which sends $b_1 \otimes b_2 \mapsto \psi(b_1)b_2$. Since $\mathrm{Spec}(B_2) \to \mathrm{Spec}(A)$ is étale and base changing preserves étale morphisms, the map $\mathrm{Spec}(B_1 \otimes_A B_2) \to \mathrm{Spec}(B_2)$ is étale. Note that the map of affine schemes corresponding to $\theta$, $\mathrm{Spec}(B_2 \otimes k) \to \mathrm{Spec}(B_1 \otimes_A B_2)$ can be regarded as a section of $\mathrm{Spec}(B_1 \otimes_A B_2) \to \mathrm{Spec}(B_2)$ defined on the fiber $\mathrm{Spec}(B_2 \otimes k)$ over the closed point of $\mathrm{Spec}(A)$. Therefore by Lemma 3.1.2, we may lift this to get a section $\mathrm{Spec}(B_2) \to \mathrm{Spec}(B_1 \otimes_A B_2)$ of the projection map $\mathrm{Spec}(B_1 \otimes_A B_2) \to \mathrm{Spec}(B_2)$. Composing this section with the other projection map $\mathrm{Spec}(B_1 \otimes_A B_2) \to \mathrm{Spec}(B_1)$, we get the morphism in $\mathrm{Hom}_A(B_1, B_2)$, whose restriction to the fiber over the closed point of $\mathrm{Spec}(A)$ gives $\psi$.

We will now show that the functor is essentially surjective. If $L/k$ is a separable algebra, then $L/k$ is a product of separable field extensions. Therefore we may write $L = k[x]/(f(x))$ for some monic, separable polynomial $f(x)$. We abuse notation and denote a lift of $f(x)$ to $A[x]$ by $f(x)$. Set $B := A[x]/(f(x))$. Note that since $f'(x)$ is coprime to $f(x)$ in $k[x]$, one can show (using Nakayma lemma) that $f(x)$ and $f'(x)$ generate the unit ideal in $A[x]$, (i.e., they are strictly coprime). Therefore $B$ is a finitely generated free $A$-module, and hence flat. By the way we defined $B$, it also turns out that $B/A$ is unramified. Thus $B$ is an étale $A$-algebra such that $B \otimes_A k \cong L$. $\square$

We will also need the following generalization of Theorem 3.1.3.

**Theorem 3.1.4.** *Let $R$ be a Noetherian domain, complete with respect to its ideal $I$. Then there is an equivalence of categories between finite étale covers of* $\mathrm{Spec}(R)$ *and finite étale covers of* $\mathrm{Spec}(R/I)$.

In its most raw form, the basic idea of the proof is the following: Note that finite étale schemes over $\mathrm{Spec}(R/I)$ are given by finite étale $R/I$-algebras $S_0$. One has a unique lift of $S_0$ to an étale $R/I^2$-algebra $S_1$. This is since $\mathrm{Spec}(S_0) \to \mathrm{Spec}(R/I)$ is étale, it satisfies the so called "topological invariance" property (see [Milne, Chapter 1, Theorem 3.23]). Continuing this process, one obtains étale $R/I^{n+1}$-algebras $S_n$ for every $n \geq 0$, together with compatible maps $\mathrm{Spec}(S_m) \to \mathrm{Spec}(S_n)$ for $m > n$. Taking the limit of all the $S_n$, one obtains the étale $R$-algebra $\widehat{S}$.

## 3.2 Simultaneous Factorization

Recall that we want to check whether giving a compatible algebraic structure on the patches is the same as giving the algebraic structure over the semi-global field. As a simple example, we can view the field $F$ as an algebraic structure over itself.

Note that for a single closed point $P$ and its complement $U$ on $\mathbb{P}^1_R$, we have the algebraic structures $F_P$ over $F_P$ and $F_U$ over $F_U$ that are compatible on the branch $\wp$ incident at $P$ and lying on $U$, i.e.,, $F_U \otimes_{F_U} F_\wp \cong F_P \otimes_{F_P} F_\wp$. Our intuition suggests that the unique algebraic structure that $F_U$ and $F_P$ determine over $F$ should be $F$ itself. If we identify $F_U$ and $F_P$ as subfields of $F_\wp$, then proving that $F$ is the required algebraic structure amounts to proving that

$$F_U \cap F_P = F. \tag{3.2}$$

Thus we would want condition (3.2) to hold.

Now consider the vector spaces $V_U/F_U$ and $V_P/F_P$, which are compatible on the branch $\wp$, i.e., there exists an isomorphism

$$\phi_\wp : V_U \otimes_{F_U} F_\wp \xrightarrow{\sim} V_P \otimes_{F_P} F_\wp. \tag{3.3}$$

We want to find a condition that guarantees the existence of a vector space $V/F$ so that $V \otimes_F F_U = V_U$ and $V \otimes_F F_P = V_P$. The following lemma finds the condition for the existence of such a vector space $V/F$ and the isomorphism.

**Lemma 3.2.1.** *Let $V_U/F_U$ and $V_P/F_P$ be vector spaces, together with an isomorphism $\phi_\wp$ as in (3.3). An $n$ dimensional vector space $V/F$ satisfying $V \otimes_F F_U \cong V_U$ and $V \otimes_F F_P \cong V_P$ exists if the following conditions are satisfied:*

1. *For every* $A_\wp$ *in* $\mathrm{GL}_n(F_\wp)$*, there exist matrices* $A_U$ *in* $\mathrm{GL}_n(F_U)$ *and* $A_P$ *in* $\mathrm{GL}_n(F_P)$ *such that* $A_\wp = A_P A_U$.

2. $F_U \cap F_P = F$.

*Proof.* Let $V_\wp := V_P \otimes F_\wp$. By the isomorphism $\phi_\wp$ one may identify $V_U$ with a subset of $V_\wp$. Let $B_U$ and $B_P$ be a basis for the vector spaces $V_U$ and $V_P$ respectively. After extending scalars to $F_\wp$, one sees that the sets $B_U$ and $B_P$ form a basis for the vector space $V_\wp$. Thus, there exists a matrix $A_\wp$ in $\mathrm{GL}_n(F_\wp)$ such that $A_\wp B_U = B_P$. Since there exist matrices $A_U$ and $A_P$ in $\mathrm{GL}_n(F_U)$ and $\mathrm{GL}_n(F_P)$ respectively such that $A_\wp = A_P A_U$, we have the following equality $B := A_U B_U = {A_P}^{-1} B_P$. Therefore $B$ serves as a basis for both $V_U/F_U$ and $V_P/F_P$. Finally, define $V$ to be the $F$ span of $B$. □

To make matters simpler, we will introduce some formalism now.

Let $\mathscr{X}/\operatorname{Spec} R$ be a normal model of $F$. Let $\mathcal{P}$ be a non-empty finite set of closed points containing the points where distinct irreducible components of the special fiber $\mathscr{X}_k$ meet. Let $\mathcal{U}$ be the set of irreducible components of $\mathscr{X}_k \smallsetminus \mathcal{P}$. Let $\mathcal{B}$ be the set of branches incident on points in $\mathcal{P}$ lying on the special fiber. The fields $\mathcal{F} := \{F_U, F_P, F_\wp\}$ form an inverse factorization system (see diagram (3.1)). Let $V_U/F_U$, $V_P/F_P$ and $V_\wp/F_\wp$ be $n$ dimensional vector spaces for every $P$ in $\mathcal{P}$, $U$ in $\mathcal{U}$ and $\wp$ in $\mathcal{B}$. Suppose there exist isomorphisms

$$V_\xi \otimes_{F_\xi} F_\wp \xrightarrow{\phi_{\xi,\wp}} V_\wp \tag{3.4}$$

for every $\xi$ in $\mathcal{P} \cup \mathcal{U}$ such that either $\wp$ is a branch incident at $\xi$ (if $\xi$ is a point in $\mathcal{P}$), or $\wp$ is a branch lying on $\xi$ and $\xi$ is an element in $\mathcal{U}$.

Define the patching problem category $\mathcal{PP}(\mathcal{F})$ whose objects are finite dimensional vector spaces over $F_\alpha$: $\{V_\alpha\}$ for $\alpha$ in $\mathcal{P} \cup \mathcal{U} \cup \mathcal{B}$, together with the isomorphism (3.4). We will denote the objects by $(\{V_\alpha\}, \phi)$. A morphism $\Theta : (\{V_\alpha\}, \phi) \to (\{W_\alpha\}, \psi)$ is a sequence of maps $\theta_\alpha : V_\alpha \to W_\alpha$ of vector spaces over $F_\alpha$, compatible with the morphisms

$\phi$ and $\psi$:

$$\begin{array}{ccc} V_\xi \otimes F_\wp & \xrightarrow{\phi_{\xi,\wp}} & V_\wp \\ \downarrow{\scriptstyle \theta_\xi} & & \downarrow{\scriptstyle \theta_\wp} \\ W_\xi \otimes F_\wp & \xrightarrow{\psi_{\xi,\wp}} & W_\wp. \end{array}$$

Note that there is a functor from the category $\mathrm{Vect}_F$ of finite dimensional vector spaces over $F$ to $\mathcal{PP}(\mathcal{F})$:

$$\beta : \mathrm{Vect}_F \to \mathcal{PP}(\mathcal{F}). \tag{3.5}$$

which takes a vector space $V/F$ to $(\{V \otimes_F F_\alpha\}, Id \otimes F_\alpha)$.

If all compatible algebraic structures are to uniquely determine an algebraic structure over $F$, we should demand that $\beta$ be an equivalence of categories. We say that $V$ is a solution to the patching problem $(\{V_\alpha\}, \phi)$ if $\beta(V) \cong (\{V_\alpha\}, \phi)$.

Lemma 3.2.1 gives some evidence that if one can factor matrices in $\mathrm{GL}_n(F_\wp)$ into matrices in $\mathrm{GL}_n(F_U)$ and in $\mathrm{GL}_n(F_P)$, then $\beta$ is essentially surjective. When we are dealing with multiple patches, for the essential surjectivity of $\beta$, we require that there should be a simultaneous factorization:

Let $(U, P, \wp)$ be a triple, with $U$ in $\mathcal{U}$, $P$ in $\mathcal{P}$, and $\wp$ a branch in $\mathcal{B}$ incident at $P$ and lying on $U$. We say that $\mathrm{GL}_n$ satisfies the *simultaneous factorization property* if for every $A_\wp$ in $\mathrm{GL}_n(F_\wp)$, there exist matrices $A_U$ in $\mathrm{GL}_n(F_U)$ and $A_P$ in $\mathrm{GL}_n(F_P)$ such that $A_\wp = A_P A_U$, for every triple $(U, P, \wp)$.

**Theorem 3.2.2** (Harbater-Hartmann)**.** *The functor $\beta$ in (3.5) is an equivalence of categories if and only if*

1. *For every $n \geq 1$, $\mathrm{GL}_n$ satisfies the simultaneous factorization property.*

2. *$\varprojlim_{\xi \in \mathcal{P} \cup \mathcal{U}} F_\xi = F$.*

Therefore in order to show that every patching problem for vector spaces has a unique solution, we need to verify that simultaneous factorization holds. We will not verify this in complete generality. Rather, following the Luxembourg notes of Harbater ([H]), we will stick to the projective line over $k[[t]]$ and show that simultaneous factorization holds in a very special case.

**Example 3.2.3.** *1. Let $U_1$ be the complement of $\{\infty\}$ on the special fiber of $\mathbb{P}^1_{k[[t]]}$ and $U_2$ be the complement of $\{0\}$. Since $U_1$ and $U_2$ cover the special fiber, one expects that $F = F_{U_1} \cap F_{U_2}$. Let $U_0 := U_1 \cap U_2$. The "overlap" between $F_{U_1}$ and $F_{U_2}$ should be given by $F_{U_0}$.*

*Note that $\widehat{R_{U_1}} = k[x][[t]]$, $\widehat{R_{U_2}} = k[x^{-1}][[t]]$ and $\widehat{R_{U_0}} = k[x,x^{-1}][[t]]$. Let $a_0$ be an element in $\widehat{R_{U_0}}$ such that $a_0 \equiv 1 \bmod t$. We will inductively show that there exist elements $a_1$ and $a_2$, respectively in $\widehat{R_{U_1}}/(t^n)$ and $\widehat{R_{U_2}}/(t^n)$ such that $a_0 = a_1 a_2$. Taking the limit, we obtain the desired factorization. Since $a_0 \equiv 1 \bmod t$, we clearly have a factorization mod $t$. Note that*

$$a_0 - 1 = t(\lambda_0 + \lambda_1 t + \cdots),$$

*where $\lambda_i$ are elements in $k[x,x^{-1}]$. Thus there exist elements $b_1$ and $b_2$ respectively in in $k[x]$ and $k[x^{-1}]$, such that $\lambda_0 = b_1 + b_2$. Therefore we have*

$$\begin{aligned} a_0 &\equiv 1 + t(b_1 + b_2) \bmod t^2 \\ &\equiv (1 + tb_1)(1 + tb_2) \bmod t^2. \end{aligned}$$

*Note that $(1+tb_i)$ lies in $F_{U_i}$ for $i = 1,2$. Replacing $a_0$ by $a_0-(1+t(b_1+b_2)+t^2 b_1 b_2)$ and going modulo $t^3$, one can find a better approximation to the factorization by a similar process. Taking the limit as $t \to \infty$, we get the desired factorization. The same process works for $n \times n$ matrices.*

*2. Let $U_1$ be the complement of $\{\infty\}$ on the special fiber $\mathbb{P}^1_k$. Recall that $\widehat{R_{U_1}} = k[x][[t]]$. Let $R_{\{\infty\}}$ be the ring of rational functions that are regular at the point $\infty$, and let $\widehat{R_{\{\infty\}}}$ be its $t$-adic completion. Let $F_{\{\infty\}}$ be the fraction field of $\widehat{R_{\{\infty\}}}$. One can compute that $\widehat{R_{\{\infty\}}} = k[x^{-1}]_{(x^{-1})}[[t]]$. These sets are disjoint, and therefore the "overlap" ring is $R_\varnothing = k(x)[[t]]$. The overlap field is its fraction field: $F_\varnothing = k(x)((t))$. We claim that for matrices $A_\varnothing$ in $\mathrm{GL}_n(\widehat{R_\varnothing})$ such that $A_\varnothing \equiv Id \bmod t$, there exist matrixes $A_{U_1}$ and $A_{\{\infty\}}$, respectively in $\mathrm{GL}_n(F_{U_1})$ and $\mathrm{GL}_n(F_{\{\infty\}})$ such that $A_\varnothing = A_{U_1} A_{\{\infty\}}$.*

*We will first show that every element in $k(x)$ can be written as a sum of elements in $k[x]$ and $k[x^{-1}]_{(x^{-1})}$. Let $g(x)/h(x)$ be an element in $k(x)$, where*

*$g(x)$ and $h(x)$ are polynomials in $x$ with degrees $g$ and $h$ respectively. If $g \leq h$, $g(x)/h(x) = x^{-h}g(x)/x^{-h}h(x)$ is an element in $k[x^{-1}]_{(x^{-1})}$. If $g > h$, then write $g(x) = p(x)h(x) + r(x)$, where $p(x)$ is a polynomial, and $r(x)$ is either $0$ or $\deg(r(x)) < h$. Thus, $g(x)/h(x)$ is either a polynomial in $x$, or $g(x)/h(x) = p(x) + r(x)/h(x)$. Since $\deg(r(x)) < h$, by the previous case, $r(x)/h(x)$ lies in $k[x^{-1}]_{(x^{-1})}$.*

*Since $A_{\varnothing} \equiv Id$ mod $t$, by a similar reasoning as before we have*

$$\begin{aligned} A_{\{\varnothing\}} &\equiv Id + t(B_1 + B_2) \text{ mod } t^2 \\ &\equiv (Id + tB_1)(Id + tB_2) \text{ mod } t^2, \end{aligned}$$

*where $B_1$ is a matrix with entries in $k[x]$, and $B_2$ is a matrix with entries in $k[x^{-1}]_{(x^{-1})}$. Replacing $A_{\{\varnothing\}}$ with $A_{\{\varnothing\}} - (Id + tB_1)(Id + tB_2)$, and going modulo $t^3$, we obtain a better approximation to the factorization. Continuing this process, one obtains the desired factorization.*

To show the intersection property, viz: $F = F_{U_1} \cap F_{U_2}$ in general, we will first prove the following proposition, and obtain the intersection property as a corollary.

**Proposition 3.2.4.** *Let $U_1$ be the complement of $\{\infty\}$ on the special fiber $\mathbb{P}^1_k$. For every $f$ in $F^{\times}_{U_1}$, there exists an element $a$ in $F^{\times}$ and $u$ in $\widehat{R_U}^{\times}$ such that $f = au$.*

*Proof.* It suffices to assume that $f$ lies in $\widehat{R_{U_1}}^{\times}$. Let $f_0$ be the constant term of $f$. Note that $f_0$ lies in $k[x]$. Note also that $f/f_0 \equiv 1 \text{mod } t$, where $f/f_0$ is seen as an element in $k(x)[[t]]$.

By Example 3.2.3, we see that $f/f_0 = f_1 f_2$, where $f_1$ is an element in $\widehat{R_{U_1}} = k[x][[t]]$, and $f_2$ an element in $\widehat{R_{\{\infty\}}} = k[x^{-1}]_{(x^{-1})}[[t]]$. Thus, $f_0 f_2 = f/f_1$. Note that $f_0 f_2$ is an element of $k[x^{-1}]_{(x^{-1})}[[t]][x]$, whereas $f/f_1$ is an element in $k[x][[t]]$. Thus this element lies in $k[[t]][x] \subset F$. Therefore $f = au$, with $a = f_0 f_2$ and $u = f_1$. □

**Corollary 3.2.5.** *If $U_1$ is the complement of $\{\infty\}$, and $U_2$ is the complement of $\{0\}$ in $\mathbb{P}^1_k$, then $F = F_{U_1} \cap F_{U_2}$.*

*Proof.* Let $F' = F_{U_1} \cap F_{U_2}$, and let $f$ be an element in $F'$. By Proposition 3.2.4, there exist elements $a_i$ in $F^\times$ and $u_i$ in $\widehat{R_{U_i}}^\times$ such that $f = a_1u_1 = a_2u_2$. Let $R' := \widehat{R_{U_1}}[x] \cap \widehat{R_{U_2}} = k[[t]][x]$. Since Frac$R' = F$, there exist elements $c_i$ and $d_i$ in $R'$ such that $a_i = c_i/d_i$. Thus, $c_1d_2u_1 = c_2d_1u_2$. Note that $c_1d_2u_1$ lies in $\widehat{R_{U_1}}[x]$ since $c_1$ and $d_2$ lie in $\widehat{R_{U_1}}[x]$ and $u_1$ is an element in $\widehat{R_{U_1}}$. Also, $c_2d_1u_2$ lies in $\widehat{R_{U_2}}$ since $c_2$, $d_1$ and $u_2$ each lie in $\widehat{R_{U_2}}$. Therefore, the common element $c_1d_2u_1 = c_2d_1u_2$ lies in $R'$. Thus, $f = c_1u_1/d_1 = c_1d_2u_1/d_1d_2$. Since both numerator and the denominator belong to $R'$, $f$ lies in $F$. □

**Proposition 3.2.6.** *Let $U_1$ and $U_2$ be open as in Corollary 3.2.5, and let $U_0 = U_1 \cap U_2$. If $A_0$ is a matrix in $\mathrm{GL}_n(F_0)$, then there exist matrices $A_1$ and $A_2$ in $\mathrm{GL}_n(F_1)$ and $\mathrm{GL}_n(F_2)$ respectively, such that $A_0 = A_1A_2$.*

*Proof.* The main idea is to first reduce to the case when $U_1$ and $U_2$ are disjoint sets. Let $U_2'$ be the complement of $U_0$ in $U_2$. Note that $F_0 \subset F_\varnothing = k(x)((t))$. Thus we may regard $A_0$ as an element of $\mathrm{GL}_n(F_\varnothing)$. Suppose that factorization holds in this disjoint situation, i.e., for $U_1$ and $U_2'$. This means that there exist matrices $A_1'$ in $\mathrm{GL}_n(F_{U_1})$ and $A_2'$ in $\mathrm{GL}_n(F_{U_2'})$ such that $A_0 = A_1'A_2'$. Note that $A_2'$ has entries in $F_{U_2}'$, and $A_2' = A_0{A_1'}^{-1}$ also has entries in $F_{U_0}$ since $A_0$ has entries in $F_{U_0}$ and ${A_1'}^{-1}$ has entries in $F_{U_1}$ and $F_{U_1} \subset F_{U_0}$. An argument similar to the one used in the proof of the above corollary, one can show that $F_{U_2'} \cap F_{U_0} = F_{U_2}$. Thus, $A_2'$ is an element in $\mathrm{GL}_n(F_{U_2})$. We have therefore reduced to the disjoint case.

