TOWARDS STABLE-STABLE TRANSFER INVOLVING SYMPLECTIC GROUPS

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ABSTRACT OF THE DISSERTATION

Towards Stable-Stable Transfer Involving Symplectic Groups

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This thesis investigates the transfer formulas for orbital integrals, in the context of the modern Langlands' program for reductive algebraic groups. In the modern theory, there are two very different transfer theorems to be accomplished. First, there is endoscopic transfer, which relates, via an appropriate embedding of their L-groups, a given group to a particular family of groups, its endoscopic groups. Here, deep theorems are known in great generality. Such a theory, however, is preliminary to the second transfer, which is much less understood. At the same time, this second transfer is generally viewed as the more fundamental of the two, involving any connected reductive group related to the given group by an L-homomorphism.

Prompted by the results for endoscopic transfer, our study focuses first on groups defined over an archimedean field. To do so, we study the geometric objects, orbital integrals, on real or complex reductive Lie groups, for which there is a basic theory due to Harish-Chandra on which to build, focusing on the split and hyperbolic symplectic groups to develop details. Concrete expressions of the final transfer formulas are notably different from those for endoscopic transfer, and the algebraicity condition on the ambient group is critical in their development.

Specifically, our main focus is on a refined version of the structure of the lattice of maximal tori and on the role this plays in developing the concrete expressions for transfer. Our structural results apply to symplectic groups of all sizes and their inner forms, and we develop an explicit transfer formula in the rank one case.

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1 Introduction

To begin, we establish the context for our work by providing a brief discussion of relevant work by Shelstad within the scope of the Langland's program. Our work is situated in the study of the geometric transfer of orbital integrals. To reach the notion of stable-stable transfer, we begin with a discussion of endoscopic transfer, following [She83] primarily and other sources where noted.

Let G denote a connected reductive linear algebraic group defined over $\mathbb{R}$, and $G(\mathbb{R})$ the group of $\mathbb{R}$-rational points on G. Let σ denote the nontrivial element of the Galois group $Gal(\mathbb{C}/\mathbb{R})$. Then, we view $G(\mathbb{R})$ as the elements of $G(\mathbb{C})$ that are fixed under the Galois action, i.e., $G(\mathbb{R}) = \{g \in G(\mathbb{C}) \ ; \ \sigma(g) = g\}$. Let T be a maximal torus in G defined over $\mathbb{R}$, so that $T(\mathbb{R})$ is a Cartan subgroup of $G(\mathbb{R})$, and every Cartan subgroup of $G(\mathbb{R})$ is realized in this sense. Denote by S_T, or S if the context is clear, the maximal $\mathbb{R}$-split torus in T and M_T, or M, the centralizer of S in G. Regard the set of roots $\Delta(G,T)$, or Δ, of T in G as a subset of the group of rational characters on T, $X^*(T)$. A root, α, is imaginary if $\sigma\alpha = -\alpha$, real if $\sigma\alpha = \alpha$, or complex if $\sigma\alpha \neq \pm\alpha$. The imaginary Weyl group, $\Omega(M,T)$, of T is the subgroup of the Weyl group, $\Omega(G,T)$, generated by the reflections with respect to the imaginary roots.

Take T and T' to be maximal tori over $\mathbb{R}$ in G, and denote by $\{T\}$ and $\{T'\}$ their stable conjugacy classes. Define $\{T\} \preceq \{T'\}$ if the unique maximal $\mathbb{R}$-split torus S_T in T is $G(\mathbb{R})$-conjugate to an $\mathbb{R}$-split torus in T', or equivalently if $\exists g \in G(\mathbb{R})$ such that $ad(g^{-1})$ maps S_T into $S_{T'}$. Then, define T and $\{T\}$ to be adjacent to T' and $\{T'\}$, respectively, if $\{T\} \preceq \{T'\}$ and $\dim S_{T'} = \dim S_T + 1$. Adjacency implies that there is an imaginary root α of T and $s \in G(\mathbb{C})$ such that $T' = s^{-1}Ts$ and $\sigma(s)s^{-1}$ realizes the Weyl reflection with respect to α. Such an s is a Cayley transform with respect to this root.

A root α determines a three-dimensional simple complex Lie algebra, $\mathbb{C}X_\alpha + \mathbb{C}H_\alpha +$

$\mathbb{C}X_{-\alpha}$, where $X_{\pm\alpha}$ are root vectors and $H_\alpha = [X_\alpha, X_{-\alpha}]$ [She79]. If α is imaginary, then this algebra is invariant under the Galois action and its σ-fixed points form a three-dimensional simple real Lie algebra that corresponds to $SU(2)$ if α is compact or $SL(2)$ if α is noncompact.

Fix a Cartan subgroup $T(\mathbb{R})$ of $G(\mathbb{R})$ and Haar measures dt and dg on each, respectively. Let G_{reg} denote the set of regular elements in G and $T(\mathbb{R})_{reg} = T(\mathbb{R}) \cap G_{reg}$. Fix $\gamma \in T(\mathbb{R})_{reg}$ and $f \in C_c^\infty(G(\mathbb{R}))$. The orbit of γ is the conjugacy class of γ in $G(\mathbb{R})$. Then the orbital integral is defined

$$\Phi_T(\gamma, f) = \int_{T(\mathbb{R})\backslash G(\mathbb{R})} f(g^{-1}\gamma g)\frac{dg}{dt}$$

Let $\mathcal{O}$ be an open subset of $G(\mathbb{R})$. A C^∞-function on $\mathcal{O}$ is a Schwartz function on $\mathcal{O}$ if all of its left and right derivatives are rapidly decreasing, in the sense of Harish-Chandra. Harish-Chandra defined the $'F_f$ transform for the Schwartz space $\mathcal{C}(G(\mathbb{R}))$. The orbital integral above, when multiplied by a suitable function of the regular element, becomes this $'F_f(\gamma)$ transform. Specifically, via [She08b], $'F_f(\gamma) = \Delta'(\gamma)\Phi_T(\gamma, f)$, where $\Delta'(\gamma)$ is a modified Weyl denominator that serves as a normalizing factor and may be taken with respect to all positive roots as

$$\Delta'(\gamma) = \prod_{\substack{\alpha \\ \text{pos.,real}}} \left|\alpha(\gamma)^{\frac{1}{2}} - \alpha(\gamma)^{\frac{-1}{2}}\right| \prod_{\substack{\alpha \\ \text{pos.,comp.}}} \left|\alpha(\gamma)^{\frac{1}{2}} - \alpha(\gamma)^{\frac{-1}{2}}\right| \prod_{\substack{\alpha \\ \text{pos.,imag.}}} (\alpha(\gamma) - 1)$$

For G semisimple and simply-connected, the stable orbit of $\gamma \in T(\mathbb{R})_{reg}$ is the intersection of $G(\mathbb{R})$ with the orbit of γ in $G(\mathbb{C})$. Define

$$\mathcal{A}(T) = \{g \in G(\mathbb{C}); gTg^{-1} \subset G(\mathbb{R})\} = G(\mathbb{R}) \cdot Norm(M, T)$$

If $g \in \mathcal{A}(T)$, then gTg^{-1} is $G(\mathbb{C})$-conjugate to T, and, moreover, this can be realized in $G(\mathbb{R})$. The elements of $\mathcal{A}(T)$ serve to establish the stable orbit. Also,

$\mathcal{D}(T) = G(\mathbb{C})\backslash\mathcal{A}(T)/T$. If γ is strongly regular, where a strongly regular element is a regular semisimple element whose centralizer is a torus, then $\mathcal{D}(T)$ parametrizes the orbits in the stable orbit of γ. Furthermore, there is a bijection between $\mathcal{D}(T)$ and $\Omega(M,T)/\Omega(M(\mathbb{R}),T(\mathbb{R}))$. Then, for $f \in \mathcal{C}(G(\mathbb{R}))$, $\gamma \in T(\mathbb{R})_{reg}$, and Haar measures dt on $T(\mathbb{R})$ and dg on $G(\mathbb{R})$, the stable orbitabl integral is given by

$$\Phi_T^{st}(\gamma, f) = \Omega(M(\mathbb{R}), T(\mathbb{R}))^{-1} \sum_{\omega \in \Omega(M,T)} \Phi_T(\gamma^\omega, f)$$

where $\gamma^\omega = w^{-1}\gamma w$, for a $w \in G(\mathbb{C})$ that realizes ω.

Then, following [She79], consider an isomorphism $\psi_{ij} : G_i \to G_j$ of connected reductive linear algebraic groups defined over $\mathbb{R}$, for which $\psi_{ij} : T_i(\mathbb{R}) \to T_j(\mathbb{R})$ is defined over $\mathbb{R}$. Assume that G_j is quasi split, G_i is an inner form of G_j, and fix ψ_{ij} so that $\sigma(\psi_{ij})\psi_{ij}^{-1}$ is inner. A suitable collection of such groups and isomorphisms is an extended group over $\mathbb{R}$. Furthermore, ψ_{ij} realizes a mapping of real (respectively, imaginary, complex) roots of $T_i(\mathbb{R})$ to real (respectively, imaginary, complex) roots of $T_j(\mathbb{R})$. Let $t_{st}(G_i)$ be the set of stable-conjugacy classes of Cartan subgroups of G_i. Then a consequence of the above is that there is a mapping $\psi_{ij}^t : t_{st}(G_i) \to t_{st}(G_j)$. The map ψ^t is order preserving in the sense of adjacency and specifically serves to map $t_{st}(G_i)$ to an initial segment of $t_{st}(G_j)$. Specifically, ψ^t is injective and maps the class of fundamental maximal tori over $\mathbb{R}$ in G_i to the corresponding class in G_j.

Rather than working with an extended group directly it is preferable to work with the L-group of the extended group. Specifically, the Weil group version of the L-group of the extended group $\{(G_i, \psi_{ij})\}$ is given by ${}^LG = G^\vee \rtimes W_\mathbb{R}$. Then the primary datum for an endoscopic transfer is the pair $(s, {}^LH \rightarrowtail {}^LG)$, where s is a semisimple element of $G^\vee$ and ${}^LH = Cent(s, G^\vee)^0 \rtimes W_\mathbb{R}$.

The set of very regular pairs forms a subset of the product of the set of strongly regular stable conjugacy classes in $H(\mathbb{R})$ with the set of strongly regular conjugacy classes in $G(\mathbb{R})$. For each regular pair $(\Gamma_H^{st}, \Gamma_G)$, we define a complex number $\Delta(\Gamma_H^{st}, \Gamma_G)$ such that for a well-chosen function f_G on $G(\mathbb{R})$ there exists a suitable function f_H on $H(\mathbb{R})$ satisfying

$$O^{st}(\Gamma_H^{st}, f_H) = \sum \Delta(\Gamma_H^{st}, \Gamma_G) O(\Gamma_G, f_G)$$

for all Γ_H^{st} contributing to a very regular pair. This leads to a well-defined geometric transfer in the endoscopic setting, and leads to the desired dual transfer in the tempered case.

With these preliminary ideas established, we defer a discussion of how endoscopic transfer points to stable-stable transfer until section 8. But, the endoscopic ideas above indicate that we should explore notions of conjugacy classes of maximal tori, their adjacency relations, and the corresponding extended groups in order to reach the novel results of this paper.

As such, the structure of this paper is as follows: Section two establishes the basic objects of our study. Section three provides a simple perspective on the establishment of the conjugacy classes of maximal tori, while deferring much of the reasoning until we establish our notion of the Cayley transforms. Section four builds on previous work to provide a general discussion on the construction of the lattices of conjugacy classes of maximal tori. Section five uses the previous sections of the paper to construct these lattices for the case C_r. In section six, we combine all of our concepts into a complete workflow, thus providing a roadmap for the reader to develop the details of a particular case they may need. Section seven establishes the extended groups for the case C_r. Finally, in section eight, we discuss some results towards stable-stable transfer involving symplectic groups and develop an explicit transfer formula in the rank one case. This principal case develops explicit aspects that will be necessary for extending our results to symplectic groups of any rank and, then, to stable-stable transfer involving symplectic groups and other algebraic groups.