Suppose $A_0$ in an element in $\mathrm{GL}_n(F_\varnothing)$. Multiplying by a sufficiently large power of $t$, one may assume that $A_0$ is an element in $\mathrm{GL}_n(\widehat{R_\varnothing})$. Since the ring $R_\varnothing \subset F$ is $t$-adically dense in $\widehat{R_\varnothing}$, one may assume that there exists $B$ in $\mathrm{GL}_n(R_\varnothing)$ such that $A_0B \equiv Id \bmod t$. By Example 3.2.3, there exist matrices $A_1$ in $\mathrm{GL}_n(F_{U_1})$ and $A_2$ in $\mathrm{GL}_n(F_{U_2'})$ such that $A_0B = A_1A_2'$. Therefore, $A_0 = A_1A_2'B^{-1}$. Setting $A_2 = A_2'B^{-1}$, we arrive at the desired conclusion. □

With a lot more additional work, one can show that such a simultaneous factorization holds in the most general case. Therefore using Theorem 3.2.2, one obtains:

**Theorem 3.2.7** (Harbater-Hartmann)**.** *The functor $\beta$ as in (3.5) is an equivalence of categories. In other words, there exists a unique (up to isomorphism) solution to a patching problem of vector spaces.*

*Proof.* See [HH, Theorem 6.4]. □

We can not only patch vector spaces, but also vector spaces with additional structure such as central simple algebras, quadratic forms, separable algebras, etc. We will show that patching holds for principal homogenous spaces under a linear algebraic group $G$. This covers the case of central simple algebras, quadratic forms and even separable algebras of a given degree, since they are in natural bijection with principal homogenous spaces under a group.

## 3.3 Local Global Principles With Respect To Patches

We want to be able to patch reasonable algebraic structures such as central simple algebras and quadratic forms. We have seen in Theorem 2.3.11 that degree $n$ central simple algebras over $F$ are classified by $\mathrm{H}^1(F, PGL_n)$. More often, algebraic structures are in a natural bijection with principal homogenous spaces under a linear algebraic group. It turns out that solutions exist to patching problems involving principal homogenous spaces under linear algebraic groups.

Let $G/F$ be a linear algebraic group. Let $\mathcal{PP}_G(\mathcal{F})$ be the patching problem category of principal homogenous spaces under $G$ for the inverse factorization system of fields: $\mathcal{F} = \{F_U, F_P, F_\wp\}$. This category is defined in a very similar manner to the patching problem category of vector spaces. The only difference is that tensor product is replaced by fiber product. Let $G-\mathrm{Tors}_F$ be the category of principal homogenous spaces under $F$. We have the following functor:

$$\gamma : G-\mathrm{Tors}_F \longrightarrow \mathcal{PP}_G(\mathcal{F}). \tag{3.6}$$

Recall that for any linear algebraic group, there exists a faithful representation $G \hookrightarrow \mathrm{GL}_n$. For any field extension $E/F$, by Theorems 2.3.16 and 2.3.13, we have the following natural bijection:

$$\mathrm{GL}_n(E)\backslash \mathrm{H}^0(E, \mathrm{GL}_n/G) \longleftrightarrow \mathrm{H}^1(E, G).$$

Therefore, a principal homogenous space under $G/E$ are given by Galois-invariant translates $hG_{E^{\mathrm{sep}}}$ for $h$ in $\mathrm{GL}_n(E^{\mathrm{sep}})$. It turns out that Galois-invariant translates also give rise to principal homogenous spaces. To see this, note first that $\mathrm{GL}_n/G$ has a structure of a quasi-projective variety (see [Sp, Corollary 5.5.6]). Second, the Galois-invariant translate $hG_{E^{\mathrm{sep}}}$ can be identified with the fiber over $\pi(h)$ under the map $\pi : \mathrm{GL}_n \to \mathrm{GL}_n/G$. Finally, since $\pi$ commutes with the action of $\mathrm{Gal}(E^{\mathrm{sep}}|E)$, $\pi(h)$ is an $E$-rational point of $\mathrm{GL}_n/G$ if and only if its fiber $hG_{E^{\mathrm{sep}}}$ is Galois-invariant. Henceforth, we will identify principal homogenous spaces by Galois-invariant translates.

**Theorem 3.3.1** (Harbater-Hartmann-Krashen)**.** *The functor $\gamma$ of (3.6) is an equivalence of categories.*

*Proof.* We will only check that $\gamma$ is essentially surjective. Suppose that for a triple $(P, U, \wp)$ we have a compatibility of principal homogenous spaces, i.e., an isomorphism $\mu_{U,P} : h_U G_{F_U^{\mathrm{sep}}} \times_{F_U} F_\wp \to h_P G_{F_P^{\mathrm{sep}}} \times_{F_P} F_\wp$ of principal homogenous spaces over $F_\wp$. Consider the element $g_\wp := \mu_{P,U}(h_U)h_U^{-1}$ which lies in $\mathrm{GL}_n(F_\wp^{sep})$. Multiplication by $g_\wp$ gives the isomorphism: $\lambda_\wp : h_U G_{F_\wp^{\mathrm{sep}}} \to g_\wp h_U G_{F_\wp^{\mathrm{sep}}}$. Note that $\lambda_\wp$ sends $h_U$ to $\mu_{P,U}(h_U)$. Therefore $\lambda_\wp = \mu_{U,P} \times_{F_\wp} F_\wp^{\mathrm{sep}}$. Since $\lambda_\wp$ is multiplication by $g_\wp$, and it also descends to $F_\wp$, $g_\wp$ lies in $\mathrm{GL}_n(F_\wp)$. Since we have a simultaneous factorization for $\mathrm{GL}_n$, there exist elements $g_U$ in $\mathrm{GL}_n(F_U)$ and $g_P$ in $\mathrm{GL}_n(F_P)$ such that $g_\wp = g_P^{-1} g_U$. Set $h'_U := g_U h_U$, and $h'_P := g_P h_P$. Consider the morphisms: $\lambda_\xi : h_\xi G_{F_\xi^{\mathrm{sep}}} \to h'_\xi G_{F_\xi^{\mathrm{sep}}}$ for $\xi$ in $\{P, U\}$. Now observe that $\lambda_P^{-1}$ and $\mu_{U,P} \circ \lambda_U^{-1}$ agree after base changing to $F_\wp$; this is essentially the identity $g_\wp g_U^{-1} = g_P^{-1}$. Consider the projection map $\pi : \mathrm{GL}_n \to \mathrm{GL}_n/G$. Note that $\pi(h'_\xi)$ gives an $F_\xi$-rational point on the quasi-projective variety $\mathrm{GL}_n/G$. We claim that the rational points $\pi(h'_U)$ and $\pi(h'_P)$ determine the same point in $(\mathrm{GL}_n/G)(F_\wp)$. As we saw before, over $F_\wp$, the isomorphism $\mu_{U,P}$ agrees with multiplication by $g_\wp$. Therefore $g_\wp h_U G_{F_\wp^{\mathrm{sep}}} = h_P G_{F_\wp^{\mathrm{sep}}}$. Multiplying both sides by $g_P$, we get $g_P g_\wp h_U G_{F_\wp^{\mathrm{sep}}} = g_P h_P G_{F_\wp^{\mathrm{sep}}} = h'_P G_{F_\wp^{\mathrm{sep}}}$. But $g_P g_\wp h_U G_{F_\wp^{\mathrm{sep}}} = g_U h_U G_{F_\wp^{\mathrm{sep}}} = h'_U G_{F_\wp^{\mathrm{sep}}}$. Since the system of rational points $\{\pi(h'_\xi)\}$ are compatible, and $F$ is the limit of the inverse system $\{F_\xi\}$, they determine a rational point over $F$. Denote the corresponding principal homogenous space by $hG_{F^{\mathrm{sep}}}$. This gives a solution to the patching problem. $\square$

Since $F$ is the limit of the system $\{F_\xi\}$, we have the following exact sequence:

$$0 \to F \to \prod_{U \in \mathcal{U}} F_U \times \prod_{P \in \mathcal{P}} F_P \rightrightarrows \prod_{\wp \in \mathcal{B}} F_\wp.$$

For a linear algebraic group $G$, we therefore have the following sequence of pointed sets:

$$0 \to G(F) \to \prod_{U \in \mathcal{U}} G(F_U) \times \prod_{P \in \mathcal{P}} G(F_P) \to \prod_{\wp \in \mathcal{B}} G(F_\wp),$$

where the right-most arrow sends a pair $(g_U, g_P)$ to $g_U g_P^{-1}$ at the components corresponding to the triple $(U, P, \wp)$ where $P$ lies in $\overline{U}$ and $\wp$ is a branch at $P$ lying on $U$.

We will say that $G$ satisfies the *simultaneous factorization property* if the rightmost map is surjective, i.e., for every triple $(U, P, \wp)$ and every $g_\wp$ in $G(F_\wp)$, there exist elements $g_U$ in $G(F_U)$ and $g_P$ in $G(F_P)$ such that $g_\wp = g_P g_U$.

Theorem 3.3.1 gives the following equalizer diagram of pointed sets:

$$\mathrm{H}^1(F, G) \to \prod_{U \in \mathcal{U}} \mathrm{H}^1(F_U, G) \times \prod_{P \in \mathcal{P}} \mathrm{H}^1(F_P, G) \rightrightarrows \prod_{\wp \in \mathcal{B}} \mathrm{H}^1(F_\wp, G).$$

One has the following relation between the two equalizer diagrams:

**Theorem 3.3.2.** *The following sequence is an exact sequence of pointed sets:*

$$\begin{array}{l} 0 \longrightarrow \mathrm{H}^0(F,G) \longrightarrow \prod_{U \in \mathcal{U}} \mathrm{H}^0(F_U, G) \times \prod_{P \in \mathcal{P}} \mathrm{H}^0(F_P, G) \longrightarrow \prod_{\wp \in \mathcal{B}} \mathrm{H}^0(F_\wp, G) \\ \longrightarrow \mathrm{H}^1(F,G) \longrightarrow \prod_{U \in \mathcal{U}} \mathrm{H}^1(F_U, G) \times \prod_{P \in \mathcal{P}} \mathrm{H}^1(F_P, G) \rightrightarrows \prod_{\wp \in \mathcal{B}} H^1(F_\wp, G). \end{array}$$

*Proof.* We only need to define the connecting map and establish exactness there. Let $(g_\wp)$ be a sequence in $\prod_{\wp \in \mathcal{B}} G(F_\wp)$. This gives rise to a principal homogenous space $T$ over $F$ in the following manner. Let $T_\xi / F_\xi$ be a trivial principal homogenous space over $F_\xi$ for every $\xi$ in $\mathcal{P} \cup \mathcal{U}$. Note that a branch $\wp$ uniquely determines a pair $(U, P)$. Now consider the isomorphisms $\mu_{U,P} : T_U \times_{F_U} F_\wp \to T_P \times_{F_P} F_\wp$ given by multiplication by $g_\wp$.

This defines a patching problem, for which we have a unique solution, say $T$ on $F$. We define the connecting map $\delta$ by sending $(g_\wp)$ to $T$. By the definition of $\delta$, the image consists of principal homogenous spaces which become trivial over $F_\xi$ for all $\xi$ in $\mathcal{P} \cup \mathcal{U}$. Thus the image of $\delta$ is contained in the kernel of the next map.

Now consider a principal homogenous space $T$ over $F$ such that $T_\xi := T \times_F F_\xi$ is trivial, i.e., isomorphic to $G$ over $F_\xi$ for every $\xi$ in $\mathcal{P} \cup \mathcal{U}$. Let $\phi_\xi$ be that isomorphism. For every $\wp$ be in $\mathcal{B}$ lying on $U$ and incident at $P$, define an isomorphism $\psi_{U,P} : T_U \times_{F_U} F_\wp \to T_P \times_{F_P} F_\wp$ compatible with the isomorphisms $\phi_U \times_{F_U} F_\wp$ and $\phi_P \times_{F_P} F_\wp$. Note that a morphism of principal homogenous spaces is given by multiplication by an element in $G$. Therefore, $\psi_{U,P}$ is given by multiplication by some $g_\wp$ in $G(F_\wp)$. This shows that $T$ is in the image of $\delta$.

Finally, suppose $(g_\wp)$ is in the kernel of $\delta$. We will show that there exist elements $g_U$ and $g_P$ in $G(F_U)$ and $G(F_P)$ respectively, such that $g_\wp = g_P g_U^{-1}$, for a triple $(U, P, \wp)$. Recall that the isomorphism $\mu_{U,P} : T_U \times_{F_U} F_\wp \to T_P \times_{F_P} F_\wp$ is given by multiplication by $g_\wp$. This patching problem gives rise to the trivial principal homogenous space over $F$ since $(g_\wp)$ lies in the kernel of $\delta$. Consider the patching problem $Id : G_{F_U} \times_{F_U} F_\wp \to G_{F_P} \times_{F_P} F_\wp$. This patching problem also gives rise to the trivial principal homogenous space over $F$. Therefore the two patching problems are isomorphic, i.e, we have the following commutative diagram for every triple $(U, P, \wp)$:

$$\begin{array}{ccc} G_{F_U} \times_{F_U} F_\wp & \xrightarrow{Id} & G_{F_P} \times_{F_P} F_\wp \\ \downarrow{\scriptstyle g_U} & & \downarrow{\scriptstyle g_P} \\ T_U \times_{F_U} F_\wp & \xrightarrow{g_\wp} & T_P \times_{F_P} F_\wp. \end{array}$$

The vertical arrows are given by multiplication by elements $g_U$ in $G(F_U)$ and $g_P$ in $G(F_P)$. Thus one obtains the factorization $g_\wp = g_P g_U^{-1}$. □

**Corollary 3.3.3** (Harbater-Hartmann-Krashen). *The kernel of the map*

$$\mathrm{H}^1(F, G) \to \prod_{U \in \mathcal{U}} \mathrm{H}^1(F_U, G) \times \prod_{P \in \mathcal{P}} \mathrm{H}^1(F_P, G)$$

*is trivial if and only if $G$ satisfies simultaneous factorization.*

Now the question becomes: which groups $G$ satisfy the simultaneous factorization property? Theorem 3.3.4 provides an answer. It does not cover all examples, but enough for our purposes. Recall that we say that $X/F$ is a *rational variety* if there exists an open subset $U$ of $X$ such that $U$ is isomorphic to an open subset $V$ of some affine space $\mathbb{A}^n_F$ over $F$.

**Theorem 3.3.4.** *A connected linear algebraic group $G/F$ that is rational as a variety satisfies the simultaneous factorization property.*

*Proof.* [HHK09, Theorem 3.2] □

Rationality of algebraic groups is a subtle question in general. Since we are interested in central simple algebras, we would like to know whether $PGL_n$ is rational. It is indeed the case.

**Example 3.3.5.** 1. *Let $A/F$ be a central simple algebra of degree $n$. We claim that $PGL(A)$ is rational and connected.*

*Note that $\mathrm{GL}(A)$ as a variety is an open subvariety of $\mathbb{A}^{n^2}$ given by the equation $Nrd_A \neq 0$. Thus, $\mathrm{GL}(A)$ is rational. Since, $PGL(A)$ is a quotient of $\mathrm{GL}(A)$ by $\mathbb{G}_m$, it is also rational. This also shows that $PGL(A)$ is connected. In particular, $PGL_n$ is also rational and connected.*

2. *$SO_n$ is rational and connected.*

*We use the well known Cayley parametrization to prove this. The Lie algebra of $SO_n$ is given by $n \times n$ skew symmetric matrices. View its Lie algebra as the affine space $\mathbb{A}^{n(n-1)/2}$. Consider the rational map:*

$$\mathbb{A}^{n(n-1)/2} \dashrightarrow SO_n,$$

*sending a skew symmetric matrix $Y$ to $(Id+Y)(Id-Y)^{-1}$. One can define its rational inverse:*

$$SO_n \dashrightarrow \mathbb{A}^{n(n-1)/2},$$

*sending a matrix $Z$ in $SO_n$ to $(Z-Id)(Z+Id)^{-1}$. Thus $SO_n$ is rational. It is connected since it is the connected component of $O_n$.*

**Theorem 3.3.6.** *Let $A/F$ be a degree $n$ central simple algebra. If $A \otimes_F F_\xi$ is split (i.e., isomorphic to $M_n(F_\xi)$) for every $\xi$ in $\mathcal{P} \cup \mathcal{U}$, then $A$ is split over $F$.*

*Proof.* Since $PGL_n$ is connected and rational, it satisfies simultaneous factorization by Theorem 3.3.4. By Corollary 3.3.3, the principal homogenous spaces satisfy a local to global principle with respect to patches. As $A$ corresponds bijectively to a principal homogenous space under $PGL_n$ by Theorem 2.3.11, it also satisfies a local to global principle. □

There is also a local global principle for homogenous varieties under $G/F$, i.e., those varieties $X/F$ for which $G(L)$ acts transitively on $X(L)$ for every $L/F$. Examples that we will be especially interested in are generalized Severi-Brauer varieties for $PGL_n$ and quadrics for $SO_n$ when $n \geq 3$.

Generalized Severi-Brauer varieties are twisted forms of Grassmannians. Let $A/F$ be a degree $n$ central simple algebra, and let $1 \leq d \leq n$ be an integer. The $d^{th}$ *generalized Severi-Brauer variety* $SB_d(A)$ is a variety parametrizing right ideals of $A$ of dimension $d \cdot \deg(A)$ (see [KMRT]).

**Theorem 3.3.7.** *Let $X/F$ be a quasi-projective variety under a rational connected linear algebraic group $G/F$. Suppose that $X(F_\xi) \neq \varnothing$ for every $\xi$ in $\mathcal{P} \cup \mathcal{U}$. Then $X(F) \neq \varnothing$.*

*Proof.* Let $\wp$ be a branch in $\mathcal{B}$. Recall that $\wp$ determines the pair $(U, P)$. Let $x_U$ be an element in $X(F_U)$ and $x_P$ in $X(F_P)$. Since $X(F_P)$ and $X(F_U)$ are subsets of $X(F_\wp)$, we may view $x_U$ and $x_P$ as elements in $X(F_\wp)$. Since $G(F_\wp)$ acts transitively on $X(F_\wp)$, there exists an element $g_\wp$ in $G(F_\wp)$ such that $g_\wp x_U = x_P$. Since $G/F$ is rational and connected, by Theorem 3.3.4, there exist elements $g_U$ in $G(F_U)$ and $g_P$ in $G(F_P)$ such that $g_\wp = g_P g_U$. Consider the elements $x'_U := g_U x_U$ and $x'_P := g_P^{-1} x_P$. Note that $x'_U$ and $x'_P$ determine the same element in $X(F_\wp)$ by construction. Denote this element in $X(F_\wp)$ by $x'_\wp$. Since $X$ is quasi-projective, the rational points $x'_U$, $x'_P$ and $x'_\wp$ are contained in an affine open set $\mathrm{Spec}(A) \subset X$ for every triple $(U, P, \wp)$. Corresponding to $x'_U$, $x'_P$ and $x'_\wp$, we have the ring homomorphisms $\theta_U : A \to F_U$, $\theta_P : A \to F_P$ and $\theta_\wp : A \to F_\wp$

respectively. Note that composing $\theta_U$ and $\theta_P$ with the inclusions $F_U \hookrightarrow F_\wp$ and $F_P \hookrightarrow F_\wp$, we obtain $\theta_\wp$. Therefore one obtains the homomorphism $\theta : A \to \varprojlim_\xi F_\xi = F$ by the universal property of inverse limits. This determines an $F$-rational point of $X$, implying that $X(F) \neq \varnothing$. □

This important theorem will allow us to obtain a formula similar to (2.17). We will also be able to compute the $u$-invariant of function fields of curves over higher local fields.

**Corollary 3.3.8.** *Let $X$ be a quadric of dimension greater than or equal to one. If $X(F_\xi) \neq \varnothing$ for every $\xi$ in $\mathcal{P} \cup \mathcal{U}$, then $X(F) \neq \varnothing$.*

*Proof.* This follows from Theorem 3.3.7 and the fact that $X$ is a homogenous quasi-projective (in fact projective) variety under the action of $SO_n$ for $n \geq 3$. □

The next corollary establishes a local global principle for indices of Brauer classes analogous to (2.17).

**Corollary 3.3.9.** *If $\alpha$ be an element in* $\mathrm{Br}(F)$*, then*

$$ind(\alpha) = \mathrm{lcm}_{\xi \in \mathcal{P} \cup \mathcal{U}}(ind(\alpha \otimes_F F_\xi)). \tag{3.7}$$

*Proof.* Let $d_\xi := ind(\alpha \otimes F_\xi)$, and $d := lcm_\xi(d_\xi)$. Note first that $d_\xi$ divides $ind(\alpha)$. Therefore, $d$ divides $ind(\alpha)$.

Let $A$ be a central simple algebra in the class of $\alpha$, with $deg(A) > d$. Let $SB_d(A)$ denote the $d^{th}$ generalized Severi-Brauer variety associated to $A$. Since $d_\xi$ divides $d$, by [KMRT, Proposition 1.17], $SB_d(A)(F_\xi) \neq \varnothing$. Since $SB_d(A)$ is a projective variety homogenous under the action of $PGL(A)$, by Theorem 3.3.7, $SB_d(A)(F) \neq \varnothing$. Again by [KMRT, Proposition 1.17], $ind(\alpha)$ divides $d$. □

# Chapter 4

# Two Applications of Patching

## 4.1 General Strategy

We now turn our attention to computing the Brauer dimension and $u$-invariant of semi-global fields. We will obtain this computation by proving local global principles for indices of Brauer classes, and isotropy of quadratic forms, with respect to discrete valuations. These two examples illustrate how using field patching one obtains local global principles with respect to discrete valuations for algebraic structures, in certain cases.

Let $F$ be a semi-global field. We can think of an algebraic structure over $F$ as an algebraic structure defined generically on a model $\mathscr{X}$ of $F$. There are finitely many closed subschemes of $\mathscr{X}$ which forms the "bad locus" or the "ramification" of the structure. Often the general strategy to obtain local-global principles with respect to discrete valuations is:

1. Use resolution of singularities to make the model $\mathscr{X}$ regular, and put this "ramification" in proper position with the special fiber $\mathscr{X}_k$.
2. Remove the points where this ramification locus meets the special fiber. Proposition 4.1.3 helps in showing that the algebraic structure is "trivial" on $F_U$ for suitable open sets $U$ of the special fiber $\mathscr{X}_k$, if it is trivial on the completions of $F$ with respect to discrete valuations coming the generic points of $\mathscr{X}_k$.
3. The complement in $\mathscr{X}_k$ of the union of open sets from the previous step is a finite collection of closed points $\mathcal{P}$. Note that for every $P$ in $\mathcal{P}$, $F_P$ is the fraction field of a two dimensional complete regular local ring. Choose a system of parameters for

this ring along the special fiber and the ramification locus (recall that the special fiber and the ramification locus are in normal crossing position). Now prove a local-global principle with respect to the completions in these two directions, and conclude that the algebraic structure is trivial on $F_P$ for every $P$ in $\mathcal{P}$.

4. Putting the previous three steps together, conclude that the algebraic structure is trivial on $F$ by field patching.

Recall the definition of a normal crossing divisor (see [Liu, Chapter 9, Definition 1.6]).

**Definition 4.1.1.** Let $\mathscr{X}$ be a regular Noetherian scheme. Let $D$ be an effective Cartier divisor of $\mathscr{X}$. We say that $D$ is a normal crossing divisor on $\mathscr{X}$ if for every $x$ in $\mathscr{X}$, there exist a system of parameters $\{f_1, \cdots, f_d\}$ for $\mathcal{O}_{\mathscr{X},x}$, integers $m$, $r_1, \cdots, r_m \geq 1$ such that the ideal sheaf $\mathcal{O}(-D)_x$ at $x$ is given by $\{{f_1}^{r_1} \cdots {f_m}^{r_m}\}$.

In other words the prime divisors in the support of $D$ intersect transversally. We will state the theorem on embedded resolution of singularities and use it in our applications and in the next chapter.

**Theorem 4.1.2.** *Let $F$ be a semi-global field, and let $\mathscr{X}/\operatorname{Spec} R$ be a normal model of $F$ as in Notation 1(see Chapter 3, section 3.1).*

1. *There exists a regular scheme $\mathscr{Y}/\operatorname{Spec} R$ and a birational morphism $\pi : \mathscr{Y} \to \mathscr{X}$ which is an isomorphism above every regular point of $\mathscr{X}$, and is obtained by a finite sequence of blowups and normalizations.*

2. *Let $D$ be an effective Cartier divisor on $\mathscr{X}$. We can choose $\mathscr{Y}/\operatorname{Spec} R$ such that the pullback $\pi^* D$ is a normal crossing divisor on $\mathscr{Y}$.*

*Proof.* See [Liu, Chapter 9, Theorem 2.26] and [Liu, Chapter 8, Theorem 3.44]. □

We will apply this theorem to a normal model $\mathscr{X}$ and to the support of the divisor given by the "ramification locus" and the special fiber. Note that $\pi^* D$ denotes the pullback of the divisor and not the divisor class of $[D]$ in $\operatorname{Pic}(\mathscr{X})$. Qing Liu (see [Liu, Chapter 7, Definition 1.34] defines the pullback $\pi^* D$ to be the image of $D \in \mathrm{H}^0(\mathscr{X}, \mathcal{K}^\times_{\mathscr{X}}/\mathcal{O}^\times_{\mathscr{X}})$ under the map $\mathcal{K}^\times_{\mathscr{X}}/\mathcal{O}^\times_{\mathscr{X}} \to f_*(\mathcal{K}^\times_{\mathscr{Y}}/\mathcal{O}^\times_{\mathscr{Y}})$.

Let $F$ be a semi-global field. The following Lemma allows us to pass from completions at generic points of the special fiber to fields of the form $F_U$, for some open set $U$ on the special fiber.

**Proposition 4.1.3.** *Let $F$ be a semi-global field, and let $H/F$ be a variety. Let $\eta$ be the generic point of an irreducible component $X_0$ of the special fiber of a normal model of $F$. Let $F_\eta$ be the completion of $F$ at $\eta$. If $H(F_\eta) \neq \varnothing$, then there exists an open affine set $U$ of $X_0$ which does not meet any other irreducible component of the special fiber, such that $H(F_U) \neq \varnothing$.*

The main idea of the proof goes as follows: standard reductions allow us to assume that the $F_\eta$-rational point of $H$ can be regarded as taking values in the completion $\widehat{R_\eta}$ of $\mathcal{O}_{\mathscr{X},\eta}$. One then uses Artin approximation to approximate this point by a rational point lying in a finite étale neighborhood $\operatorname{Spec} S$ of $\mathcal{O}_{\mathscr{X},\eta}$. Spreading out $\operatorname{Spec} S$, we may assume the rational point is in an étale neighborhood of some Zariski open $\operatorname{Spec} A$ of $\mathscr{X}$ containing $\eta$. Since our variety has an $S$-rational point, and this point is an approximation to the original $\widehat{R_\eta}$-rational point, we obtain a section of $H$ over the generic point $\eta$, or in other words a section over some open set $U$ of the special fiber of $\operatorname{Spec} A$. Now using a variant of Hensel's lemma (see [HHK09, Lemma 4.5]), we can lift this section to get a $\widehat{R_U}$-rational point of $H$.

## 4.2 Brauer Dimension Of Semi-global Fields

We now define the ramification locus of Brauer classes, in fact Galois cohomology classes in general:

**Definition 4.2.1.** Let $\ell$ be a prime not equal to $char(k)$ in the situation of Notation 1 in Chapter 3, Section 3.1. Let $B$ be a finite subset of $\mathrm{H}^n(F, \mu_\ell^{\otimes m})$. Let $\mathscr{X}$ be a normal model of $F$. Note that every prime divisor $D$ of $\mathscr{X}$ gives rise to a discrete valuation of $F$, which we denote by $v_D$.

We define the *ramification locus* of $B$ on $\mathscr{X}$ to be the union of the supports of prime divisors $D$ on $\mathscr{X}$ for which $\partial_{v_D}(\alpha) \neq 0$ for some $\alpha$ in $B$. We will denote it by $\mathrm{Ram}(B)$, and sometimes abusively refer to it as the *ramification divisor*.

We say that the ramification of $B$ is split (or $B$ is *unramified*) if $\partial_v(\alpha) = 0$ for all $\alpha$ in $B$ and for every discrete valuation $v$ in $\Omega_F$ which comes from a codimension one point on some regular projective model $\mathscr{Y}$ of $F$.

We will need the following theorem, due to Saltman, to describe Brauer classes on a two dimensional complete regular local ring. It can be seen as a two dimensional analogue of the Witt exact sequence. Saltman shows this for any excellent two dimensional regular local ring. We will only apply it to complete rings.

**Theorem 4.2.2.** *Let $R$ be a two dimensional regular local ring with fraction field $F$ and residue field $k$. Assume that char$(k)$ is not equal to $\ell$. Then the following sequence is exact:*

$$0 \to {}_\ell\operatorname{Br}(R) \to {}_\ell\operatorname{Br}(F) \xrightarrow{\oplus\partial_\mathfrak{p}} \bigoplus_{ht(\mathfrak{p})=1} \mathrm{H}^1(k(\mathfrak{p}), \mathbb{Z}/\ell\mathbb{Z}) \to \mathrm{H}^0(k, \mu_\ell^{-1}) \to 0.$$

*Proof.* See [Salt2, Theorem 5.2]. □

**Remark 4.2.3.** *If $R$ is a complete two dimensional regular local ring, then by [A-G, Corollary 6.2], we have: ${}_\ell\operatorname{Br}(R) \cong {}_\ell\operatorname{Br}(k)$. Thus we may replace ${}_\ell\operatorname{Br}(R)$ by ${}_\ell\operatorname{Br}(k)$ in the above sequence when $R$ is complete.*

Note that the map $\oplus\partial_\mathfrak{p}$ is the direct sum of ramification maps. These maps factor through the completions of $F$ at height one primes $\mathfrak{p}$ followed by the ramification map of the Witt exact sequence. Let $\pi$ be a prime generating a height one prime ideal of $R$. Assuming that $\mu_\ell$ is contained in $F$, we can identify the Galois modules $\mathbb{Z}/\ell\mathbb{Z}$ and $\mu_\ell$, and therefore $\mathrm{H}^1(k(\pi), \mathbb{Z}/\ell\mathbb{Z}) \cong k(\pi)^\times/(k(\pi)^\times)^\ell$. Now consider the cyclic algebra $(u, \pi)$ for $u$ in $R_\pi^\times$. Then $\partial_\pi(u, \pi) = (\overline{u}) \in \mathrm{H}^1(k(\pi), \mathbb{Z}/\ell\mathbb{Z})$.

**Proposition 4.2.4.** *Let $R$ be a complete two dimensional regular local ring with fraction field $F$ and residue field $k$, and let $\ell$ be a prime not equal to char$(k)$. Let $\{\pi, \delta\}$ form a regular system of parameters of $R$. We will assume that the $\ell^{th}$ roots of unity $\mu_\ell$, are contained in $k$. If $\alpha$ is a class in ${}_\ell\operatorname{Br}(F)$, unramified on $R$ except possibly at $\pi$ and $\delta$, then there exist units $u$, $v$ in $R^\times$, an integer $s$, and the lift $\alpha_0$ of a Brauer class in*

${}_\ell\operatorname{Br}(k)$ *to* ${}_\ell\operatorname{Br}(F)$*, such that*

$$\alpha = \alpha_0 + (u, \pi) + (v, \delta) + s(\pi, \delta).$$

*Proof.* Set $\theta_\pi := \partial_\pi(\alpha)$ and $\theta_\delta := \partial_\delta(\alpha)$. Note that $\theta_\pi$ is an element in $k(\pi)^\times/(k(\pi)^\times)^\ell$ and $\theta_\delta$ an element in $k(\delta)^\times/(k(\delta)^\times)^\ell$. Note also that $k(\pi)$ and $k(\delta)$ are complete discretely valued fields with respective parameters $\overline{\delta}$ and $\overline{\pi}$. Since $\alpha$ is unramified everywhere except possibly at $\pi$ and $\delta$, we have the following equality: $\partial_{\overline{\delta}}(\theta_\pi) + \partial_{\overline{\pi}}(\theta_\delta) = 0$ by Theorem 4.2.2. Define $s := \partial_{\overline{\delta}}(\theta_\pi) = -\partial_{\overline{\pi}}(\theta_\delta)$. Note that $s$ lies in $\mathrm{H}^0(k, \mu_\ell^{-1}) = \mathbb{Z}/\ell\mathbb{Z}$. We have: $\theta_\pi = \overline{u}\overline{\delta}^s$ and $\theta_\delta = \overline{v\pi^{-s}}$. Define $\beta := (u, \pi) + (v, \delta) + s(\pi, \delta)$ and $\alpha_0 := \alpha - \beta$. Now $\partial_\pi(\alpha_0) = \theta_\pi(\overline{u}\overline{\delta}^s)^{-1} = (1)$ and $\partial_\delta(\alpha_0) = \theta_\delta(\overline{v\pi^{-s}})^{-1} = (1)$. If $\rho$ generates any other height one prime of $R$, then $\partial_\rho(\alpha_0) = 0$. Thus $\alpha_0$ comes from a unique Brauer class in ${}_\ell\operatorname{Br}(k)$. □

The next proposition helps us in computing the indices of Brauer classes with ramification in proper position on a two dimensional complete regular local ring. This proposition together with Proposition 4.1.3 will help us in obtaining a local global principle for indices of Brauer classes.

**Proposition 4.2.5.** *Let $R$ be a complete two dimensional regular local ring with fraction field $F$, residue field $k$ and system of parameters $\{\pi, \delta\}$. Let $F_v$ be the completion of $F$, either at $\pi$ or $\delta$. Let $\ell$ be a prime not equal to char$(k)$. We will assume that $\mu_\ell$ is contained in $k$. Let $\alpha$ be a class in ${}_\ell\operatorname{Br}(F)$. If $\alpha$ is unramified on $R$, except possibly at $\pi$ or $\delta$, then $ind(\alpha) = ind(\alpha \otimes F_v)$.*

*Proof.* If $\alpha$ is only ramified at one of $\pi$ or $\delta$, say $\pi$, we may write $\alpha = \alpha_0 + (u, \pi)$. By Proposition 2.4.28, we have:

$$\begin{aligned} ind(\alpha \otimes F_\pi) &= ind(\alpha_0 \otimes k(\pi))[k(\pi)(\sqrt[\ell]{u}) : k(\pi)] \\ &= ind(\alpha_0 \otimes F_\pi(\sqrt[\ell]{u}))[F_\pi(\sqrt[\ell]{u}) : F_\pi]. \end{aligned}$$

Since $\alpha_0$ is unramified on $F$, its reduction on $k(\pi)$ is also unramified and therefore $ind(\alpha_0 \otimes F_\pi(\sqrt[\ell]{u})) = ind(\alpha_0 \otimes F(\sqrt[\ell]{u}))$. Note that $ind(\alpha)$ divides $ind(\alpha_0)[F(\sqrt[\ell]{u}) : F]$.

Thus, $ind(\alpha)$ divides $ind(\alpha \otimes F_\pi)$. Since $ind(\alpha \otimes F_\pi)$ divides $ind(\alpha)$, we get that $ind(\alpha) = ind(\alpha \otimes F_\pi)$.

Now suppose $\alpha$ is ramified at both $\pi$ and $\delta$. Then by Proposition 4.2.4, we have the following presentation: $\alpha = \alpha_0 + (u,\pi) + (v,\delta) + (\pi^s,\delta) = \alpha_0 + (u,\pi) + (v\pi^s,\delta)$. Set $\alpha_0' := \alpha_0 + (u,\pi)$. Note that $\alpha_0'$ is unramified on $\delta$. Applying Proposition 2.4.28 to $\alpha \otimes F_\delta = \alpha_0' + (v\pi^s,\delta)$, we get that:

$$\begin{aligned} ind(\alpha \otimes F_\delta) &= ind(\alpha_0' \otimes k(\delta)(\sqrt[\ell]{v\pi^s}))[k(\delta)(\sqrt[\ell]{v\pi^s}) : k(\delta)] \\ &= ind(\alpha_0' \otimes F_\delta(\sqrt[\ell]{v\pi^s}))[F_\delta(\sqrt[\ell]{v\pi^s}) : F_\delta] \\ &= ind(\alpha_0' \otimes F(\sqrt[\ell]{v\pi^s}))[F(\sqrt[\ell]{v\pi^s}) : F]. \end{aligned}$$

Set $\pi' := v\pi$. Now notice that $\alpha_0' \otimes k(\delta)(\sqrt[\ell]{v\pi^s}) = \alpha_0 - (u,v) + (u,\pi')$. Set $\alpha_0'' := \alpha_0 - (u,v)$. Note that $\alpha_0''$ is unramified on the complete discretely valued field $k(\delta)(\sqrt[\ell]{v\pi^s})$ whose residue field is $k$. Thus $ind(\alpha_0' \otimes k(\delta)(\sqrt[\ell]{v\pi^s})) = ind(\alpha_0'' \otimes k(\sqrt[\ell]{u}))[k(\sqrt[\ell]{u}) : k]$. Therefore, we obtain:

$$\begin{aligned} ind(\alpha \otimes F_\delta) &= ind(\alpha_0'' \otimes k(\sqrt[\ell]{u}))[k(\sqrt[\ell]{u}) : k][k(\delta)(\sqrt[\ell]{v\pi^s}) : k(\delta)] \\ &= ind(\alpha_0'' \otimes k(\sqrt[\ell]{u}))[F_\delta(\sqrt[\ell]{v\pi^s}) : F_\delta][F_\delta(\sqrt[\ell]{v\pi^s}, \sqrt[\ell]{u}) : F_\delta(\sqrt[\ell]{v\pi^s})] \\ &= ind(\alpha_0'' \otimes k(\sqrt[\ell]{u}))[F_\delta(\sqrt[\ell]{v\pi^s}, \sqrt[\ell]{u}) : F_\delta] \\ &= ind(\alpha_0'' \otimes k(\sqrt[\ell]{u}))[F(\sqrt[\ell]{v\pi^s}, \sqrt[\ell]{u}) : F]. \end{aligned}$$

Note that $ind(\alpha)$ divides $ind(\alpha_0'' \otimes k(\sqrt[\ell]{u}))[F(\sqrt[\ell]{v\pi^s}, \sqrt[\ell]{u}) : F]$, and thus divides $ind(\alpha \otimes F_\delta)$. Since $ind(\alpha \otimes F_\delta)$ divides $ind(\alpha)$, we get that $ind(\alpha) = ind(\alpha \otimes F_\delta)$. $\square$

We are finally in a position to prove a local global principle for Brauer classes of prime period $\ell$. This was first obtained by Reddy-Suresh in [Re-Su].