2 Algebraic Groups, Algebraic Tori, and Real Forms

We concern ourselves with linear algebraic groups G defined over $\mathbb{R}$. Hence, there is a Galois, $Gal(\mathbb{C}/\mathbb{R})$, action on G. G possesses a group of complex points, $G(\mathbb{C})$, and $G(\mathbb{C})$ contains a subgroup, $G(\mathbb{R})$, of real points of G, the latter of which we view as the fixed points under the Galois action. If G is connected and reductive, then $G(\mathbb{C})$ is a connected reductive complex Lie group, which is an algebraic subgroup of $SL(n,\mathbb{C})$ for some n, while $G(\mathbb{R})$ is a reductive real Lie group, which is not necessarily a subgroup of $SL(n,\mathbb{R})$.

If G is also abelian, then we will write T instead and call it an algebraic torus. Then $T(\mathbb{C})$ is isomorphic to the product of several copies of $\mathbb{C}^\times$,

$$T(\mathbb{C}) \cong \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times = (\mathbb{C}^\times)^r.$$

We refer to r, necessarily positive integral, as the rank of the torus. However, $T(\mathbb{R})$ is isomorphic to the product of several copies of $\mathbb{R}^\times$,

$$T(\mathbb{R}) \cong \mathbb{R}^\times \times \cdots \times \mathbb{R}^\times = (\mathbb{R}^\times)^r,$$

only in certain, but crucial, cases. In these cases, we will say that T is **split** over $\mathbb{R}$, $\mathbb{R}$-split, or simply split if the context is clear, and that the rank r is equal to the $\mathbb{R}$-split rank of T. The real points on a split torus are not connected unless the torus is trivial. If T is a split torus, this, in turn, indicates that any G in which T is maximal is also split over $\mathbb{R}$. The rank of any split group must necessarily match the split rank of the maximal split torus. In the event that a torus defined over $\mathbb{R}$ has split rank zero, we call such a torus **anisotropic** over $\mathbb{R}$. Then, $T(\mathbb{R})$ is necessarily compact and connected. [BT65] is a helpful resource for these ideas. As we will leverage in section 3, there are tori that are neither anisotropic nor split and possess, in a sense we will describe, $\mathbb{R}$-split ranks in between these two extremes.

In general, a torus T defined over $\mathbb{R}$ contains a maximal torus split over $\mathbb{R}$, T_s, and a maximal torus anisotropic over $\mathbb{R}$, T_{an}. Furthermore,

$$T = T_{an} \cdot T_s, \tag{2.1}$$

and this product need not be direct on either real or complex points.

Proposition [BT65]: Let k be a field, T a k-torus, and S a k-subtorus. There exists a k-subtorus S' of T such that T is the almost direct product of S and S'. If S is the unique split-torus of T, then S' is the unique anisotropic torus of T.

Proof: for example, in [Bor62]

We are concerned with the case where G is a symplectic group. These come in two types.

For the first, $G = Sp(n, -)$, we denote $G(\mathbb{C}) = Sp(n, \mathbb{C})$ and $G(\mathbb{R}) = Sp(n, \mathbb{R})$, which agree with the usual Lie group notation. The latter is the split real form, which has split rank $\frac{n}{2}$.

For the second, $G = Sp(p, q)$, we denote $G(\mathbb{C}) = Sp(n, \mathbb{C})$ and $G(\mathbb{R}) = Sp(p, q)(\mathbb{R})$. The latter corresponds to the usual Lie group notation $Sp(p, q)$. It is necessarily the case that $p + q = n$ and the group, again, has rank $\frac{n}{2}$. For a given rank, there are

$$\left\lfloor \frac{\text{rank}}{2} \right\rfloor + 1$$

nonisomorphic real forms of this type [Sug59]. If we take $K_{p,q} = diag(I_{\frac{p}{2}}, -I_{\frac{q}{2}}, I_{\frac{p}{2}}, -I_{\frac{q}{2}})$, then the inner automorphism group of $K_{p,q}$ is equivalent to that of $K_{q,p}$. In addition, because all automorphisms of the symplectic group are inner, we view $Sp(p, q)$ as equivalent to $Sp(q, p)$. As such, we will work with $Sp(p, q)$ and insist that $p > q$. We realize the compact symplectic group as $Sp(n, 0)$.

In what follows, we will develop low-rank cases explicitly. For these cases, the complex and real points are:

$$
\begin{aligned}
\text{rank 1,}\quad & G(\mathbb{C}) = Sp(2,\mathbb{C})\\
& G(\mathbb{R}) = Sp(2,\mathbb{R}) \text{ or } Sp(2,0)(\mathbb{R})\\
\text{rank 2,}\quad & G(\mathbb{C}) = Sp(4,\mathbb{C})\\
& G(\mathbb{R}) = Sp(4,\mathbb{R}) \text{ or } Sp(2,2)(\mathbb{R}) \text{ or } Sp(4,0)(\mathbb{R})\\
\text{rank 3,}\quad & G(\mathbb{C}) = Sp(6,\mathbb{C})\\
& G(\mathbb{R}) = Sp(6,\mathbb{R}) \text{ or } Sp(4,2)(\mathbb{R}) \text{ or } Sp(6,0)(\mathbb{R})\\
\text{rank 4,}\quad & G(\mathbb{C}) = Sp(8,\mathbb{C})\\
& G(\mathbb{R}) = Sp(8,\mathbb{R}) \text{ or } Sp(4,4)(\mathbb{R}) \text{ or } Sp(6,2)(\mathbb{R}) \text{ or } Sp(8,0)(\mathbb{R})\\
\text{rank 5,}\quad & G(\mathbb{C}) = Sp(10,\mathbb{C})\\
& G(\mathbb{R}) = Sp(10,\mathbb{R}) \text{ or } Sp(6,4)(\mathbb{R}) \text{ or } Sp(8,2)(\mathbb{R}) \text{ or } Sp(10,0)(\mathbb{R})
\end{aligned}
$$

3 Conjugacy Classes of Maximal Tori

As we will leverage heavily in this section, the real points of any algebraic torus may be expressed as

$$T(\mathbb{R}) \cong (\mathbb{R}^{\times})^{c_1} \cdot (\mathbb{C}^{\times})^{c_2} \cdot (S^1)^{c_3} \text{ , } c_i \in \mathbb{Z}_{\geq 0} \tag{3.1}$$

The sum $c_1 + c_2$ indicates the $\mathbb{R}$-split rank. Any two elements $t_1, t_2 \in T(\mathbb{R})$ that adhere to the same instance of 3.1 are stably conjugate. As such, we will use various instances of this expression to track the stable conjugacy classes in which we are interested.