**Theorem 4.2.6** (Suresh, Reddy)**.** *Let $F$ be a semi-global field with residue field $k$. Assume that $\ell$ is a prime not equal to $char(k)$, and that $F$ contains the group of $\ell^{th}$ roots of unity. If $\alpha$ is a Brauer class in ${}_\ell\mathrm{Br}(F)$, then there exists a discrete valuation $v$ coming from a codimension one point of some regular model of $F$ such that*

$$ind(\alpha) = ind(\alpha \otimes F_v).$$

*Proof.* Using Theorem 4.1.2, we may choose a regular projective model $\mathscr{X}/\operatorname{Spec} R$ such that the union of the ramification divisor $\operatorname{Ram}(\alpha)$ and the support of the special fiber of $\mathscr{X}$ is in normal crossing position.

Let $\{X_1, \cdots, X_k\}$ be the irreducible components of the special fiber of $\mathscr{X}_k$ with respective generic points $\{\eta_1, \cdots, \eta_k\}$. Let $d := \operatorname{lcm}_v\{ind(\alpha \otimes F_v)\}$, and let $A/F$ be a central simple algebra in the Brauer class of $\alpha$ with $d < deg(A)$. We will denote the completion of $F$ at the codimension one point $\eta_i$ by $F_{\eta_i}$. Since $ind(\alpha \otimes F_{\eta_i})$ divides $d$, by [KMRT, Proposition 1.17], the $d^{th}$ generalized Severi-Brauer variety has an $F_{\eta_i}$-rational point, i.e., $SB_d(A)(F_{\eta_i}) \neq \varnothing$. By Proposition 4.1.3, there exists an affine open set $U_i$ of $X_i$ which does not meet any other irreducible component $X_i$ such that $SB_d(A)(F_{U_i}) \neq \varnothing$. Let $U_i'$ be the complement in $U_i$ of the set of closed points where horizontal components (i.e., components flat over $\operatorname{Spec} R$) of $\operatorname{Ram}(\alpha)$ intersects $U_i$. Since $U_i' \subseteq U_i$, observe that $F_{U_i} \subseteq F_{U_i'}$. Since $SB_d(A)(F_{U_i}) \neq \varnothing$, we also have $SB_d(A)(F_{U_i'}) \neq \varnothing$. Therefore $ind(A \otimes F_{U_i'})$ divides $d$.

Let $\mathcal{P}$ be the complement of $\bigcup U_i$ in the special fiber $\mathscr{X}_k$ including the points where the ramification divisor of $\alpha$ meets the special fiber. Let $P$ be a point in $\mathcal{P}$. If $P$ does not lie in the ramification locus of $\alpha$, then $\alpha$ is unramified on $F_P$. Suppose that $\pi$ cuts out one component of the special fiber, say $X_i$. Note that the completion of $F_P$ at $\pi$, denoted by $F_{P,\pi}$ contains $F_{\eta_i}$. Since, $SB_d(A)(F_{\eta_i}) \neq \varnothing$, $SB_d(A)(F_{P,\pi}) \neq \varnothing$. Therefore, $ind(A \otimes F_{P,\pi})$ divides $d$. By Proposition 4.2.5, $ind(A \otimes F_P)$ divides $d$. Thus $SB_d(A)(F_P) \neq \varnothing$.

If $P$ lies on the ramification locus of $\alpha$, recall that the ramification of $\alpha$ on $\widehat{R_P}$, forms a normal crossing divisor with the special fiber. Choose a system of parameters $\{\pi, \delta\}$ for $\widehat{R_P}$, where $\pi$ cuts out a component $X_i$ of the special fiber. By an argument similar to the one in the previous paragraph, $SB_d(A)(F_{P,\pi}) \neq \varnothing$ and $SB_d(A)(F_{P,\delta}) \neq \varnothing$. Therefore by Proposition 4.2.5, $ind(A \otimes F_P)$ divides $d$.

Finally, using Theorem 3.3.9, we conclude that $ind(\alpha)$ divides $d$. Since $\alpha$ has prime period $\ell$, its index is a power of $\ell$. Therefore, there exists a discrete valuation $v$ such that $ind(\alpha \otimes F_v) = d$. Since $ind(\alpha \otimes F_v)$ divides $ind(\alpha)$, we conclude that $ind(\alpha) = d$. $\square$

The proof of the theorem above is a slight modification of that of Reddy-Suresh. The basic idea however remains the same. As a corollary, we can provide the following upper bound to the Brauer $\ell$-dimension of semi-global fields, first obtained in [HHK09].

**Corollary 4.2.7** (Harbater-Hartmann-Krashen)**.** *Let $F$ be a semi-global field with residue field $k$ with char$(k)$ not equaling $\ell$. If* $\mathrm{Brd}_\ell(k) \le d$ *and* $\mathrm{Brd}_\ell(k(t)) \le d+1$*, then*

$$\mathrm{Brd}_\ell(F) \le d+2.$$

*Proof.* By Lemma 2.4.8, it is enough to find an upper bound for the index of a Brauer class having period $\ell$. We will first assume that $\mu_\ell \subset F$ in order to apply Theorem 4.2.6. Let $\alpha$ be a Brauer class of period $\ell$. Let $v$ be a discrete valuation corresponding to a codimension one point of a regular projective model $\mathscr{X}$, where $\mathscr{X}$ is as in the proof of Theorem 4.2.6, and such that $ind(\alpha) = ind(\alpha \otimes F_v)$. Let $\pi_v$ be a parameter of $F_v$. Note that the residue field of $F_v$ is either a finite extension of $K$, if the codimension one point gives a divisor flat over $\operatorname{Spec} R$ (horizontal), or a finite extension of $k(t)$ if the codimension one point gives a divisor on the special fiber (vertical).

If the residue field is a finite extension of $k(t)$, then $\alpha \otimes F_v = \alpha_0 + (u, \pi_v)$. Therefore, $ind(\alpha) = ind(\alpha \otimes k(\pi)(\sqrt[\ell]{u})) \cdot \ell$ by Proposition 2.4.28. Therefore, $ind(\alpha) \le \ell^{d+2}$. If the residue field is a finite extension of $K$, we have $\alpha = \alpha_0 + (u, \pi_v)$. Thus $ind(\alpha) = ind(\alpha_0 \otimes k(\pi)(\sqrt[\ell]{u})) \cdot \ell$. Note that $k(\pi)$ is also a complete discretely valued field, with residue field a finite extension of $k$. Let $\pi'$ be a parameter of $k(\pi)(\sqrt[\ell]{u})$. We may thus write $\alpha_0$ as $\alpha_0 = \alpha_0' + (v, \pi')$. Therefore

$$ind(\alpha_0 \otimes k(\pi)(\sqrt[\ell]{u})) = ind(\alpha_0' \otimes k(\pi)(\sqrt[\ell]{u})(\sqrt[\ell]{v})) \cdot \ell \le \ell^{d+1}.$$

Hence it follows that $ind(\alpha) \le \ell^{d+2}$. Since $ind(\alpha) = ind(\alpha \otimes F_v)$, $ind(\alpha) \le \ell^{d+2}$. This shows that $\mathrm{Brd}_\ell(F) \le d+2$.

We will now remove the assumption on the $\ell^{th}$ roots of unity. Let $F$ be a semi-global field not containing $\mu_\ell$, and $\alpha$ a class in ${}_\ell\mathrm{Br}(F)$. Let $F' := F(\mu_\ell)$. By what we have proved in the previous paragraph, $ind(\alpha \otimes F') \le \ell^{d+2}$. Thus there exists a field extension $L/F'$ of degree dividing $\ell^{d+2}$ which splits $\alpha \otimes F'$. Recall that the degree of $F'/F$ is coprime to $\ell$. Thus, $ind(\alpha)$ divides $[F' : F]\ell^{d+2}$. Since $\alpha$ has period $\ell$, $ind(\alpha)$ is also a power of $\ell$. Therefore, $ind(\alpha)$ divides $\ell^{d+2}$. This shows the inequality: $\mathrm{Brd}_\ell(F) \leq d + 2$, in general. ☐

**Definition 4.2.8.** We say that $K$ is a 0-*local field*, if it is either a finite field, or a finite extension of a complete discretely valued field with separably closed residue field.

We say that $K$ is an $m$-*local field* (for $m > 0$) if it is a complete discretely valued field with residue field an $(m-1)$-local field. Thus there is a sequence of $n$-local fields for $n < m$, associated to $K$.

We will denote the corresponding 0-local field by $k_0$ and call it the *smallest residue field*.

If $F$ is the function field of a curve over a finite field, then by Corollary 2.4.34 $\mathrm{Brd}_\ell(F) = 1$. When $F$ is the function field of a curve over a complete discretely valued field with separably closed residue field, one sees by Corollary 4.2.7 that $\mathrm{Brd}_\ell(F) \leq 2$. This is because the Brauer group of the residue field $k$ is trivial, and so is the Brauer group of any finite extension of $k(t)$. We may improve this bound to the following:

**Corollary 4.2.9.** *Let $F$ be the function field of a curve over a complete discretely valued field $K$ with separably closed residue field $k$, and $\ell$ be a prime co-prime to $char(k)$. We have the following upper bound for the Brauer $\ell$-dimension:*

$$\mathrm{Brd}_\ell(F) \leq 1.$$

*Proof.* By Lemma 2.4.8, it suffices to show that the index of Brauer classes with period dividing $\ell$ is at most $\ell$. Let $\alpha$ be a class in ${}_\ell\mathrm{Br}(F)$. Let $v$ be a discrete valuation of $F$ coming from a codimension one point of some regular projective model of $F$. Note that the residue field of $v$ is either a function field of a curve over $k$, or a finite extension of $K$. Thus the Brauer group of the residue field of $v$ is trivial. By the Witt exact sequence (see Theorem 2.4.24), $\alpha \otimes F_v$ may be identified with a character of the residue field $k(v)$ of period dividing $\ell$. Splitting $\alpha \otimes F_v$ therefore amounts to splitting this character. Since a non-trivial period $\ell$ character can be split by an extension of degree $\ell$, one sees that $ind(\alpha \otimes F_v) \leq \ell$. By Theorem 4.2.6, it follows that $ind(\alpha) \leq \ell$, and hence

$\mathrm{Brd}_\ell(F) \leq 1$. □

This Corollary, together with Corollary 2.4.34 shows that if $F$ is the function field of a curve over a 0-local field, then $\mathrm{Brd}_\ell(F) \leq 1$.

**Corollary 4.2.10.** *Let $F$ be the function field of a curve over an $m$-local field, and $\ell$ be a prime not equal to the characteristic of the smallest residue field. We have the following upper bound for the Brauer $\ell$-dimension:*

$$\mathrm{Brd}_\ell(F) \leq m + 1.$$

*Proof.* We will show this by induction on $m$. If $m = 0$, note that $\mathrm{Brd}_\ell(F) \leq 1$ by Corollary 4.2.9 and Corollary 2.4.34.

If $m \geq 1$, the residue field $k$ is an $(m-1)$ local field. Repeatedly using Proposition 2.4.28, one can show that $\mathrm{Brd}_\ell(k) \leq m - 1$. Since $k(t)$ is a semi-global field, by the induction hypothesis, $\mathrm{Brd}_\ell(k(t)) \leq m$. Therefore by Corollary 4.2.7, we conclude that $\mathrm{Brd}_\ell(F) \leq m + 1$. □

The above corollary recovers the result proved by Saltman [Salt], that $\mathrm{Br}_\ell(\mathbb{Q}_p(t)) \leq 2$ for $\ell \neq p$. Harbater-Hartmann-Krashen ([HHK09]) directly show this using Corollary 3.3.6.

## 4.3 Local Global Principle For Quadratic Forms

We now turn our attention to obtain a local-global principle similar to Theorem 4.2.6 for isotropy of quadratic forms over semi-global fields. This was first obtained by Colliot-Thélène, Parimala and Suresh in [CT-Par-Su]. The general strategy remains the same as outlined in the introduction of the chapter: first, put ramification in proper position. Then show that there exist affine open sets $U$ of the special fiber such that the quadratic form becomes isotropic over $F_U$ using Proposition 4.1.3. Then establish isotropy over $F_P$ for the remaining closed points $P$. Now using Corollary 3.3.8, conclude that the form is isotropic over $F$. This will enable us to compute the $u$-invariant, a result first obtained by Harbater-Hartmann-Krashen in [HHK09].

We first need an appropriate definition of ramification locus of quadratic forms. This will turn out to be easy. As we mention above, the ramification locus of an algebraic structure is the "bad locus" where the algebraic structure is not defined. For example: consider the quadratic form over $\mathbb{Q}$: $q = \langle 1, 8, 3, 24\rangle$. This form is defined on $\operatorname{Spec}\mathbb{Z}$ almost everywhere, i.e., we can specialize the form to a non-degenerate form at almost all primes in $\mathbb{Z}$ except at 2 and 3. Thus the ramification locus of $q$ on $\operatorname{Spec}\mathbb{Z}$ should be the union of the support of the divisors $(2)$ and $(3)$. With this example in mind, we define the ramification divisor of a quadratic form as follows:

**Definition 4.3.1.** Let $F$ be a semi-global field with residue field $k$. Let $\mathscr{X}$ be a normal projective model for $F$. Suppose that $char(k) \neq 2$. Let $q = \langle a_1, \cdots, a_n\rangle$ be a non-degenerate quadratic form. We define the *ramification divisor* or *ramification locus* of $q$ to be the union of the supports of prime divisors $D$ on $\mathscr{X}$ such that $v_D(a_i)$ is odd for some $i$. We will denote it by $\operatorname{Ram}(q)$.

**Theorem 4.3.2** (Colliot-Thélène, Parimala, Suresh)**.** *Let $F$ be a semi-global field with residue field $k$, and $char(k) \neq 2$. Let $q/F$ be a quadratic form of dimension $\geq 3$. If $q \otimes F_v$ is isotropic for every divisorial discrete valuation $v$ of $F$, then $q/F$ is isotropic.*

*Proof.* Let $\mathscr{X}$ be a regular projective model of $F$. Using Theorem 4.1.2, one can replace $\mathscr{X}$ by another regular model such that the union of the support of the special fiber and $\operatorname{Ram}(q)$ are in normal crossing position. We will abuse notation and also denote this new regular model by $\mathscr{X}$.

Let $\{X_1, \cdots, X_k\}$ be the irreducible components of the special fiber $\mathscr{X}_k$ with respective generic points $\{\eta_1, \cdots, \eta_k\}$. Let $F_{\eta_i}$ be the completion of $F$ at $\eta_i$. By hypothesis, $q \otimes F_{\eta_i}$ is isotropic. Thus by Proposition 4.1.3, there exist affine open sets $U_i$ of $X_i$ such that $q \otimes F_{U_i}$ is isotropic for every $i = 1, \cdots, k$, and $U_i$ does not meet any other irreducible component of $\mathscr{X}_k$. Let $U_i'$ be the complement in $U_i$ of the closed points where horizontal components of $\operatorname{Ram}(q)$ intersects $U_i$. Observe that $F_{U_i} \subseteq F_{U_i'}$. Since $q \otimes F_{U_i}$ is isotropic, so is $q \otimes F_{U_i'}$.

Let $\mathcal{P}$ be the complement of $\bigcup U_i'$ in the special fiber $\mathscr{X}_k$, and let $P$ be in $\mathcal{P}$. Suppose that $P$ does not lie on the ramification locus of $q$. In this case, we may write $q \otimes F_P = \langle u_1, \cdots, u_n \rangle$, where $u_i$ lies in $\widehat{R_P}^{\times}$ for all $i = 1, \cdots, n$. Let $\{\pi, \delta\}$ be a regular system of parameters for $\widehat{R_P}$ where $\pi$ cuts out one component of the special fiber, say $X_i$. Note that $F_{\eta_i} \subset F_{P,\pi}$. Since $q \otimes F_{\eta_i}$ is isotropic, $q \otimes F_{P,\pi}$ is also isotropic. Let $\widehat{R_{P,\pi}}$ denote the completion of the localization of $\widehat{R_P}$ at $\pi$; it is the ring of integers of $F_{P,\pi}$. Since $q \otimes F_{P,\pi}$ is isotropic, so is its reduction $q \otimes k(\pi)$. Note that $k(\pi)$ is a complete discretely valued field with residue field $k(P)$. Thus again, the reduction of $q \otimes k(\pi)$ with respect to its parameter $\overline{\delta}$ is isotropic. This shows that $q \otimes k(P)$ is isotropic. By Hensel's lemma, it follows that $q \otimes F_P$ is isotropic.

Now suppose that $P$ lies on only one irreducible component of the ramification locus. Choose a regular system of parameters $\{\pi, \delta\}$ for $\widehat{R_P}$ such that $\pi$ cuts out that component of the ramification locus. One may therefore write $q \otimes F_P = \langle u_1, \cdots, u_m \rangle \perp \pi \langle u_{m+1}, \cdots, u_n \rangle$. Let $F_\pi$ denote the completion of $F$ at the codimension one point given by the component of the ramification locus. Observe that $F_\pi \subset F_{P,\pi}$. Since, $q \otimes F_\pi$ is isotropic, so is $q \otimes F_{P,\pi}$. Therefore by Springer's theorem (see Proposition 2.5.6), either the reduction of $\langle u_1, \cdots, u_m \rangle$ is isotropic on its residue field $k(\pi)$ or the reduction of $\langle u_{m+1}, \cdots, u_n \rangle$ is isotropic on $k(\pi)$. Again, $k(\pi)$ is a complete discretely valued field with residue field $k(P)$. Therefore the reduction of either of those forms is isotropic over $k(P)$. Thus by Hensel's lemma, $q$ is isotropic over $F_P$.

Let $P$ lie on two irreducible components of the ramification locus. Choose a regular system of parameters $\{\pi, \delta\}$ for $\widehat{R_P}$ such that $\pi$ cuts out one component and $\delta$ cuts out the other component on $\widehat{R_P}$. We may write $q = q_1 \perp \pi q_2 \perp \delta q_3 \perp \pi\delta q_4$, where the entries of $q_i$ are units in $\widehat{R_P}$. Let $F_\pi$ and $F_\delta$ denote the completions of $F$ at the respective components. Note since $q \otimes F_{P,\pi}$ is isotropic, the reduction of $q_1 \perp \delta q_3$ or the reduction of $q_2 \perp \delta q_4$ is isotropic. As before, the residue field is a complete discretely valued field with residue field $k$ and parameter $\overline{\delta}$. Therefore the reduction of one of $q_1$, $q_2$, $q_3$ or $q_4$ is isotropic on $k(P)$. Hensel's lemma then shows that $q \otimes F_P$ is isotropic.

Finally, using Corollary 3.3.8, we conclude that $q/F$ is isotropic. $\square$

We are now in a position to compute the $u$-invariant of function fields of curves over $m$-local fields.

**Corollary 4.3.3** (Harbater-Hartmann-Krashen)**.** *Let $F$ be the function field of a curve over an $m$-local field with characteristic of the smallest residue field unequal to 2. The $u$-invariant of $F$ satisfies:*

$$u(F) \leq 2^{m+2}.$$

*Proof.* Let $q/F$ be a quadratic form of dimension greater than $2^{m+2}$. Let $v$ be a divisorial discrete valuation of $F$. Note that the completion of $F$ at $v$ is an $m+1$ local field. We will show that $q \otimes F_v$ is isotropic by induction on $m$. For the base case $m = 0$: $F_v$ is a 1-local field, with residue field a 0-local field. Since 0-local fields are $C_1$ fields, their $u$-invariants are equal to 2. Let $\pi_v$ denote the parameter of $F_v$. We may write $q \otimes F_v = q_1 \perp \pi_v q_2$, where the entries of $q_1$ and $q_2$ are units in the ring of integers. Since the dimension of $q \otimes F_v$ is greater than 4, the dimension of either $q_1$ is greater than 2, or the dimension of $q_2$ is greater than 2. Thus the reduction of $q_1$ or $q_2$ is isotropic. By Springer's theorem, $q \otimes F_v$ is isotropic for every divisorial discrete valuation $v$. Thus by Theorem 4.3.2, $q/F$ is isotropic.

When $m \geq 1$: $F_v$ is an $(m+1)$-local field with residue field an $m$-local field. We may write $q \otimes F_v = q_1 \perp \pi_v q_2$, where the entries of $q_1$ and $q_2$ are units. Since the dimension of $q \otimes F_v$ is greater than $2^{m+1}$, either the dimension of the reduction of $q_1$ is greater than $2^m$, or the reduction of $q_2$ is greater than $2^m$. By the induction hypothesis, either of those reductions is isotropic. Therefore $q \otimes F_v$ is isotropic for every divisorial discrete valuation $v$. Again by Theorem 4.3.2, $q/F$ is isotropic. □

The $u$-invariant of function fields of curves over a $p$-adic field (denoted by $F$) was first computed by Parimala-Suresh (see [Pa-Su1] and [Pa-Su2]), thereby settling a long standing question. They used the results of Saltman to show that every element in $\mathrm{H}^3(F, \mathbb{Z}/\ell\mathbb{Z})$ is a symbol, and obtain the computation of the $u$-invariant using that. Their techniques do not obviously generalize to the case of function fields over higher local fields. Using field patching we can not only recover their result, but also obtain a local-global principle for isotropy of quadratic forms.

# Chapter 5

# Generalized Brauer Dimension

## 5.1 Introduction

Over global fields, we can simultaneously split a finite collection of Brauer classes of prime period $\ell$ by making a degree $\ell$ extension. To see this, all we need to do is split the ramification of this finite collection of Brauer classes simultaneously as in the proof of Theorem 2.4.34 for function fields of curves over finite fields, and Theorem 2.4.40 for number fields. The Hasse principle for Brauer classes is what makes things work.

Let $F$ be a semi-global field. We might expect that if there is a uniform bound to split an arbitrary number of Brauer classes over the residue field $k$ and over $k(t)$, there will be a uniform bound to split an arbitrary collection of Brauer classes simultaneously on $F$. This expectation is guided by Theorem 4.2.7. This theorem informs us that some arithmetic properties of a field $k$ and $k(t)$ transmit to $F$. If there is such a uniform bound for simultaneously splitting any finite collection of Brauer classes, we obtain another arithmetic invariant of semi-global fields. Let us define it formally at this point.

**Definition 5.1.1.** Let $F$ be a field, and let $n$ be a natural number. Let $B$ be a finite subset in ${}_n\operatorname{Br}(F)$. The *index of* $B$, denoted by $ind(B)$, is the minimum of the degrees of field extensions $L/F$ such that $\alpha \otimes L = 0$ for every $\alpha$ in $B$.

The *Generalized Brauer* $n$*-dimension* $\operatorname{GBrd}_n(F)$ is the supremum of $ind(B)$ as $B$ ranges over finite subsets of ${}_n\operatorname{Br}(L)$ and $L/F$ ranges over finite degree field extensions.

**Remark 5.1.2.** *1. We defined the* Brauer $n$-dimension *to be the largest exponent appearing in the index (which is a power of $n$) rather than the largest index (see Definition 2.4.6). We do not define the* Generalized Brauer $n$-dimension *in this* *way because there is no reason for the index of a finite subset* $B$ *of* ${}_n\mathrm{Br}(F)$ *to be a power of* $n$.

*Let us define the index of* $B$ *somewhat differently: we define the index of* $B$ *to be the gcd of degrees of field extensions which split all elements in* $B$. *We will denote this alternative index by* $ind'(B)$. *If* $B$ *is a finite subset of* ${}_n\mathrm{Br}(F)$, $ind'(B)$ *is indeed a power of* $n$. *While* $ind'(B)$ *divides* $ind(B)$, *they need not be equal. Note that* $ind'(B)$ *equals the minimum of degrees of effective* $0$*-cycles on the variety* $\prod_{\alpha\in B} SB(D_\alpha)$, *i.e., the product of Severi Brauer varieties of division algebras associated to Brauer classes in* $B$, *whereas* $ind(B)$ *is the minimum of degrees of closed points in* $\prod_{\alpha\in B} SB(D_\alpha)$. *The question whether* $ind(B) = ind'(B)$ *has been asked by Totaro in a more general context, namely for principal homogenous spaces under connected linear algebraic groups. (see [Tot]). More precisely, Totaro asked whether the existence of an effective* $0$*-cycle of degree dividing* $d$ *on a principal homogenous space implies the existence of a (étale) closed point of degree dividing* $d$. *We now know that this is not true in general (see [GS-S]). We still however do not know whether Totaro's question has an affirmative answer for* $\prod_\alpha SB(D_\alpha)$.

2. *As we will see later, computing* $ind(B)$ *is a difficult problem. The present state of our understanding only allows us to obtain a uniform upper bound for the index of any finite subset* $B$ *over semi-global fields. Since* $ind'(B) \leq ind(B)$, *this uniform upper-bound for* $ind(B)$ *also provides us with an upper-bound for* $ind'(B)$.

As we mention in the beginning of this section, when $F$ is a global field, the same argument that shows that $\mathrm{Brd}_\ell(F) = 1$ also shows that $\mathrm{GBrd}_\ell(F) = \ell$. In other words, there exists a degree $\ell$ extension which splits any finite collection of Brauer classes in the $\ell$-torsion part of the Brauer group of $F$ simultaneously. This begs the question whether splitting a single Brauer class is as hard as splitting any finite collection of Brauer classes, at least over nice fields. If that were true, we would have the following:

$$\mathrm{GBrd}_n(F) = n^{\mathrm{Brd}_n(F)}.$$

However the following proposition tells us this need not be true in general.

**Proposition 5.1.3.** *Let $F = \mathbb{Q}_p((t))$, with $p \neq 2$. Consider the set of nontrivial quaternion algebras $B = \{(u,p),(p,t),(u,t)\}$, where $u$ is a unit in $\mathbb{Z}_p$. Then* $\mathrm{Brd}_2(F) = 1$, *but there does not exist a quadratic field extension splitting all the classes in $B$ simultaneously, i.e.,* $\mathrm{GBrd}_2(F) \geq 4$.

*Proof.* We will first show that $\mathrm{Brd}_2(F) = 1$. Let $\alpha$ be an element in ${}_2\operatorname{Br}(F)$. Using the Witt exact sequence (see Theorem 2.4.24), one can write $\alpha = (u,p) + (\chi,t)$ or $\alpha = (\chi,t)$ for some $\mathbb{Z}/2$ character of the Galois group of $\mathbb{Q}_p$. Note that such characters uniquely determine a square class in $\mathbb{Q}_p^{\times}$. If $\alpha = (\chi,t)$, clearly there is a quadratic extension splitting it. If $\chi$ corresponds to the square classes $(u)$ or $(p)$, $\alpha = (u,pt)$ or $\alpha = (ut,p)$. In either case, there exists a quadratic extension splitting $\alpha$. Thus, $\mathrm{Br}_2(F) = 1$.

Suppose that $L/F$ is a quadratic extension splitting all elements of $B$. There are only eight square classes in $F$: $\{1,u,p,t,up,pt,ut,upt\}$. Therefore $L$ is generated by the square root of one of these non-trivial square classes. Note that $F(\sqrt{u})$ does not split $(p,t)$, because $F(\sqrt{u})/F$ is unramified, with residue field extension $\mathbb{Q}_p(\sqrt{u})/\mathbb{Q}_p$. Thus $\partial_t(p,t) = (p)$ is a non-square in the residue field of $F(\sqrt{u})$. Note also that $F(\sqrt{p})$ does not split $(u,t)$ by a similar reasoning. Third, $F(\sqrt{t})/F$ does not split $(u,p)$, because $(u,p)$ is an unramified Brauer class of $F(\sqrt{t})$. The residue field of $F(\sqrt{t})$ is $\mathbb{Q}_p$, and the class $(u,p)$ is non-zero in ${}_2\operatorname{Br}(\mathbb{Q}_p)$.

$F(\sqrt{up})/F$ does not split $(p,t)$. Note that $F(\sqrt{up})/F$ is unramified with residue field extension $\mathbb{Q}_p(\sqrt{up})/\mathbb{Q}_p$. Further, $\partial_t(p,t) = (p)$ is a non-square in $\mathbb{Q}_p(\sqrt{up})$. Observe also that $F(\sqrt{pt})/F$ does not split $(u,p)$, because $(u,p)$ is unramified on $F(\sqrt{pt})$, and $F(\sqrt{pt})/F$ is totally ramified and hence has $\mathbb{Q}_p$ as its residue field. Since $(u,p)$ is non-zero in ${}_2\operatorname{Br}(\mathbb{Q}_p)$, it follows that $(u,p)\otimes_F F(\sqrt{pt})$ is non-split. By a similar reasoning, one can show also that $F(\sqrt{ut})/F$ does not split $(p,t)$, and $F(\sqrt{upt})/F$ does not split $(u,p)$. $\square$

**Remark 5.1.4.** *There is a better example of the phenomena occurring in the proposition above. When $F = \mathbb{C}(x,y)$, Chapman and Tignol in [C-Tig] show that the set of quaternion algebras $\{(x,y),(x,y+1),(y,x+1),(y,xy+1)\}$ is not split by a quadratic field extension* *of $F$. However since $F$ is a $C_2$ field (see Theorem 2.4.11), it has $u$-invariant equal to* $4$*. Thus $F$ does not admit Brauer classes of index greater than* $2$*. Thus* $\mathrm{Brd}_2(F) = 1$*, but* $\mathrm{GBrd}_2(F) \geq 4$.

Proposition 5.1.3 and the above remark show that the Generalized Brauer $\ell$-dimension is another arithmetic invariant, and one can therefore ask a more general version of Question 1 (see Subsection 2.4.1):

**Question 2.** *Let $F$ be the function field of a variety over either an algebraically closed field, finite field, non-archimedean local field or a global field (any reasonably "nice" field). Is* $\mathrm{GBrd}_n(F)$ *finite? If it is finite, compute* $\mathrm{GBrd}_n(F)$*.*

For most of these nice fields, even finiteness is not known. Even when we know what the Brauer dimension is, in some cases, there are no known upper bounds for the Generalized Brauer dimension. For example, Lieblich has shown that when $F = \mathbb{F}_p(x,y)$, $\mathrm{Brd}_\ell(F) \leq 2$ (see [Lie]). However we do not even know if $\mathrm{GBrd}_\ell(F)$ is finite. We also do not know if $\mathrm{GBrd}_\ell(\mathbb{C}(x,y))$ is finite.

The most hopeful case in tackling Question 2 is when $F$ is a semi-global field. In [Pa-Su3], Parimala-Suresh call the Generalized Brauer $\ell$-dimension "$(2,\ell)$-uniform bound". Their goal is to show finiteness of the $u$-invariant, in the mixed characteristic 2 situation. Along the way, they obtain the following upper bound for the Generalized Brauer $\ell$-dimension of a semi-global field.

**Theorem 5.1.5** (Parimala-Suresh)**.** *Let $F$ be a semi-global field with residue field $k$ such that char$(k) \neq \ell$. We have the following upper bound for the Generalized Brauer $\ell$-dimension.*

$$\mathrm{GBrd}_\ell(F) \leq \ell^3[\mathrm{GBrd}_\ell(k(t))]![\mathrm{GBrd}_\ell(k)]!$$

While this shows finiteness of $\mathrm{GBrd}_\ell(F)$ for a nice semi-global field, the bound seems far from optimal.

To answer Question 2 for global fields, one uses the local-global principle for Brauer classes. While we do have a local-global principle for Brauer classes on semi-global fields, and we can split them locally at every completion by making a field extension there, we may not be able to put them together to get a global extension. This is because we might have to make infinitely many local extensions where the Brauer classes are non-trivial, and may not be able to approximate all of them simultaneously to get an extension of the semi-global field.

Another approach could be to obtain a statement similar to Theorem 3.3.9 for index of a finite collection of Brauer classes. The index of a Brauer class is detected by an appropriate generalized Severi-Brauer variety of a central simple algebra in the Brauer class: i.e., $ind(\alpha)$ divides $d$ if and only if $SB_d(A)(F) \neq \varnothing$ for $A$ in the Brauer class of $\alpha$. These varieties are homogenous under the action of a rational, connected linear algebraic group, and therefore satisfy a local global principle with respect to patches. Let $B$ be a finite subset of ${}_{\ell}\mathrm{Br}(F)$. Suppose that there exists a variety $X/F$ which is index detecting in the following sense: $X_d(F) \neq \varnothing$ if and only if there exists a field extension of degree dividing $d$ which splits all $\alpha$ in $B$. If further, $X$ is homogenous under the action of a connected linear algebraic group, that would make it easier to compute $ind(B)$. We do not know if there exists such a variety. While one could construct an index detecting variety, it is not at all clear if it is homogenous.

The most promising approach is first to split ramification. Let us consider the case of $F = K(t)$, where $K$ is a complete discretely valued field with residue field $k$. Let $B$ be a finite subset in ${}_{\ell}\mathrm{Br}(F)$. Let $\Omega_{F/K}$ be the set of discrete valuations trivial on $K$. Recall that they correspond to codimension one points on $\mathbb{P}^1_K$. The ramification locus of $B$ consists of a finite collection of points on $\mathbb{P}^1_K$. It suffices to make a degree $\ell$ extension $L/K(t)$ to split the ramification. Taking the normalization of $\mathbb{P}^1_K$ in $L$, one obtains a curve $C/K$ which possibly has genus $g > 1$. Thus all Brauer class in $B$ are unramified with respect to the valuations trivial on $K$ on this curve $C/K$. One has a "Fadeev like" exact sequence (see Theorem 2.4.31) for $C/K$, but the first map need not be injective. Using the Hochschild-Serre spectral sequence, one can uniquely associate to each class $\alpha$ in $B$ an element in $\mathrm{Br}(K) \oplus \mathrm{H}^1(K, \mathrm{Pic}^0(C))$. There are known bounds for indices of principal homogenous spaces under Jacobian varieties over certain fields $K$ (see [Clark]), but they depend on the genus of $C$, which in turn depends upon the number of places where $B$ is ramified. This is far from optimal. For the same reason, this strategy does not work for computing the Brauer dimension of function fields of curves over number fields.

Splitting ramification with respect to valuations trivial on the field of constants is thus clearly not enough. On the other hand, if we also split ramification with respect to valuations coming from the parameter of $R$ (recall Notation 1 from Section 3.1), we might hope that this should make it simpler to split the Brauer classes further. In other words, we should split the ramification on a two dimensional model of $F$. Clearing out the ramification on a two dimensional model allows us to specialize the Brauer classes on the special fiber. The function fields of curves in the special fiber are fields of lower arithmetic complexity. For example, the function field of the special fiber of $\mathbb{P}^1_{\mathbb{Z}_p}$ is $\mathbb{F}_p(t)$ where we know what the Brauer dimension is. Saltman uses this idea to compute the Brauer dimension of function fields of $p$-adic curves.

## 5.2 Splitting Ramification

We will start by gathering some evidence which makes us more confident about the approach outlined in the previous paragraph.

**Theorem 5.2.1** (Grothendieck)**.** *Let $F$ be the function field of a curve over a $p$-adic field, and $\alpha$ be an element in* $\mathrm{Br}(F)$ *of period coprime to $p$. If $\alpha$ is unramified (see Definition 4.2.1), then $\alpha = 0$.*

*Proof.* Let $\mathscr{X}$ be a regular projective model of $F$. Let $\{X_1, \cdots, X_k\}$ be the irreducible components of the special fiber $\mathscr{X}_k$. We may also assume that each $X_i$ is a regular curve over the residue field $k$. Let $\{\eta_i, \cdots, \eta_k\}$ denote the corresponding generic points, and let $F_{\eta_i}$ be the completion of $F$ at $\eta_i$. Since $\alpha \otimes F_{\eta_i}$ is unramified, by the Witt exact sequence (see Theorem 2.4.24) it follows that $\alpha \otimes F_{\eta_i}$ restricts uniquely to a Brauer class on the residue field $k(X_i)$ of $F_{\eta_i}$. Denote this restriction by $\alpha_i$. Note that $k(X_i)$ is the function field of a curve over a finite field. We will show that $\alpha_i = 0$ by showing that it is unramified on the curve $X_i$. Let $P$ be a closed point of $X_i$. Note that $\alpha \otimes F_P$

is unramified on $\widehat{R_P}$. Thus by Theorem 4.2.2, $\alpha$ comes from a unique Brauer class on the residue field $k(P)$. But since $k(P)$ is a finite field, $\alpha \otimes F_P = 0$. Let $\pi$ be the parameter defining one irreducible component $X_i$ of $\mathscr{X}_k$ containing $P$. Let $F_{P,\pi}$ denote the completion of $F_P$ at $\pi$. Note that the residue field $k(\pi)$ is the completion of $k(X_i)$. Since $\alpha \otimes F_{P,\pi} = 0$, we conclude that $\alpha_i \otimes k(\pi) = 0$. Therefore by Theorem 2.16, it follows that $\alpha_i = 0$. This also means that $\alpha \otimes F_{\eta_i} = 0$. By Proposition 4.1.3, there exists an affine open set $U_i$ which does not meet any other irreducible component such that $\alpha \otimes F_{U_i} = 0$.

Let $\mathcal{P}$ be the complement of $\bigcup U_i$ in $\mathscr{X}_k$. If $P$ is any closed point on $\mathscr{X}_k$ we have seen above that $\alpha \otimes F_P = 0$. In particular, for every $P$ in $\mathcal{P}$, $\alpha \otimes F_P = 0$. By Theorem 3.3.6, we conclude that $\alpha = 0$. □

Therefore if we split the ramification of a Brauer class or even a finite collection of Brauer classes on regular two dimensional models of function fields of curves over $p$-adic fields, we would be done. We do that first by having a local description of the ramification locus by two functions. We first need the following lemma. This was proved by [AAIKL] to split the ramification in a controlled manner. The original idea goes back to Pirutka (See [Pir]).