Take σ to be the nontrivial element of $Gal(\mathbb{C}/\mathbb{R})$. Then a root α of T in G is classified as **real** if $\sigma(\alpha) = \alpha$, **imaginary** if $\sigma(\alpha) = -\alpha$, or **complex** if $\sigma(\alpha) \neq \pm\alpha$. We will denote the set of imaginary positive roots as Δ^{im} and the set of real positive roots Δ^{re}.

This section, guided by (3.1), the split rank, and dimensional considerations, establishes the basic structure of and basic facts about the conjugacy classes of maximal tori. We will develop the details of these ideas as we increase the rank of the group. The two sections subsequent to this will establish how these conjugacy classes relate to one another. Specifically, this section presents an unsophisticated perspective of the relevant conjugacy classes, while acknowledging that these conjugacy classes are, more precisely, classified by the appropriate Cayley transforms. This latter perspective will reach full fruition in subsequent sections.

3.1 Rank 1 Groups

We begin with the split form, $Sp(2, \mathbb{R})$, which has two conjugacy classes of maximal tori, corresponding to the anisotropic and split tori.

We take the real points of the anisotropic torus to be

$$T_{an}(\mathbb{R}) = \left\{ \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} ; \theta \in \mathbb{R} \right\}$$

As such, $T_{an}(\mathbb{R}) \cong S^1$ and has $\mathbb{R}$-split rank zero. Should we, in the subsequent, need to think of the matrix associated to an element of S^1, we do so in terms of this rotational matrix, observing that it occupies two rows and columns. As will be the case for all anisotropic tori, the set of positive imaginary roots is equivalent to the root system of $G(\mathbb{C})$ and the set of positive real roots is empty. Here, specifically,

$$\Delta^{im} = A_1 \ , \ \Delta^{re} = \emptyset$$

Proposition: Any anisotropic torus contains only imaginary roots.

Proof: Viewing the roots as unitary characters, this is immediate (cf. [War12] for arguments from the Lie algebra perspective).

We take the real points of the split torus to be

$$T_s(\mathbb{R}) = \left\{ \begin{bmatrix} x & \\ & x^{-1} \end{bmatrix} ; x \in \mathbb{R}^\times \right\}$$

As such, $T_s(\mathbb{R}) \cong \mathbb{R}^\times$ and has split rank equivalent to that of the group, i.e., rank one. Should we, in subsequent sections, need to think of the matrix associated to an element of $\mathbb{R}^\times$, we do so in terms of this diagonal matrix, observing that it occupies two rows and columns. As will be the case for all split tori, the set of positive imaginary roots is empty and the set of positive real roots is equivalent to the root system of $G(\mathbb{C})$. Here, specifically,

$$\Delta^{im} = \emptyset \ , \ \Delta^{re} = A_1$$

Proposition: Any split torus contains only real roots.

Proof: Viewing the roots as real-valued characters, this is immediate.

The real form $Sp(2,0)(\mathbb{R})$ is compact and, hence, there is only one conjugacy class of maximal tori, as will be the case for all subsequent compact forms. Here, this is represented by S^1, which has split rank zero, and

$$\Delta^{im} = A_1 \ , \ \Delta^{re} = \emptyset.$$

We end this section, and all subsequent sections, by summarizing our findings, which may be found, here, in table 1.

	Split Rank	
	0	1
$Sp(2,\mathbb{R})$	S^1	$\mathbb{R}^\times$
$Sp(2,0)(\mathbb{R})$	S^1	-

Table 1: Rank 1 Conjugacy Classes

3.2 Rank 2 Groups

We begin with the split form, $Sp(4,\mathbb{R})$, starting with the anisotropic and split tori, which build off of the work in the rank one case above.

Take the real points of the anisotropic torus to be

$$T_{an}(\mathbb{R}) = \left\{ \begin{bmatrix} \cos(\theta_1) & & \sin(\theta_1) & \\ & \cos(\theta_2) & & \sin(\theta_2) \\ -\sin(\theta_1) & & \cos(\theta_1) & \\ & -\sin(\theta_2) & & \cos(\theta_2) \end{bmatrix} ; \theta_i \in \mathbb{R} \right\}$$

This is such that $T_{an}(\mathbb{R}) \cong (S^1)^2$ and has split rank zero. Additionally,

$$\Delta^{im} = C_2 \ , \ \Delta^{re} = \emptyset$$

Take the real points of the split torus to be

$$T_s(\mathbb{R}) = \left\{ \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & x_1^{-1} & \\ & & & x_2^{-1} \end{bmatrix} ; x_i \in \mathbb{R}^\times \right\}$$

This is such that $T_s(\mathbb{R}) \cong (\mathbb{R}^\times)^2$ and has split rank two. Additionally,

$$\Delta^{im} = \emptyset \ , \ \Delta^{re} = C_2$$

This establishes, what we think of, as the minimum and maximum split rank cases, in a sense that will reach full fruition in subsequent sections. There are, additionally, two conjugacy classes of maximal tori corresponding to split rank one.