**Lemma 5.2.2.** *Let $\mathscr{X}$ be a two dimensional regular noetherian scheme. Let $D$ be a normal crossing divisor on $\mathscr{X}$. Then there exists a sequence of blowups at closed points $f : \mathscr{Y} \to \mathscr{X}$ such that $f^{-1}(D)$ can be expressed as the union of two regular, not necessarily connected divisors.*

*Proof.* Let $\{C_1, \cdots, C_n\}$ be the prime divisors in the support of $D$. By blowing up if necessary, we may assume that each $C_i$ is regular. Define a graph $\Gamma$ with vertices $v_i$ corresponding to $C_i$; an edge exists between two vertices $v_i$ and $v_j$ if $C_i$ and $C_j$ intersect. Since $\mathscr{X}$ is two dimensional and $D$ is normal crossing, no three curves $C_i, C_j, C_k$ intersect at a point. Therefore each graph gives rise to a unique configuration of $D$. The partition of $D$ into $D_1$ and $D_2$ can be translated into a coloring problem for $\Gamma$: we want to color the vertices in two different colors, say blue and green, such that no two vertices which share an edge have the same color. We could then define $D_1$ as the union of the divisors corresponding to blue and $D_2$ as the union of divisors corresponding to green. Since no irreducible component of $D_i$ intersect any other irreducible component, and each component is regular, both $D_1$ and $D_2$ are regular since they are the disjoint union of regular curves.

If every connected component of $\Gamma$ is a tree, we can alternately color the vertices blue and green, and we are done.

Suppose that every connected component of $\Gamma$ consists of only even cycles (i.e., cycles with even number of vertices), then we can alternately color each vertex blue and green and obtain a graph with no two adjacent vertices having the same color, and we are done.

Note that blowing up at an intersection of two given divisors creates a new divisor between them. That is, we obtain a new configuration of the divisors, namely the original divisors and the exceptional curve. Furthermore, the two intersecting divisors now intersect on this exceptional curve (recall that the divisor is normal-crossing). The new graph of this configuration is obtained by adding an additional vertex between the two adjacent edges.

Now suppose that there is an odd cycle in a connected component of $\Gamma$. All we do is blow up at any intersecting point and introduce an additional vertex. If we do this process whenever we have an odd cycle, the new graph obtained in this manner will only have even cycles. Thus we can alternately color each vertex and obtain the desired coloring. Therefore there is a sequence of blowups such that $f : \mathscr{Y} \to \mathscr{X}$ such that $f^{-1}(D) = D_1 \cup D_2$, with $D_1$ and $D_2$ regular. □

The authors in [AAIKL] in fact show that on any $d$ dimensional regular scheme, one may write a divisor as a union of $d$ regular divisors. The proof is also pretty combinatorial.

**Theorem 5.2.3.** *Let $F$ be the function field of a curve over a $p$-adic field. Let $B$ be a finite subset in ${}_\ell \mathrm{Br}(F)$. Suppose that $\ell \neq p$, then there exists a field extension $L/F$ of degree $\ell^2$ such that $\alpha \otimes L$ is unramified on every regular model of $L$ for every $\alpha$ in $B$.*

*Proof.* Let $\mathscr{X}$ be a regular projective model of $F$. Using Theorem 4.1.2, we may assume that $\operatorname{Ram}(B)$ is a normal crossing divisor. Using Lemma 5.2.2, after replacing $\mathscr{X}$ by another regular model, we may express $\operatorname{Ram}(B)$ as a union of two regular divisors $D_1$ and $D_2$. We will abusively denote this new model of $F$ by $\mathscr{X}$. Let $\mathcal{P}$ be the set of points on every irreducible component of $\operatorname{Ram}(B)$ including the intersection points of $D_1$ and $D_2$. Since $\mathscr{X}$ is quasi-projective, there exists an affine open set $U$ containing $P$. Let $A$ be the ring obtained by semi-localizing $U$ at the points in $\mathcal{P}$. Since $A$ is a semi-local ring, $\operatorname{Pic}(A)$ is trivial. This means that there exists an $f_1$ in $F$ such that $\operatorname{div}_A(f_1) = D_1 + D_2$. Therefore, $\operatorname{div}_{\mathscr{X}}(f_1) = D_1 + D_2 + E$ where $E$ is a divisor that does not pass through any intersection points of $D_1$ and $D_2$. By the same argument as above, we may find $f_2$ in $F$ such that $\operatorname{div}_{\mathscr{X}}(f_2) = D_1 + F$ and $f_2$ is a unit at closed points in $\operatorname{Supp}(D_2) \bigcap \operatorname{Supp}(E)$ . Using the Chinese Remainder Theorem, we may also assume that $f_2$ is not an $\ell^{th}$ power in $k(x)$ for every $x$ in $\operatorname{Supp}(D_2) \bigcap \operatorname{Supp}(E)$.

We claim that $L = F(\sqrt[\ell]{f_1}, \sqrt[\ell]{f_2})$ splits the ramification of $B$. Let $v$ be a discrete valuation of $L$, and $w$ be the restriction of $v$ to $F$. Since $\mathscr{X}/\operatorname{Spec} R$ is proper, $w$ has a unique center $x$ on $\mathscr{X}$. Note that since $\mathcal{O}_{\mathscr{X},x}$ dominates $R_w$, for every height one prime ideal $\mathfrak{p}$ of $\mathcal{O}_{\mathscr{X},x}$, we have the following containments of residue fields: $k(\mathfrak{p}) \subset k(w) \subset k(v)$. Therefore we have the following two commutative squares by (2.15):

$$\begin{tikzcd}
\mathrm{H}^2(F,\mu_\ell) \arrow[r, "\partial_{\mathfrak{p}}"] \arrow[d, "Id"] & \mathrm{H}^1(k(\mathfrak{p}),\mathbb{Z}/\ell\mathbb{Z}) \arrow[d] \\
\mathrm{H}^2(F,\mu_\ell) \arrow[r, "\partial_w"] \arrow[d, "res_{L/F}"] & \mathrm{H}^1(k(w),\mathbb{Z}/\ell\mathbb{Z}) \arrow[d] \\
\mathrm{H}^2(L,\mu_\ell) \arrow[r, "\partial_v"] & \mathrm{H}^1(k(v),\mathbb{Z}/\ell\mathbb{Z}).
\end{tikzcd}$$

If $x$ lies on a codimension one point of either $D_1$ or $D_2$, or is a codimension two point, lying on $\operatorname{Supp}(D_1) \bigcap \operatorname{Supp}(F)$, $\operatorname{Supp}(D_2) \bigcap \operatorname{Supp}(F)$ or $\operatorname{Supp}(D_1) \bigcap \operatorname{Supp}(E)$, we may express $\alpha$ as $\alpha = \alpha_0 + (u, f_i)$, where $i$ lies in $\{1, 2\}$, and $\alpha_0$ is unramified. Note that by the two commutative squares above, $\partial_v(\alpha \otimes L) = \partial_v(\alpha_0 \otimes L) = \partial_{\mathfrak{p}}(\alpha_0) \otimes k(v) = 0$.

If $x$ is a codimension two point lying on the intersection of $D_1$ and $D_2$, then by Theorem 4.2.2, we may write $\alpha$ as $\alpha = \alpha_0 + (u, f_2) + (u, f_1/f_2) + s(f_2, f_1/f_2)$, where $\alpha_0$ is unramified on $\mathcal{O}_{\mathscr{X},x}$ and $s$ is some integer. Again we see that $\alpha \otimes L = \alpha_0 \otimes L$. By the two commutative squares, $\alpha \otimes L$ splits the ramification of $\alpha$.

The only remaining case is when $x$ lies on $\mathrm{Supp}(D_2) \cap \mathrm{Supp}(E)$. We may express $\alpha$ as $\alpha = \alpha_0 + (u, \pi_2)$, where $\alpha_0$ is unramified on $O_{\mathscr{X},x}$ and $\pi_2$ is a local parameter of $D_2$. On $O_{\mathscr{X},x}$, we may write $f_1 = \lambda\pi_2\delta$, where $\lambda$ is a unit and $\delta$ is a local parameter for $E$. We may therefore rewrite $\alpha$ as $\alpha = \alpha_0 - (u, \delta) + (u, f_1)$ for some unramified class $\alpha_0$ and a unit $u$. Restricting to $K = F(\sqrt[\ell]{f_1})$, $\alpha \otimes K = \alpha_0 \otimes K + (u, \delta)$. Note that $\partial_v(\alpha \otimes L) = \bar{u}^{v(\delta)}$, where $\bar{u}$ is an $\ell^{th}$ power class in the residue field $k(v)$.

Let $\mathscr{Y} \to \mathscr{X}$ be the normalization of $\mathscr{X}$ in $L$, and let $y$ be the center of $v$ on $\mathscr{Y}$. Note first that $k(x)$ is a finite field since $x$ is a closed point. Second, the residue field extension of $k(y)/k(x)$ contains a degree $\ell$ extension. Thus, every element in $k(x)$ becomes an $\ell^{th}$ power in $k(y)$. In particular, $\bar{u}$ in $k(x)$ becomes an $\ell^{th}$ power in $k(y)$. Since $k(y) \subset k(v)$, $\bar{u}$ is an $\ell^{th}$ power in $k(v)$. Thus, $\partial_v(\alpha \otimes L) = 0$. Therefore, $L/F$ splits the ramification of $\alpha$ for every $\alpha$ in $B$. □

As a corollary, we obtain Saltman's result on the Brauer dimension of $F$ a function field of a $p$-adic curve. In fact, we can also show that $\mathrm{GBrd}_\ell(F) \leq \ell^2$ for $\ell \neq p$.

**Corollary 5.2.4.** *Let $F$ be the function field of a curve over a $p$-adic field. Suppose that $\ell$ is a prime, not equal to $p$. Then, we have*

$$\mathrm{GBrd}_\ell(F) \leq \ell^2.$$

*Proof.* Let $B$ be a finite subset of ${}_\ell\mathrm{Br}(F)$. By Theorem 5.2.3, there exists a field extension $L/F$ of degree $\ell^2$ which splits the ramification of $B$. Therefore by Theorem 5.2.1, $\alpha \otimes L = 0$ for every $\alpha$ in $B$. □

Notice that in the proof of Theorem 5.2.3, we have used that $\mathscr{X}$ is a model of a $p$-adic curve only in the last paragraph. That is the only place where we use the arithmetic of $\mathbb{Z}_p$. We use that fact that the residue fields of closed points are finite fields. This suggests that there should be another way to split ramification of Brauer classes, on a regular surface. Just as in the proof of Theorem 5.2.3, if we are able to locally describe the ramification divisor by combination of two functions, we should be able to split the ramification by extracting $\ell^{th}$ roots. While this may not be enough to split Brauer classes on semi-global fields, we expect that splitting their ramification should make it easier to split them.

**Theorem 5.2.5.** *Let $F$ be a semi-global field. Let $B$ be a finite subset of Brauer class in ${}_\ell\mathrm{Br}(F)$. If $\ell \neq 2$, there exists a field extension $L/F$ of degree $\ell^2$ such that $\alpha \otimes L$ is unramified with respect to every discrete valuation of $L$ for every $\alpha$ in $B$. If $\ell = 2$, then there exists a degree 8 field extension $L/F$ such that $\alpha \otimes L$ is unramified with respect to every discrete valuation of $L$, for every $\alpha$ in $B$.*

*Proof.* We claim that it suffices to assume that $\mu_\ell \subset F$. Suppose that $F$ does not contain $\mu_\ell$. Let $K := F(\mu_\ell)$. Let $v$ be a non-trivial discrete valuation of $K$ and let $w := v|F$. Note that since the degree of $K/F$ is coprime to $\ell$, the ramification index $e_{w/v}$ and the degree of the residue field extension $f_{w/v}$ are coprime to $\ell$. Let $\alpha$ be in $B$. By the commutative diagram (2.15), notice that $\partial_w(\alpha \otimes K) = 0$ if $\partial_v(\alpha) = 0$. If $\partial_w(\alpha \otimes K) = 0$, then $e_{w/v} Res_{k(w)/k(v)}(\partial_v(\alpha)) = 0$. Since $e_{w/v}$ is coprime to $\ell$, $Res_{k(w)/k(v)}(\partial_v(\alpha)) = 0$. Since the residue field extension has degree coprime to $\ell$, a standard restriction-corestriction argument shows that $\partial_v(\alpha) = 0$. Thus we may assume that $F$ contains a primitive $\ell^{th}$ root of unity.

Let $\mathscr{X}$ be a regular projective model of $F$. Using Theorem 4.1.2, we may assume that $\mathrm{Ram}(B)$ is a normal crossing divisor on $\mathscr{X}$. Lemma 5.2.2 further allows us to assume that $\mathrm{Ram}(B)$ can be written as a union of two regular, not necessarily connected divisors, $D_1$ and $D_2$. Let $\mathcal{P}$ be a finite set of points on each irreducible component of $D_1 \cup D_2$, including the intersection points of $D_1$ and $D_2$. Since $\mathscr{X}$ is quasi-projective, one may find an affine open set $U$ containing all the points in $\mathcal{P}$. Let $A$ be the semi-local ring obtained by semi-localizing $U$ at the points in $\mathcal{P}$. Note that $\mathrm{Pic}(A)$ is trivial.

We will first consider the case $\ell \neq 2$. Consider the divisors $\widetilde{D_1} = D_1 + 2D_2$ and $\widetilde{D_2} = D_1 + D_2$ on $\mathrm{Spec}\, A$. Since $\mathrm{Pic}(A)$ is trivial, there exists a rational function $f_1$ such that $\mathrm{div}_A(f_1) = D_1 + 2D_2$. Therefore on $\mathscr{X}$, $\mathrm{div}_{\mathscr{X}}(f_1) = D_1 + 2D_2 + E$ where $E$ does not pass through any points in $\mathcal{P} := \mathrm{Supp}(D_1) \bigcap \mathrm{Supp}(D_2)$. Let $\mathcal{P}_1$ be the points of intersection of $\mathrm{Supp}(E)$ with $\mathrm{Supp}(D_1) \bigcup \mathrm{Supp}(D_2)$. Let $A'$ be the semi-local ring at the points $\mathcal{P} \cup \mathcal{P}_1$. Consider the rational function $f_2$ such that $\mathrm{div}_{A'}(f_2) = D_1 + D_2$. Therefore on $\mathscr{X}$, $\mathrm{div}_{\mathscr{X}}(f_2) = D_1 + D_2 + G$ where $G$ does not pass through any points in $\mathcal{P} \bigcup \mathcal{P}_1$.

Consider the field extension $L = F(\sqrt[\ell]{f_1}, \sqrt[\ell]{f_2})$. We claim that $L/F$ splits the ramification of $B$. Let $v$ be a divisorial discrete valuation of $L$. Let $x$ be the unique center of $v$ on $\mathscr{X}$. We may assume that $x$ lies on the ramification divisor of $B$. Let $\alpha$ be any non-trivial element in $B$. Just as in the proof of Theorem 5.2.3, it suffices to show that the residue of $\alpha$ at each height one prime ideal of $\mathcal{O}_{\mathscr{X},x}$ is split by $L$.

If $x$ is a codimension one point lying on any of the $\mathrm{Supp}(D_i)$, or a codimension two point on $\mathrm{Supp}(D_2) \bigcap \mathrm{Supp}(E)$, $\mathrm{Supp}(D_1) \bigcap \mathrm{Supp}(E)$ or $\mathrm{Supp}(D_1) \bigcap \mathrm{Supp}(G)$, then we may express $\alpha = \alpha_0 + (u, f_i)$, where $\alpha_0$ is unramified on $\mathcal{O}_{\mathscr{X},x}$ and $u$ is a unit in $\mathcal{O}_{\mathscr{X},x}$. The subextension $F(\sqrt[\ell]{f_i})$ is totally ramified on the ramification locus of $\alpha$ in $O_{\mathcal{X},x}$. Therefore, we see that $\partial_v(\alpha \otimes L) = 0$. Thus a degree $\ell^2$ extension splits the ramification of $B$.

If $x$ lies on $\mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2)$, then the equation for $D_1$ on $O_{\mathcal{X},x}$ is given by $f_2^2/f_1$, and for $D_2$ is given by $f_1/f_2$. Since $L = F(\sqrt[\ell]{f_2^2/f_1}, \sqrt[\ell]{f_1/f_2})$ is totally ramified at the local parameters for $D_1$ and $D_2$ on $O_{\mathcal{X},x}$, we are done. Finally, if $x$ lies on $\mathrm{Supp}(D_2) \cap \mathrm{Supp}(G)$, the local equation for $2D_2$ is given by $f_1$. But because $\ell \neq 2$, the local parameter for $D_2$ is totally ramified in the extension $F(\sqrt[\ell]{f_1})$. Therefore $\alpha \otimes L$ is unramified.

For the prime $\ell = 2$: consider three functions $f_1$, $f_2$ and $f_3$ chosen as in the case for $l \neq 2$ such that $\mathrm{div}_{\mathscr{X}}(f_1) = D_1 + D_2 + E$, $\mathrm{div}_{\mathscr{X}}(f_2) = D_1 + G$ and $\mathrm{div}_{\mathscr{X}}(f_3) = D_2 + H$ where the support of no three divisors among $D_1$, $D_2$, $E$, $G$ and $H$ intersect. Just as before, we show that the possible local parameters for $D_i$ are totally ramified by the extension $L = F(\sqrt{f_1}, \sqrt{f_2}, \sqrt{f_3})$. We will show this in one case; the rest are similar. If $x$ lies in $\mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2)$, the local equations for $D_1$ and $D_2$ are given by $f_2$ and $f_3$. Thus, the subextension $F(\sqrt{f_2}, \sqrt{f_3})$ splits the ramification. Thus a degree 8 extension splits the ramification of $B$. $\square$

In particular, if $B$ is a finite subset in ${}_\ell\operatorname{Br}(F)$ for a semi-global field $F$, then we can split the ramification of $B$ my either making a degree $\ell^2$ extension or a degree 8 extension.

## 5.3 Generalized Brauer Dimension Of Semi-global Fields

Now that we can split the ramification of an arbitrary finite collection $B$ of Brauer classes in a controlled manner, we are one step closer to give a uniform upper-bound for $ind(B)$. Observe that we can specialize the classes in $B$ on the special fiber and split them, at least generically on the special fiber. We can then make extensions locally to split the Brauer classes on every component of the special fiber. The following lemma allows us to construct a global extension inducing the local extensions.