First, consider $GL_1 \cdot SL_2$, for which there is a subgroup of real points of the form

$$\begin{bmatrix} * & & & \\ & * & & * \\ & & * & \\ & * & & * \end{bmatrix}$$

We realize this subgroup of real points as

$$T_1(\mathbb{R}) = \left\{ \begin{bmatrix} x & & & & \\ & \cos(\theta) & & \sin(\theta) \\ & & x^{-1} & \\ & -\sin(\theta) & & \cos(\theta) \end{bmatrix} ; x \in \mathbb{R}^\times \ , \ \theta \in \mathbb{R} \right\}$$

This is such that $T_1(\mathbb{R}) \cong \mathbb{R}^\times \cdot S^1$ and has split rank one. Additionally,

$$\Delta^{im} = A_1 \ , \ \Delta^{re} = A_1$$

Next, consider GL_2, for which there is a subgroup of real points of the form

$$\begin{bmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{bmatrix}$$

We realize this subgroup of real points, in block diagonal form, as

$$T_2(\mathbb{R}) = \left\{ \begin{bmatrix} c \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} & \\ & c^{-1} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \end{bmatrix} ; c \in \mathbb{R}_{>0} \ , \ \theta \in \mathbb{R} \right\}$$

This is such that $T_2(\mathbb{R}) \cong \mathbb{C}^\times$. Note, as we indicated in (2.1), every torus defined over $\mathbb{R}$ is such that $T = T_{an} \times T_s$. This idea is realized here as

$$\begin{aligned} T_2 &= (T_2)_{an} \times (T_2)_s \\ &\cong (T_2)_{an} \times (T_2)_s / \{(I, I), (-I, -I)\}. \end{aligned}$$

This indicates that $T_2(\mathbb{R})$, and by extension the $\mathbb{C}^\times$ terms that will appear in higher group rank cases, has split rank one. Observe that, for this instance of split rank one, it occupies four rows and columns. Additionally,

$$\Delta^{im} = A_1 \ , \ \Delta^{re} = A_1$$

Having established this last case, we detour to make important general remarks. First, in completing the above, we have produced the basic building blocks of our starting point for this section, namely

$$T(\mathbb{R}) \cong (\mathbb{R}^\times)^{c_1} \cdot (\mathbb{C}^\times)^{c_2} \cdot (S^1)^{c_3}$$

These will combine in straightforward ways for the remainder of our constructions. Specifically, their repeated contributions will be as follows:

i) $\mathbb{R}^\times$ has split rank **one** and occupies **two** rows and columns in the relevant matrices

ii) $\mathbb{C}^\times$ has split rank **one** and occupies **four** rows and columns

iii) S^1 has split rank **zero** and occupies **two** rows and columns

As such, we will de-emphasize the explicit matrices moving forward, noting that they may be easily constructed, as need be, from the discussion above.

Secondly, two additional restrictions will aid our constructions of the conjugacy classes in the real forms that are not split. The split forms contain all possible conjugacy classes for a given group rank, i.e.,

$$\left\{\begin{matrix}\text{conj. classes of max.}\\ \text{tori of } Sp(p,q)(\mathbb{R})\end{matrix}\right\} \subset \left\{\begin{matrix}\text{conj. classes of max.}\\ \text{tori of } Sp(n,\mathbb{R})\end{matrix}\right\}$$

We delay formally establishing this result until we realize our Cayley transforms. Furthermore, no conjugacy class of a real form that is not split may contain $\mathbb{R}^\times$ terms. This is because $Sp(n,0)$ and $Sp(p,q)$ may be realized as subgroups of the special unitary group, for which S^1 is isomorphic to a subgroup of $SU(2)$ for which there exists an embedding into $Sp(2)$ and $\mathbb{C}^\times$ embeds into $SU(2,2)$ which in turn embeds into $Sp(2,2)$.

With these general comments established, we continue our discussion with the real form $Sp(2,2)(\mathbb{R})$. This contains only two conjugacy classes of maximal tori. The first corresponding to its anisotropic torus, for which we represent the real points as

$$T_{an}(\mathbb{R}) \cong (S^1)^2,$$

has split rank zero. Additionally,

$$\Delta^{im} = C_2 \ , \ \Delta^{re} = \emptyset.$$

For the second conjugacy class, we represent the real points as

$$T_1(\mathbb{R}) \cong \mathbb{C}^\times,$$

which has split rank one. Additionally,

$$\Delta^{im} = A_1 \ , \ \Delta^{re} = A_1.$$

Note, this real form is nonsplit. As such, it does not contain a torus of split rank 2.

Finally, the compact real form $Sp(4,0)(\mathbb{R})$ has one conjugacy class of maximal tori. We realize the real points as

$$T_{an}(\mathbb{R}) \cong (S^1)^2,$$

which has split rank zero. Additionally,

$$\Delta^{im} = C_2 \ , \ \Delta^{re} = \emptyset.$$

And, we summarize our work for this section in table 2.

	Split Rank		
	0	1	2
$Sp(4,\mathbb{R})$	$(S^1)^2$	$\mathbb{R}^\times \cdot S^1$; $\mathbb{C}^\times$	$(\mathbb{R}^\times)^2$
$Sp(2,2)(\mathbb{R})$	$(S^1)^2$	$\mathbb{C}^\times$	-
$Sp(4,0)(\mathbb{R})$	$(S^1)^2$	-	-

Table 2: Rank 2 Conjugacy Classes

3.3 Rank 3 Groups

We begin with the split real form $Sp(6,\mathbb{R})$, which has six conjugacy classes of maximal tori. The anisotropic and split cases amount to the ideas of the previous cases, with an extra copy of S^1 and $\mathbb{R}^\times$, respectively, added. Specifically, the anisotropic torus has real points represented by

$$T_{an}(\mathbb{R}) \cong (S^1)^3,$$

which has split rank zero. Additionally,

$$\Delta^{im} = C_3 \ , \ \Delta^{re} = \emptyset.$$

The split torus has real points represented by

$$T_s(\mathbb{R}) \cong (\mathbb{R}^\times)^3,$$

which has split rank three. Additionally,

$$\Delta^{im} = \emptyset \ , \ \Delta^{re} = C_3.$$

There are two conjugacy classes corresponding to split rank one. The first has real points represented by

$$T_1(\mathbb{R}) \cong \mathbb{R}^\times \cdot (S^1)^2.$$

Recall that the $\mathbb{R}^\times$ contributes two rows and columns to the attached matrices and that two copies of S^1 each contribute two rows and columns, giving the full six rows and columns needed in this case. Additionally,

$$\Delta^{im} = C_2 \ , \ \Delta^{re} = A_1.$$

The second has real points represented by

$$T_2(\mathbb{R}) \cong \mathbb{C}^\times \cdot S^1.$$

Considering again the attached matrices, recall that the $\mathbb{C}^\times$ contributes four rows and columns and the S^1 contributes two rows and columns, giving the full requisite six rows and columns. Additionally,

$$\Delta^{im} = A_1 \times A_1 \ , \ \Delta^{re} = A_1.$$

There are two conjugacy classes corresponding to split rank two. The first has real points represented by

$$T_3(\mathbb{R}) \cong (\mathbb{R}^\times)^2 \cdot S^1.$$

Additionally,

$$\Delta^{im} = A_1 \ , \ \Delta^{re} = C_2.$$

The second has real points represented by

$$T_4(\mathbb{R}) \cong \mathbb{R}^\times \cdot \mathbb{C}^\times.$$

Additionally,

$$\Delta^{im} = A_1 \ , \ \Delta^{re} = A_1 \times A_1.$$

Because of the repetitive nature of these results, it is worth making a potentially obvious comment on the thinking underlying this work. If one is not being careful, it is tempting to conclude that, for example, $(\mathbb{C}^\times)^2$ has the correct split rank and must be an option here. But, this would require a matrix of eight rows and columns and, hence, cannot occur in the rank three case, but will indeed occur in the rank four case of the next section. The perspective we develop here requires agreement both in split rank and the size of the attached matrices.

Next, the real form $Sp(4,2)(\mathbb{R})$ contains only two conjugacy classes of maximal tori. The first corresponding to its anisotropic torus, for which we represent the real points as

$$T_{an}(\mathbb{R}) \cong (S^1)^3,$$

has split rank zero. Additionally,

$$\Delta^{im} = C_3 \ , \ \Delta^{re} = \emptyset.$$

For the second conjugacy class, we represent the real points as

$$T_1(\mathbb{R}) \cong \mathbb{C}^{\times} \cdot S^1,$$

which has split rank one. Additionally,

$$\Delta^{im} = A_1 \times A_1 \ , \ \Delta^{re} = A_1.$$

Finally, the compact real form $Sp(6,0)(\mathbb{R})$ has one conjugacy class of maximal tori. We realize the real points as

$$T_{an}(\mathbb{R}) \cong (S^1)^3,$$

which has split rank zero. Additionally,

$$\Delta^{im} = C_3 \ , \ \Delta^{re} = \emptyset.$$

And, we summarize our work for this section in table 3.

	Split Rank			
	0	1	2	3
$Sp(6,\mathbb{R})$	$(S^1)^3$	$\mathbb{R}^\times \cdot (S^1)^2$; $\mathbb{C}^\times \cdot S^1$	$(\mathbb{R}^\times)^2 \cdot S^1$; $\mathbb{R}^\times \cdot \mathbb{C}^\times$	$(\mathbb{R}^\times)^3$
$Sp(4,2)(\mathbb{R})$	$(S^1)^3$	$\mathbb{C}^\times \cdot S^1$	-	-
$Sp(6,0)(\mathbb{R})$	$(S^1)^3$	-	-	-

Table 3: Rank 3 Conjugacy Classes

3.4 Rank 4 Groups

We begin with the split real form $Sp(8,\mathbb{R})$, which has nine conjugacy classes of maximal tori. Much as before, the anisotropic torus has real points represented by

$$T_{an}(\mathbb{R}) \cong (S^1)^4,$$

which has split rank zero. Additionally,

$$\Delta^{im} = C_4 \ , \ \Delta^{re} = \emptyset.$$

The split torus has real points represented by

$$T_s(\mathbb{R}) \cong (\mathbb{R}^\times)^4,$$

which has split rank three. Additionally,

$$\Delta^{im} = \emptyset \ , \ \Delta^{re} = C_4.$$

There are two conjugacy classes corresponding to split rank one. The first has real points represented by

$$T_1(\mathbb{R}) \cong \mathbb{R}^\times \cdot (S^1)^3.$$

Additionally,

$$\Delta^{im} = C_3 \ , \ \Delta^{re} = A_1.$$

The second has real points represented by

$$T_2(\mathbb{R}) \cong \mathbb{C}^\times \cdot (S^1)^2.$$

Additionally,

$$\Delta^{im} = A_1 \times C_2 \ , \ \Delta^{re} = A_1.$$

We observe that, in general for any higher group rank, the split rank one conjugacy classes will always amount to a copy of $\mathbb{R}^\times$ or $\mathbb{C}^\times$ combined with a sufficient number of copies of S^1 to "fill-in" the underlying dimensional considerations.

There are three conjugacy classes corresponding to split rank two. The first has real points represented by

$$T_3(\mathbb{R}) \cong (\mathbb{R}^\times)^2 \cdot (S^1)^2.$$

Additionally,

$$\Delta^{im} = C_2 \ , \ \Delta^{re} = C_2.$$

The second has real points represented by

$$T_4(\mathbb{R}) \cong \mathbb{R}^\times \cdot \mathbb{C}^\times \cdot S^1.$$

Additionally,

$$\Delta^{im} = A_1 \times A_1 \ , \ \Delta^{re} = A_1 \times A_1.$$

The third has real points represented by

$$T_5(\mathbb{R}) \cong (\mathbb{C}^\times)^2.$$

Additionally,

$$\Delta^{im} = A_1 \times A_1 \ , \ \Delta^{re} = A_1 \times A_1.$$

Finally, there are two conjugacy classes corresponding to split rank three. The first has real points represented by

$$T_6(\mathbb{R}) \cong (\mathbb{R}^\times)^3 \cdot S^1.$$

Additionally,

$$\Delta^{im} = A_1 \ , \ \Delta^{re} = C_3.$$

The second has real points represented by

$$T_7(\mathbb{R}) \cong (\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times.$$