**Lemma 5.3.1.** *In the situation of Notation 1 (see Section 3.1), let $F$ be a semi-global field, and let $\mathscr{X}/\operatorname{Spec}(R)$ be a regular model of $F$. Let $\{X_1, \cdots, X_n\}$ be the irreducible components of the special fiber, and let $\eta_i$ denote the generic points of $X_i$, and let $F_{\eta_i}$ be the completion of $F$ at the discrete valuation given by $\eta_i$. Suppose that $L_{\eta_i}/F_{\eta_i}$ are separable field extensions of degree $d$. Then there exists a field extension $L/F$ of degree $d$ such that $L \otimes_F F_{\eta_i} \cong L_{\eta_i}$.*

*Proof.* Since $L_{\eta_i}/F_{\eta_i}$ are separable, $L_{\eta_i} \cong F_{\eta_i}[x]/\langle f_{\eta_i}(x)\rangle$. By weak approximation we may find a polynomial $f(x)$ in $F[x]$ sufficiently close to $f_{\eta_i}(x)$. Therefore by Krasner's Lemma, we have $L_{\eta_i} \cong F[x]/\langle f(x)\rangle \otimes_F F_{\eta_i}$. Let $L := F[x]/\langle f(x)\rangle$. Thus we see that $L \otimes_F F_{\eta_i} \cong L_{\eta_i}$. □

There is another issue of splitting $B$ on a finite set of points on $\mathscr{X}_k$. Since $B$ is unramified, we can specialize the classes in $B$ on closed points and split them there. But we need to build a global extension out of these extensions. The following lemma allows us to do just that:

**Lemma 5.3.2.** *Let $\mathscr{X}/\operatorname{Spec}(R)$ be a normal projective model of $F$. Let $\mathcal{P}$ be a finite non empty set of closed points on the special fibre $X$ of $\mathcal{X}$ which includes points where irreducible component of $X$ meet. For each point $P$ in $\mathcal{P}$, let $l(P)/k(P)$ be a degree $d$ separable extension of the residue field $k(P)$. For each $P$ in $\mathcal{P}$, let $L_P/F_P$ be the* *unramified lift of $l(P)/k(P)$. There exists a field extension $L/F$ of degree $d$ such that $L \otimes_F F_P \cong L_P$.*

*Proof.* Let $P$ be in $\mathcal{P}$ and $\wp$ be a branch incident at $P$. We will denote the ring of integers of $F_\wp$ by $\widehat{R_\wp}$ for every branch $\wp$. Observe that $L_P \otimes_{F_P} F_\wp \cong \prod_i L_{\wp_i}$, where $L_{\wp_i}/F_\wp$ are finite field extensions of $F_\wp$. Since $L_P/F_P$ is unramified, so are $L_{\wp_i}/F_\wp$ for every $i$. Let $l(\wp_i)/k(\wp)$ be the corresponding residue field extensions. Set $l(\wp) \coloneqq \prod_i l(\wp_i)$ and $L_\wp \coloneqq \prod L_{\wp_i}$. Thus we have that $L_P \otimes_{F_P} F_\wp \cong L_\wp$.

Now to obtain an extension $L/F$, we need to construct a separable algebra $L_V/F_U$ for suitable open sets $U$ such that $L_V \otimes_{F_U} F_\wp \cong L_P \otimes_{F_P} F_\wp$ for every triple $(P, U, \wp)$.

Let $\mathcal{U}$ be the set of irreducible components of the the complement of $\mathcal{P}$ on the special fiber. For each $U$ in $\mathcal{U}$, let $\mathcal{B}_U$ denote the set of branches lying on $U$. Note that the function field $k(U)$ is dense in $\prod_{\wp \in \mathcal{B}_U} k(\wp)$ for every $\wp$ in $\mathcal{B}_U$. Using Krasner's Lemma and weak approximation, there exists a separable algebra $l_U/k(U)$ such that $l_U \otimes_{k(U)} k(\wp) \cong l(\wp)$ for every branch $\wp$ in $\mathcal{B}_U$.

Let $V \to U$ be the normalization of $U$ in $l_U$. After shrinking $U$ if necessary, we may assume that the map is étale. Abusing notation, we write $U$ for this new open set. Let $\mathcal{P} \cup \mathcal{P}_1$ be the complement of these new open sets on the special fiber $\mathscr{X}_k$. By Theorem 3.1.4, we may uniquely lift $V \to U$ to get an étale algebra $\widehat{S_V}/\widehat{R_U}$. Note that $\widehat{S_V}$ is a product of domains. Let $L_V$ be the product of their fraction fields. We claim that $L_V \otimes F_\wp \cong L_\wp$ for each triple $(U, P, \wp)$, such that $\wp$ is a branch incident at $P$, lying on $U$ and $P$ lies in $\overline{U}$.

For every branch $\wp$ incident at $P$ in $\mathcal{P}$, we have the following sequence of isomorphisms:

$$\begin{aligned} \widehat{S_V} \otimes_{\widehat{R_U}} \widehat{R_\wp} \otimes_{\widehat{R_\wp}} \widehat{R_\wp}/\wp &\cong \widehat{S_V} \otimes_{\widehat{R_U}} \widehat{R_U} \otimes_{\widehat{R_U}} k(\wp) \\ &\cong \widehat{S_V} \otimes_{\widehat{R_U}} k(U) \otimes_{k(U)} k(\wp) \\ &\cong l_U \otimes_{k(U)} k(\wp) \\ &\cong l(\wp). \end{aligned}$$

Let $\widehat{S_\wp}$ be the integral closure of $\widehat{R_\wp}$ in $L_\wp$. Thus the $\widehat{R_\wp}$-algebras, $\widehat{S_\wp}$ and $\widehat{S_V} \otimes_{\widehat{R_U}} \widehat{R_\wp}$, induce the same algebra over the residue field $k(\wp)$. By Theorem 3.1.3, there is an equivalence of categories between étale algebras over $\widehat{R_\wp}$ and étale algebras over the residue field $k(\wp)$. Thus we have that $\widehat{S_V} \otimes_{\widehat{R_U}} \widehat{R_\wp} \cong \widehat{S_\wp}$ and also $L_V \otimes_{F_U} F_\wp \cong L_\wp$.

The algebras $L_V/F_U$ induce algebras of the same dimension on the branches incident at the points in $\mathcal{P}_1$. Using weak approximation again, there exist compatible algebras at all the closed points in $\mathcal{P}_1$. All this patches together to give an algebra $L/F$ such that $L \otimes_F F_P \cong L_P$. Since $L_P/F_P$ is a field extension of degree $d$, so is $L/F$. $\square$

In general, it is not true that if $L_P/F_P$ are finite degree field extensions of the same degree at a finite collection of closed points $P$, then they induce a global extension $L/F$. When $L_P/F_P$ is ramified on one branch incident on $P$ and not the other, one can show that such a global $L/F$ does not exist (see [HHKPS, Remark 2.7(b)]). The authors in [HHKPS] show that general local extensions $L_P/F_P$ at closed points indeed induce a global extension $L/F$, but only when the points $P$ are unibranched. The fact that $L_P/F_P$ are unramified extensions in Lemma 5.3.2 allows us to drop the hypothesis that the points $P$ are unibranched.

We are finally in a position to prove our main theorem.

**Theorem 5.3.3.** *Let $F$ be a semi-global field with residue field $k$, and $\ell$ be a prime not equal to char$(k)$.*

1. *We have the following upper bound for the Generalized Brauer 2-dimension:*

$$\mathrm{GBrd}_2(F) \le 2^3 \cdot \mathrm{GBrd}_2(k(t)) \cdot \mathrm{GBrd}_2(k).$$

2. *If $\ell \neq 2$, we obtain*

$$\mathrm{GBrd}_\ell(F) \le \ell^2 \cdot \mathrm{GBrd}_\ell(k(t)) \cdot \mathrm{GBrd}_\ell(k).$$

*Proof.* If $\ell \neq 2$, let $K/F$ be a degree $\ell^2$ extension that splits the ramification of $B$, as chosen in Theorem 5.2.5 or if $\ell = 2$, let $K/F$ be a degree 8 extension as chosen in Proposition 5.2.5 which splits the ramification of $B$. Let $\mathscr{X}$ be a regular projective model of $K$ with $\{X_1, \cdots, X_n\}$ being the irreducible components of its special fiber.

We prove the statement in two steps. In Step 1, we show that there exists a field extension $M/K$ of degree $\mathrm{GBrd}_\ell(k(t))$ with some normal model $\mathscr{Y}$ such that $B$ is split on all but finitely many closed points of the special fibre of $\mathscr{Y}$. After that, in Step 2, we construct another extension $L/M$ of degree $\mathrm{GBrd}_\ell(k)$ which finally splits $B$.

**Step 1:** Let $\eta_i$ be the generic points of $X_i$ and let $K_{\eta_i}$ denote the completion of $K$ at $\eta_i$. Since $B$ is unramified, every element $\alpha$ in $B$ comes from a unique element in ${}_\ell\mathrm{Br}(k(X_i))$ by Theorem 2.4.24. We will denote this element by $\alpha_{k(X_i)}$. Note that there exists a separable field extension $m_i/k(X_i)$ of degree at most $\mathrm{GBrd}_\ell(k(t))$ splitting $\alpha_{k(X_i)}$. We may as well assume that $m_i/k(X_i)$ has degree $\mathrm{GBrd}_\ell(k(t))$. Let $M_i/K_{\eta_i}$ denote the unramified lifts of the extensions $m_i/k(X_i)$. By (2.15), $\alpha \otimes M_i$ is split. Thus by Lemma 5.3.1, there exists a field extension $M/K$ of degree $\mathrm{GBrd}_\ell(k(t))$ such that $M \otimes_K K_{\eta_i} \cong M_i$.

Now let $f : \mathscr{Y} \to \mathscr{X}$ be the normalization of $\mathscr{X}$ in $M$. In view of our choice of $M/K$, note that for $i = 1, \cdots, n$, $\eta_i' := f^{-1}(\eta_i)$ are the generic points of $Y_i := f^{-1}(X_i)$, the irreducible components of the special fibre of $\mathscr{Y}$ and $M_i$ are the respective completions of $M$ at $\eta_i'$. Since $\alpha \otimes M_i$ is split, by Proposition 4.1.3, there exist non-empty dense affine open subsets $U_i \subset Y_i$ which do not meet any other component and such that $\alpha \otimes M_{U_i}$ is split. Thus $\alpha$ is split everywhere on the special fibre, except possibly at the complement of the union of the open sets $U_i$.

**Step 2**: Let $\mathcal{U}$ be the set of the open sets $U_i$ from Step 1. Let $\mathcal{P}$ be the complement of $\bigcup_i U_i$ on the special fibre $Y$ of $\mathcal{Y}$. Because $\alpha$ is unramified on the regular local ring $\widehat{\mathcal{O}_{\mathscr{X},f(P)}}$, by Theorem 4.2.2, it comes from a unique class on the residue field $k(f(P))$. Therefore, $\alpha \otimes M_P$ comes from a class on the residue field $k(P)$ of $\widehat{\mathcal{O}_{\mathcal{Y},P}}$, for all points $P$ in $\mathcal{P}$; we will denote this class by $\alpha_{k(P)}$. Let $l(P)/k(P)$ be separable field extensions of degree $\mathrm{GBrd}_\ell(k)$ splitting $\alpha_{k(P)}$. Let $L_P/M_P$ be the unramified lift of $l(P)/k(P)$. By [A-G, Corollary 6.2], $\alpha \otimes L_P$ is split. By Lemma 5.3.2, there exists a field extension $L/M$ of degree $\mathrm{GBrd}_\ell(k)$ inducing $l(P)/k(P)$.

We claim that $\alpha \otimes L$ is split. Let $g : \mathscr{Z} \to \mathscr{Y}$ be the normalization of $\mathscr{Y}$ in $L$. Let $\mathcal{P}'$ be the inverse images of the points $P$ in $\mathcal{P}$ under the normalization map $g$. Let $\mathcal{U}'$ be the set of irreducible components of the complement of $\mathcal{P}'$ in the special fibre of $\mathscr{X}$. Note that for each $U'$ in $\mathcal{U}'$, there exists some $U_i$ in $\mathcal{U}$ such that $M_{U_i} \subset L_{U'}$. Since $\alpha \otimes M_{U_i}$ is split, so is $\alpha \otimes L_{U'}$. Furthermore for each $P'$ in $\mathcal{P}'$, $L_{P'}$ is an unramified extension of $F_P$ for some $P$ induced by a residue field extension $l(P)/k(P)$ as constructed in the previous paragraph. Since $l(P)/k(P)$ splits $\alpha \otimes k(P)$, $\alpha \otimes L_{P'}$ is split. Thus, using Theorem 3.3.6, $\alpha \otimes L$ is split for every $\alpha$ in $B$. □

A standard argument as in the proof of Lemma 2.4.8 shows:

**Corollary 5.3.4.** *Let $F$ be a semi-global field with residue field $k$, and let $\ell$ be a prime not equal to char$(k)$.*

1. *Let $m \geq 1$; we obtain the following upper bound for the Generalized Brauer $2^m$-dimension:*

$$\mathrm{GBrd}_{2^m}(F) \leq 8^m \cdot [\mathrm{GBrd}_2(k(t))]^m \cdot [\mathrm{GBrd}_2(k)]^m.$$

2. *Let $m \geq 1$; for $\ell \neq 2$, we have*

$$\mathrm{GBrd}_{\ell^m}(F) \leq (\ell^2)^m \cdot [\mathrm{GBrd}_\ell(k(t))]^m \cdot [\mathrm{GBrd}_\ell(k)]^m.$$

*Proof.* We show this by induction on $m$. We show this only for $\ell \neq 2$ since the case for $\ell = 2$ is similar. The base case, $m = 1$, follows from Theorem 5.3.3. Suppose that the statement holds for $m-1$. Let $B$ be a finite subset of ${}_{\ell^m}\mathrm{Br}(F)$. We denote the subset of ${}_{\ell}\mathrm{Br}(F)$ obtained by multiplying each element in $B$ by $\ell^{m-1}$ by $\ell^{m-1}B$. If $L/F$ is a field extension, the subset of ${}_{\ell^m}\mathrm{Br}(L)$ obtained by restricting each element in $B$ to the field $L$ will be denoted by $B_L$. By Theorem 5.3.3, it follows that there exists a field extension $L/F$ of degree at most $\ell^2\,\mathrm{GBrd}_\ell(k(t))\,\mathrm{GBrd}_\ell(k)$ splitting $\ell^{m-1}B$. Thus $B_L$ is a subset of ${}_{\ell^{m-1}}\mathrm{Br}(F)$. By the induction hypothesis, there exists a field extension $M/L$ of degree at most $(\ell^2)^{m-1}[\mathrm{GBrd}_\ell(k(t))]^{m-1}[\mathrm{GBrd}_\ell(k)]^{m-1}$ splitting $B_L$. Therefore it follows that the extension $M/F$ splits all elements in $B$. Since the degree of $M/F$ is at most $(\ell^2)^m[\mathrm{GBrd}_\ell(k(t))]^m[\mathrm{GBrd}_\ell(k)]^m$, we prove the claim. □

As a corollary, we can give upper-bounds to the Generalized Brauer dimension of function fields of curves over $m$-local fields.

**Corollary 5.3.5.** *Let $F$ be the function field of a curve over an $m$-local field where $m \geq 1$.*

1. *We obtain the following upper bound for the Generalized Brauer 2-dimension:*

$$\mathrm{GBrd}_2(F) \leq 2^{(m^2+5m-2)/2}.$$

2. *For $\ell \neq 2$, we have:*

$$\mathrm{GBrd}_\ell(F) \leq \ell^{(m^2+3m)/2}.$$

*Proof.* Let $a_n$ be the Generalized Brauer $\ell$-dimension of function fields of curves over $n$-local fields. Let $k$ be the residue field of the field of constants of $F$. Note that $k$ is an $(m-1)$-local field. The Generalized Brauer $\ell$-dimension of an $(m-1)$ local field will be denoted by $b_{m-1}$. By Theorem 5.3.3, we have $a_m \leq \ell^2 a_{m-1} b_{m-1}$. Since $b_{m-1} \leq \ell b_{m-2}$, $b_{m-1} \leq \ell^{m-1}$. Therefore it follows that $a_m \leq \ell^{(m^2+3m)/2}$.

For $\ell = 2$: $a_m \leq 8a_{m-1}b_{m-1} = 8a_{m-1}2^{m-1} = 2^{m+2}a_{m-1}$. Therefore, $a_m \leq a_1 2^{(m^2+5m-6)/2}$. By Theorem 5.2.4, we have that $a_1 \leq 4$. As a result, it follows that $a_m \leq 2^{(m^2+5m-2)/2}$. ☐

**Remark 5.3.6.**

1. *It is difficult to say if this is the best bound. Obtaining lower bounds for Generalized Brauer dimension is not very easy. Even when $F = \mathbb{C}(x,y)$, the only known lower bound in the period 2 situation is due to Tignol-Chapman ([C-Tig]: $\mathrm{GBrd}_2(F) \geq 4$. It is not clear how to improve this, and their methods which use some quadratic form theory, do not seem to obviously generalize to other prime periods.*

2. *In the mixed, bad characteristic situation, i.e., when the characteristic of the residue field $k$ is $\ell$ and $char(F) = 0$, Parimala and Suresh in [Pa-Su3] obtain that $\mathrm{GBrd}_\ell(F) \leq (\ell-1)\ell^{4d+2}$, where $d$ is a non-negative integer such that $[k : k^\ell] = \ell^d$. In another paper, in the mixed, bad characteristic situation, they show that $\mathrm{Brd}_\ell(F) \leq 2d$ (see [Pa-Su2]), i.e., we can split a single Brauer class of period $\ell$ by*

*making a field extension of degree at most $\ell^{2d}$. This suggests that the bound for the Generalized Brauer dimension, even in the good characteristic case, should be a linear function of the Brauer dimension in the exponent.*

## 5.4 Splitting Index Of Quadratic Forms

The Generalized Brauer dimension is related to other measures of complexity of the field. One of these measures come from quadratic forms, namely the $u$-invariant. The finiteness of Generalized Brauer dimension implies the finiteness of the $u$-invariant. This is not at all obvious to see. One needs the Milnor conjectures and a result of Krashen (see [Kra]) to establish this. We will freely use the Milnor conjectures, and record the observation in Theorem 5.4.16.

The other measure of complexity comes from Galois cohomology classes, namely the symbol length. Again, the finiteness of the Generalized Brauer dimension implies the finiteness of mod $\ell$-symbol length. This also needs the norm residue isomorphism theorem, a highly non-trivial theorem and a theorem of Krashen proved in [Kra].

One could define another notion of dimension coming from quadratic forms, inspired by the Brauer dimension.

**Definition 5.4.1.** Let $q/F$ be a quadratic form of dimension $n > 1$. We define the splitting index $i_s(q)$ of $q$ to be the minimum of the degrees of field extensions $[L : F]$ such that $q \otimes_F L$ has Witt index $\lfloor \frac{n}{2} \rfloor$.

The splitting dimension of a field $F$, $i_s(F)$ is the supremum of the $i(q)$ as $q$ ranges over quadratic forms over $L$ and $L/F$ ranges over finite degree field extensions.

Recall that any even dimensional form $q/F$ with trivial discriminant lies in the square of the fundamental ideal $I^2F$ in the Witt ring $W(F)$. Thus one may write $q = \sum_i p_i$ in $W(F)$, where each $p_i$ is a two-fold Pfister form, up to a sign. A naïve way to obtain a bound on the splitting index $i_s(q)$ is to find an extension $L/F$ which splits all the $p_i$ simultaneously. Note that two-fold Pfister forms are norms of Quaternion algebras. Thus splitting all the $p_i$ simultaneously amounts to splitting the corresponding Brauer classes simultaneously. Therefore the Generalized Brauer 2-dimension is also related to the splitting dimension of the field.

If the $u$-invariant of a field is finite, there is an obvious, crude upper bound on the splitting dimension:

**Proposition 5.4.2.** *Let $F$ be a field with $char(F) \neq 2$. If the $u$-invariant of $F$ equals $N$, then $i_s(F) \leq 2^{\lfloor N/2 \rfloor}$.*

*Proof.* Let $q$ be a non-degenerate quadratic form of dimension $m$. One may express

$$q = \perp_{i=1}^{\lfloor m/2 \rfloor} \langle a_i, b_i \rangle \perp \epsilon,$$

where $\epsilon$ is either the 0 form or a one dimensional form. Observe that the field extension $F(\sqrt{-a_1 b_1}, \cdots, \sqrt{-a_{\lfloor m/2 \rfloor} b_{\lfloor m/2 \rfloor}})$ splits $q$. Therefore $i_s(q) \leq 2^{\lfloor m/2 \rfloor}$. Further, since $u(F) = N$, any quadratic form $p$ of dimension greater than $N$ may be written as $p = p_{an} \perp t\mathbb{H} \perp \epsilon$, where $p_{an}$ is the anisotropic part of $p$ and $\epsilon$ is either a 0 form or a one dimensional form. If $L/F$ splits $p_{an}$, it certainly splits $p$. One also sees that if $L/F$ splits $p$, then by the Witt cancellation theorem, $L/F$ splits $p$. Notice therefore, $i_s(p) = i_s(p_{an})$. Note also that $\dim(p_{an}) \leq N$. Thus, the splitting dimension of $F$ is at most the splitting index of anisotropic forms of dimension $N$. Thus, $i_s(F) \leq 2^{\lfloor N/2 \rfloor}$ □

The following question was asked in the conference on Brauer groups and deformation theory in 2011 (see [AimPL]):

**Question 3.** *Does there exist a field $F$ with $char(F) \neq 2$ and $u$-invariant $u(F)$ such that the splitting dimension*

$$i_s(F) \leq 2^{(u(F)/2-1)}?$$

By Proposition 5.4.2, the inequality $i_s(F) \leq 2^{u(F)/2}$ always holds. The inequality in Question 3 is slightly better than this owing to the fact that largest anistropic Pfister forms over a field $F$ are split by any quadratic field extension of $F$. One sees this being used in Proposition 5.4.9.

As an application of Theorem 5.3.5, we will show that $i_s(F)$ is considerably smaller than $2^{(u(F)-1)/2}$ when $F$ is the function field of a curve over an $m$-local fields, thereby providing a somewhat non-trivial answer to the question. But before that, we record the following computation of the splitting dimension of global fields:

**Proposition 5.4.3.** *If $F$ is a totally imaginary number field or a global field of characteristic not equal to 2, then*

$$i_s(F) = 4.$$

*Proof.* If $F$ is a totally imaginary number field or a function field of a curve over a finite field, it is well known that $u(F) = 4$. Writing an anisotropic four dimensional quadratic form as a sum of two binary forms, we see that $i_s(F) \leq 4$.

To construct a quadratic form with splitting index at least 4, we use the Albert-Brauer-Hasse-Noether theorem (see Theorem 2.4.35). Let $\alpha$ be a non-trivial element in ${}_2\operatorname{Br}(F)$, and let $v$ be a place where $\alpha$ is ramified. Construct a quadratic field extension $L/F$ where $v$ is totally split using Weak approximation and Krasner's lemma. By the Albert-Brauer-Hasse-Noether theorem, $\alpha \otimes L$ is non-trivial. Let $p$ be the norm form of $L/F$ and $p_\alpha$ be the norm form of a quaternion algebra in the class of $\alpha$. Consider the quadratic form $q = p \perp p_\alpha$. If $M/F$ splits $q$, $M$ contains $L$. Since $M/L$ splits $q \otimes L$, it splits $\alpha \otimes L$. Therefore, 2 divides $[M : L]$. Therefore $[M : F] \geq 4$. □

We will adopt the same strategy as in the proof of Proposition 5.4.3 to give lower bounds for splitting dimension in Propositions 5.4.11 and 5.4.13.

**Proposition 5.4.4.** *If $K$ be a complete discretely valued field with parameter $t$ and residue field $k$ of characteristic not equal to 2, then $i_s(K) \leq 2i_s(k)$.*

*Proof.* Let $q/K$ be a quadratic form. Then $q = q_1 \perp tq_2$, where the entries of $q_1$ and $q_2$ are units in its ring of integers. Consider the ramified extension $L = K(\sqrt{t})$. The residue field of $L$ is also $k$. Then $q \otimes L \cong q_1 \perp q_2$. Let $m/k$ be a field extension of degree at most $i_s(k)$ splitting the reduction $\overline{q_1 \perp q_2}$. Let $M/K$ be the unramified lift of the extension $m/k$. By Hensel's lemma, $(q_1 \perp q_2) \otimes L$ is split by $M/L$. Therefore, $i_s(K)$ is at most $2i_s(k)$. $\square$

Note that the above bound is not tight. When $F = \mathbb{Q}_p$, one can show that $i_s(F) = 2$. This follows from the fact that every quaternion algebra is split by any quadratic extension. Such a phenomena only happens over 1-local fields.

**Corollary 5.4.5.** *Let $F$ be the function field of a curve over an $n$-local field. Then*

$$i_s(F) \leq 2^{(n^2+5n)/2}.$$

*Proof.* Let $L/F$ be a finite field extension and let $q/L$ be a quadratic form. We may assume that it is even dimensional. Let $K/L$ be a quadratic extension splitting its discriminant. One may write $q \otimes K = \sum_{i=1}^{m} \epsilon_i \langle\langle a_i, b_i \rangle\rangle$ in the Witt group $W(F)$ where $\epsilon_i$ is in $\{1, -1\}$. Applying Corollary 5.3.5 to the subset $B = \{(a_1, b_1), \cdots, (a_n, b_n)\}$ of ${}_2\operatorname{Br}(K)$, there exists a field extension of degree $2^{(n^2+5n-2)/2}$ splitting $q \otimes K$. Therefore, $q$ is split by an extension of degree $2^{(n^2+5n)/2}$. $\square$

**Remark 5.4.6.** *By Corollary 4.3.3, note that $u(F) \leq 2^{n+2}$. Thus by Proposition 5.4.2, we have the following crude bound for the splitting dimension: $i_s(F) \leq 2^{2^{(n+1)}}$. The bound in Corollary 5.4.5 is significantly better, and thus answers Question 3.*

We will now find upper bounds for the splitting dimension for other "higher dimensional fields", namely those satisfying the hypothesis of Proposition 5.4.9. Examples of such fields include $\mathbb{Q}_p(t)$ and $\mathbb{F}_p(x, y)$. We will start by proving some useful lemmas:

**Lemma 5.4.7.** *Let $\{\phi_1, \phi_2, \cdots, \phi_n\}$ be a set of anisotropic Pfister forms over a field $F$. Let $\phi_i'$ denote their pure subforms for $i = 1, \cdots, n$. For every $i$, $\phi_i$ do not share a common quadratic splitting field if and only if $\bigcap_{i=1}^{n} D_F(\phi_i') = \varnothing$.*

*Proof.* Suppose all the Pfister forms are split by $F(\sqrt{c})$. In that case $\langle 1, -c \rangle$ is a subform of all the $\phi_i$. In which case $-c \in \bigcap_{i=1}^{n} D_F(\phi_i')$. Conversely, if $c \in \cap_{i=1}^{n} D_F(\phi_i')$, all $\phi_i$ have $\langle 1, c \rangle$ as a subform. If $c \in F^{\times 2}$, all the Pfister forms are split by $F(\sqrt{-1})$. If $c \notin F^{\times 2}$, then all $\phi_i$ are split by $F(\sqrt{-c})$. $\square$

**Corollary 5.4.8.** *Let $\phi$ and $\gamma$ be $m$ and $n$-fold anisotropic Pfister forms respectively over a field $F$, and let $\phi'$ and $\gamma'$ denote their pure subforms. Then $\phi$ and $\gamma$ share a common quadratic splitting field if and only if $\gamma' \perp \langle -1\rangle\phi'$ is isotropic.*

*Proof.* Note that $\gamma' \perp \langle -1\rangle\phi'$ is anisotropic if and only if $D(\gamma')\bigcap D(\phi') = \varnothing$. The rest follows by Lemma 5.4.7. $\square$

**Proposition 5.4.9.** *Let $F$ be a field with $char(F) \neq 2$. Suppose that $\mathrm{Brd}_2(F) \leq 2$ and for every finite extension $L/F$, $u(L) \leq 8$. Then*

$$i_s(F) \leq 8.$$

*Proof.* Let $q$ be a quadratic form over $F$. We may assume it is even dimensional by replacing the form by a codimension one subform if necessary. Let $L/F$ be a quadratic extension which splits its discriminant. Thus in the Witt ring $W(L)$, $q \otimes L$ lies in $I^2(L)$. Let $\alpha := e_2(q \otimes L)$ be its $e_2$ invariant. By our assumption on the index of $\alpha$, it follows by Albert's theorem (see [Lam, Chapter 4, Theorem 4.8]) that $\alpha = \alpha_1 + \alpha_2$, where $\alpha_i$ are symbols. We will abuse notation and not distinguish the norm forms of symbols (which are Pfister forms) and the symbols themselves. Suppose that $M/L$ is quadratic extension splitting $\alpha_1$. Thus, $q \otimes M - \alpha_2$ lies in $I^3(M)$. Thus $q \otimes M = \alpha_2 + \beta$, where $\beta$ lies in $I^3(M)$. Because $u(M) \leq 8$, $\beta$ is similar to a three-fold Pfister form by the Arason-Pfister Hauptsatz (see, for example [Lam, Theorem 5.6]). By Corollary 5.4.8, there is a quadratic extension $K/M$ splitting $\alpha_2$ and $\beta$ simultaneously. Thus $M/F$ splits $q$ and its degree is 8. $\square$

**Remark 5.4.10.** *1. In view of results of Saltman [Salt] on period-index bounds for p-adic curves, and of Parimala and Suresh [Pa-Su1] on the u-invariant, the splitting dimension of function fields of curves over p-adic fields is at most 8.*

*2. The splitting dimension of function fields of surfaces over finite fields such as $\mathbb{F}_p(x,y)$, and that of fraction fields of complete two dimensional regular local rings with finite residue field, for example $\mathbb{F}_p((x,y))$, is also 8. Note that such fields satisfy the $C_3$ property (see Theorem 2.4.10 and Theorem 2.4.11). Therefore the*

*$u$-invariant of these fields is at most 8. The period-index bound for surfaces over finite fields was proved by Lieblich [Lie].*

For function fields of varieties over an algebraically closed fields, and function fields of curves over complete discretely valued fields, we record the following lower-bounds in Proposition 5.4.11 and Proposition 5.4.13 respectively.

**Proposition 5.4.11.** *If $F$ is the function field of a smooth $d$-dimensional variety over an algebraically closed field $K$ of characteristic zero, then $i_s(F) \geq 2^d$.*

*Proof.* We first show that there exists a Brauer class $\alpha$ over $F$ of index $2^{d-1}$ and a quadratic extension $L/F$ such that $ind(\alpha \otimes L) = 2^{d-1}$. Let $p$ be the norm form of $L/F$ and $p_\alpha$ be a form with Clifford invariant $\alpha$. Then $q = p \perp p_\alpha$ has spitting index at least $2^d$.

We construct the class and the field extension by induction on the dimension. The base case $d = 1$ follows from the fact that all Brauer classes on function fields of curves over algebraically closed fields are trivial.

Let $X/K$ be a smooth affine variety of dimension $k$, and let $Y \hookrightarrow X$ be a smooth codimension 1 subvariety given by $\pi$, obtained by shrinking $X$ and $Y$ if necessary. By the induction hypothesis, there exists $\alpha_0$ in ${}_2\operatorname{Br}(K(Y))$ of index $2^{k-2}$ and an extension $L := K(Y)(\sqrt{u})$ such that $ind(\alpha_0 \otimes L) = 2^{k-2}$. Suppose that $Y$ is given by the function $\pi$ in $K[X]$, and let $F_\pi$ denote the completion of $K(X)$ at $\pi$. The residue field of $F_\pi$ is $K(Y)$. Consider the class $\alpha = \alpha_0 + (u, \pi)$ in $\operatorname{Br}(F_\pi)$. This class descends to $\operatorname{Br}(F)$. By Proposition 2.4.28, $ind(\alpha \otimes F_\pi) = 2ind(\alpha_0 \otimes F_\pi(\sqrt{u})) = 2ind(\alpha_0 \otimes L) = 2^{k-1}$. Note also that the extension, $F(\sqrt{\pi+1})/F$ splits in $F_\pi$. Thus, $ind(\alpha \otimes F(\sqrt{\pi+1})) = 2^{k-1}$. □

**Remark 5.4.12.** *As an explicit example, consider the quadratic form over $\mathbb{C}(x, y, z)$:*

$$q = \langle x, y, xy, -(y+1), -z, z(y+1)\rangle \perp \langle 1, -(z+1)\rangle.$$

*One can check that $q$ cannot be split by an extension of degree less than 8.*

**Proposition 5.4.13.** *Let $F$ be the function field of a curve over a complete discretely valued field. If the Brauer 2-dimension of $F$ is $n$, then*

$$i_s(F) \geq 2^{n+1}.$$

*Proof.* Let $\alpha$ be in ${}_2\operatorname{Br}(F)$ such that $ind(\alpha) = 2^n$. By Merkurjev's theorem (see 2.5.9), there is a quadratic form $p_\alpha$ of trivial discriminant such that $e_2(p_\alpha) = \alpha$. By Theorem 4.2.6, there exists a non-trivial discrete valuation $v$ such that $ind(\alpha \otimes F_v) = 2^n$. Let $L/F$ be a quadratic extension which is split over $v$. Let $p$ be the norm form of $L/F$. Then $q = p \perp p_\alpha$ is our required quadratic form. To split the discriminant of $q$, we need to make the extension $L/F$. Since $L$ is split at $v$, $ind(\alpha \otimes L_v) = 2^n$. □

We will now establish a relationship between three arithmetic invariants of fields: the Generalized Brauer 2-dimension, mod-2 symbol length and the $u$-invariant. First, note that if the Generalized Brauer dimension of a field is finite, then using the norm residue isomorphism theorem, it follows that there is a uniform bound on the index of mod-2 Galois cohomology classes. A theorem proved in [Kra] implies that the mod-2 symbol length is finite. As a result, one can show that the $u$-invariant is also finite. Therefore the finiteness of the Generalized Brauer 2-dimension implies the finiteness of symbol length and the finiteness of the $u$-invariant. We will start by recalling the Milnor conjecture, which was proved in 2007 by Orlov-Vishik-Voevodsky in [OVV]. This is an example of one of the many theorems in mathematics, which are easy to state and difficult to prove.

**Theorem 5.4.14** (Orlov-Vishik-Voevodsky)**.** *Let $F$ be a field with $char(F) \neq 2$. Then, we have the following isomorphisms of graded rings which sends the class $\otimes_{i=1}^{n}\langle 1, -a_i\rangle$ mod $I^{n+1}F$ to the cup-product $(a_1) \cup \cdots \cup (a_n)$.*

$$\bigoplus_{n=0}^{\infty} \frac{I^n F}{I^{n+1}F} \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} \mathrm{H}^n(F, \mathbb{Z}/2\mathbb{Z}).$$

Therefore, every element $\zeta$ in $\mathrm{H}^n(F, \mathbb{Z}/2\mathbb{Z})$ can be expressed as

$$\zeta = \sum_{i=1}^{m} (a_{i1}) \cup \cdots \cup (a_{in}). \tag{5.1}$$

**Definition 5.4.15.** An element $\zeta$ in $\mathrm{H}^n(F,\mathbb{Z}/2\mathbb{Z})$ of the form $\zeta = (a_1)\cup\cdots\cup(a_n)$ is called a *symbol*.

Let $\zeta$ be an element in $\mathrm{H}^n(F,\mathbb{Z}/2\mathbb{Z})$. We define the *symbol length* of $\zeta$, denoted by $\lambda(\zeta)$, to be the minimum positive integer $m$ such that $\zeta$ can be expressed as sum of $m$ symbols as in (5.1).

We define the mod-2, $n$-symbol length of $F$, denoted by $\lambda_2^n(F)$, to be the supremum of $\lambda(\zeta)$ as $\zeta$ varies over $\mathrm{H}^n(L,\mathbb{Z}/2\mathbb{Z})$, and $L/F$ varies over finite field extensions.

We will end by proving a relationship between Generalized Brauer dimension, $u$-invariant and symbol length.

**Theorem 5.4.16.** *Let $F$ be a field of characteristic not equal to 2. Suppose that the 2-cohomological dimension $\mathrm{cd}_2(F)$ is finite. Consider the following statements:*

1. $\mathrm{GBrd}_2(F)$ *is finite;*
2. *For every $n \geq 1$, $\lambda_2^n(F)$ is finite;*
3. *For every finite degree field extension $L/F$, $u(L)$ is finite.*

*Then* $(1) \implies (2) \implies (3)$.

*Proof.* For $(1) \implies (2)$: Let $L/F$ be a finite field extension. Recall that by Example 2.3.4, we have the isomorphism $\mathrm{H}^1(L,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} L^\times/(L^\times)^2$. Thus every element $\zeta$ in $\mathrm{H}^1(L,\mathbb{Z}/2\mathbb{Z})$ is a square class in $L$. Therefore, $\lambda_2^1(F) = 1$. Now consider the element $\zeta$ in $\mathrm{H}^n(L,\mathbb{Z}/2\mathbb{Z})$ for $n \geq 2$. By Theorem 5.4.14, we may write $\zeta$ as $\zeta = \sum_{i=1}^m \beta_i \cup \gamma_i$, where $\beta_i$ are elements in ${}_2\mathrm{Br}(L)$ and $\gamma_i$ lie in $\mathrm{H}^{n-2}(L,\mathbb{Z}/2\mathbb{Z})$. Consider $B = \{\beta_1,\cdots,\beta_m\} \subset {}_2\mathrm{Br}(L)$. Since, $\mathrm{GBrd}_2(F)$ is finite, $ind(B)$ is finite. Therefore, there exists a finite degree field extension $M/L$ such that $\zeta \otimes L = 0$ for every $\zeta$ in $\mathrm{H}^n(L,\mathbb{Z}/2\mathbb{Z})$. Finally, by [Kra, Theorem 4.2], it follows that $\lambda_2^n(F)$ is finite.

For $(2) \implies (3)$: Let $L/F$ be a finite field extension. Since $\mathrm{cd}_2(F) < \infty$, there exists some $M$ such that $\mathrm{H}^M(L,\mathbb{Z}/2\mathbb{Z}) = 0$. By Theorem 5.4.14, $I^M(L) = I^{M+1}(L)$. By the Arason-Pfister Hauptsatz (see [Lam, Corollary 5.2]), it follows that $I^M(L) = 0$. Let $N$

be the smallest positive integer such that $I^N(L) = 0$. Let $q/L$ be an even dimensional, anisotropic form. Recall that $q$ lies in the fundamental ideal $I(L)$. Let $d$ be the discriminant of $q$. Note that the form $q - \langle 1, -d\rangle$ has trivial discriminant, and hence lies in $I^2(L)$. Since $I^2(L)$ is additively generated by 2-fold Pfister forms, modulo $I^3(L)$, $q - \langle 1, -d\rangle$ is a sum of 2-fold Pfister forms up to signs, i.e., $q - \langle 1, -d\rangle = \sum_{i=1}^{k_2} \epsilon_i \pi_2^{(i)}$ up to $I^3(L)$, where $k_2 = \lambda_2^2(F)$, and $\pi_2^{(i)}$ are 2-fold Pfister forms, and $\epsilon_i$ is in $\{-1, 1\}$. Continuing in this fashion, we have

$$q = \langle 1, -d\rangle + \sum_{i=1}^{k_2} \epsilon_2^{(i)} \pi_2^{(i)} + \cdots + \sum_{i=1}^{k_{N-1}} \epsilon_{N-1}^{(i)} \pi_{N-1}^{(i)},$$

where $\pi_i^{(j)}$ are $i$-fold Pfister forms, $k_2 = \lambda_2^i(F)$, and $\epsilon_i^{(j)}$ lies in $\{-1, 1\}$. We will denote the quadratic form on the right hand side by $p$. Since $q$ is anisotropic, we have $q \perp n\mathbb{H} = p$, for some $n$. Therefore, $\dim(q) \leq \dim(p)$. Since $\dim(p)$ is expressible in terms of the constants $k_1, \cdots, k_{N-1}$ and $N$, we see that the dimension of every even dimensional anisotropic form is bounded. Therefore, $u(L)$ is finite for every finite field extension $L/F$. $\square$

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