Additionally,

$$\Delta^{im} = A_1 \ , \ \Delta^{re} = C_2 \times A_1.$$

Next, the real form $Sp(4,4)(\mathbb{R})$ contains only three conjugacy classes of maximal tori. The first corresponding to its anisotropic torus, for which we represent the real points as

$$T_{an}(\mathbb{R}) \cong (S^1)^4,$$

has split rank zero. Additionally,

$$\Delta^{im} = C_4 \ , \ \Delta^{re} = \emptyset.$$

For the second conjugacy class, we represent the real points as

$$T_1(\mathbb{R}) \cong \mathbb{C}^\times \cdot (S^1)^2,$$

which has split rank one. Additionally,

$$\Delta^{im} = C_2 \times A_1 \ , \ \Delta^{re} = A_1.$$

For the third conjugacy class, we represent the real points as

$$T_2(\mathbb{R}) \cong (\mathbb{C}^\times)^2,$$

which has split rank two. Additionally,

$$\Delta^{im} = A_1 \times A_1 \ , \ \Delta^{re} = A_1 \times A_1.$$

Next, the real form $Sp(6,2)(\mathbb{R})$ contains only two conjugacy classes of maximal tori. The first corresponding to its anisotropic torus, for which we represent the real points as

$$T_{an}(\mathbb{R}) \cong (S^1)^4,$$

has split rank zero. Additionally,

$$\Delta^{im} = C_4 \ , \ \Delta^{re} = \emptyset.$$

For the second conjugacy class, we represent the real points as

$$T_1(\mathbb{R}) \cong \mathbb{C}^\times \cdot (S^1)^2,$$

which has split rank one. Additionally,

$$\Delta^{im} = C_2 \times A_1 \ , \ \Delta^{re} = A_1.$$

Finally, the compact real form $Sp(6,0)(\mathbb{R})$ has one conjugacy class of maximal tori. We realize the real points as

$$T_{an}(\mathbb{R}) \cong (S^1)^4,$$

which has split rank zero. Additionally,

$$\Delta^{im} = C_4 \ , \ \Delta^{re} = \emptyset.$$

And, we summarize our work for this section in table 4.

	Split Rank				
	0	1	2	3	4
$Sp(8,\mathbb{R})$	$(S^1)^4$	$\mathbb{R}^\times \cdot (S^1)^3$; $\mathbb{C}^\times \cdot (S^1)^2$	$(\mathbb{R}^\times)^2 \cdot (S^1)^2$; $\mathbb{R}^\times \cdot \mathbb{C}^\times \cdot S^1$; $(\mathbb{C}^\times)^2$	$(\mathbb{R}^\times)^3 \cdot S^1$; $(\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times$	$(\mathbb{R}^\times)^4$
$Sp(4,4)(\mathbb{R})$	$(S^1)^4$	$\mathbb{C}^\times \cdot (S^1)^2$	$(\mathbb{C}^\times)^2$	-	-
$Sp(6,2)(\mathbb{R})$	$(S^1)^4$	$\mathbb{C}^\times \cdot (S^1)^2$	-	-	-
$Sp(8,0)(\mathbb{R})$	$(S^1)^4$	-	-	-	-

Table 4: Rank 4 Conjugacy Classes

3.5 Rank 5 Groups

As the discussion of the preceding ranks has firmly established all potential patterns, for this case we list our results directly in table 5.

	Split Rank					
	0	1	2	3	4	5
$Sp(10,\mathbb{R})$	$(S^1)^5$	$\mathbb{R}^\times \cdot (S^1)^4$; $\mathbb{C}^\times \cdot (S^1)^3$	$(\mathbb{R}^\times)^2 \cdot (S^1)^3$; $\mathbb{R}^\times \cdot \mathbb{C}^\times \cdot (S^1)^2$; $(\mathbb{C}^\times)^2 \cdot S^1$	$(\mathbb{R}^\times)^3 \cdot (S^1)^2$; $(\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times \cdot S^1$; $\mathbb{R}^\times \cdot (\mathbb{C}^\times)^2$	$(\mathbb{R}^\times)^4 \cdot S^1$; $(\mathbb{R}^\times)^3 \cdot \mathbb{C}^\times$	$(\mathbb{R}^\times)^5$
$Sp(6,4)(\mathbb{R})$	$(S^1)^5$	$\mathbb{C}^\times \cdot (S^1)^3$	$(\mathbb{C}^\times)^2 \cdot S^1$	-	-	-
$Sp(8,2)(\mathbb{R})$	$(S^1)^5$	$\mathbb{C}^\times \cdot (S^1)^3$	-	-	-	-
$Sp(10,0)(\mathbb{R})$	$(S^1)^5$	-	-	-	-	-

Table 5: Rank 5 Conjugacy Classes

3.6 Comments on the General Case

In the above discussion, we aimed to draw attention to the notion that, once one has come to understand the details of the lower rank cases, the work involved in higher rank cases amounts to repeated application of the same details. We did so because this same notion applies to cases of any general rank. In this final section, we aim to draw out some of these patterns in more detail, so that our work may be applied as needed.

We begin with one of the more elegant patterns. In split real forms, the number of conjugacy classes of each split rank form grow in a organized arrangement. First, Let N represent the total number of conjugacy classes of maximal tori, and the results of our above work are summarized in table 6.

	Number of Conj. Classes of Max. Tori of Split Rank						
	0	1	2	3	4	5	N
$Sp(2,\mathbb{R})$	1	1	-	-	-	-	2
$Sp(4,\mathbb{R})$	1	2	1	-	-	-	4
$Sp(6,\mathbb{R})$	1	2	2	1	-	-	6
$Sp(8,\mathbb{R})$	1	2	3	2	1	-	9
$Sp(10,\mathbb{R})$	1	2	3	3	2	1	12

Table 6: Number of Low-Split-Rank Conjugacy Classes

Then, for r equal to the rank of the group, we may express

$$N = \begin{cases} \frac{(r+2)^2}{4} & , r \text{ even} \\ \frac{(r+1)(r+3)}{4} & , r \text{ odd} \end{cases}$$

which is consistent with the results of Sugiura [Sug59]. And, the number of conjugacy classes of maximal tori of each split rank follows the symmetric ascending then descending pattern shown. For example, the rank ten split real form would entail the details of table 7.

	Number of Conj. Classes of Max. Tori of Split Rank											
	0	1	2	3	4	5	6	7	8	9	10	N
$Sp(20,\mathbb{R})$	1	2	3	4	5	6	5	4	3	2	1	36

Table 7: Number of Conjugacy Classes, $Sp(20,\mathbb{R})$, by Split Rank

In the case of a real form that is not split, $Sp(p,q)(\mathbb{R})$, there are $\frac{q}{2}+1$ conjugacy classes of maximal tori [Sug59]. And, the number of conjugacy classes for each relevant split rank is always one, as we have seen.

We have already included methods in the discussion above for finding a representative of each of these conjugacy classes.

Next, we look to the positive imaginary and positive real root systems for each of the conjugacy classes, but delay the reasoning for a subsequent section. Of course, the root system for the complex points, $Sp(n,\mathbb{C})$, and all of its real forms is given by $C_{\frac{n}{2}}$. For easy reference, we collect in table 8 some of the results of our above work.

		Δ^{im}	Δ^{re}
Group Rank 1	S^1	A_1	$\emptyset$
	$\mathbb{R}^\times$	$\emptyset$	A_1
Group Rank 2	$(S^1)^2$	C_2	$\emptyset$
	$\mathbb{R}^\times \cdot S^1$	A_1	A_1
	$\mathbb{C}^\times$	A_1	A_1
	$(\mathbb{R}^\times)^2$	$\emptyset$	C_2
Group Rank 3	$(S^1)^3$	C_3	$\emptyset$
	$\mathbb{R}^\times \cdot (S^1)^2$	C_2	A_1
	$\mathbb{C}^\times \cdot S^1$	$A_1 \times A_1$	A_1
	$(\mathbb{R}^\times)^2 \cdot S^1$	A_1	C_2
	$\mathbb{R}^\times \cdot \mathbb{C}^\times$	A_1	$A_1 \times A_1$
	$(\mathbb{R}^\times)^3$	$\emptyset$	C_3

		Δ^{im}	Δ^{re}
Group Rank 4	$(S^1)^4$	C_4	$\emptyset$
	$\mathbb{R}^\times \cdot (S^1)^3$	C_3	A_1
	$\mathbb{C}^\times \cdot (S^1)^2$	$A_1 \times C_2$	A_1
	$(\mathbb{R}^\times)^2 \cdot (S^1)^2$	C_2	C_2
	$\mathbb{R}^\times \cdot \mathbb{C}^\times \cdot S^1$	$A_1 \times A_1$	$A_1 \times A_1$
	$(\mathbb{C}^\times)^2$	$A_1 \times A_1$	$A_1 \times A_1$
	$(\mathbb{R}^\times)^3 \cdot S^1$	A_1	C_3
	$(\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times$	A_1	$C_2 \times A_1$
	$(\mathbb{R}^\times)^4$	$\emptyset$	C_4

Table 8: Δ^{im} and Δ^{re} for Low-Split-Rank

Furthermore, once again recall the fundamental expression that has underpinned our work thus far:

$$(\mathbb{R}^\times)^{c_1} \cdot (\mathbb{C}^\times)^{c_2} \cdot (S^1)^{c_3}$$

The exponents c_i for each term determine a straightforward contribution to the two root systems with which we are concerned. Specifically, these are as follows:

i) c_1 indicates a contribution of C_{c_1} to Δ^{re}

ii) c_2 indicates a contribution of c_2 copies of A_1 to each of Δ^{im} and Δ^{re}

iii) c_3 indicates a contribution of C_{c_3} to Δ^{im}

Noting that, in the previous, we have simply written A_1 for C_1.

Finally, it is interesting to observe that our work does imply a notion of working the other direction. If one were to choose specific values of c_i for the expression

$$(\mathbb{R}^{\times})^{c_1} \cdot (\mathbb{C}^{\times})^{c_2} \cdot (S^1)^{c_3},$$

then there exists a symplectic group of some rank in which this expression would demarcate a conjugacy class of maximal tori for the split real form of that group. For example, choose

$$(\mathbb{R}^{\times})^5 \cdot (\mathbb{C}^{\times})^5 \cdot (S^1)^3.$$

This has split rank ten and requires a matrix with 36 rows and columns, and does occur in $Sp(36, \mathbb{R})$

4 General Discussion of Lattice Construction

Guided by Shelstad [She79], this section develops the general notions of how the conjugacy classes of maximal tori relate to each other. The next section develops the details for symplectic groups.

Take G to be a connected reductive linear algebraic group defined over $\mathbb{R}$ and T to be a maximal torus in G and defined over $\mathbb{R}$. The Weyl group of this torus is given by

$$\Omega_G(T) := Norm(T(\mathbb{C}), G(\mathbb{C}))/T(\mathbb{C}).$$

Each $gT(\mathbb{C}) \in \Omega_G(T)$ acts on $T(\mathbb{C})$ by

$$gT(\mathbb{C}) \cdot t = gtg^{-1},$$

for $t \in T(\mathbb{C})$ and $g \in Norm(T(\mathbb{C}), G(\mathbb{C}))$. However, it is not necessarily the case that every element of the normalizer of the complex points $T(\mathbb{C})$ also normalizes the real points $T(\mathbb{R})$. To identify which elements have this additional property, we must check for a correspondence between their direct action on a real point of the torus and their action composed with a Galois action on a real point. Precisely, g normalizes $T(\mathbb{R})$ if and only if

$$\begin{aligned} gtg^{-1} &= \sigma(gtg^{-1}) \ , \ t \in T(\mathbb{R}) \ , \ \sigma \in Gal(\mathbb{C}/\mathbb{R}) \\ gtg^{-1} &= \sigma(g)t\sigma(g)^{-1} \\ t &= g^{-1}\sigma(g)t\sigma(g)^{-1}g \\ t &= (g^{-1}\sigma(g))t(g^{-1}\sigma(g))^{-1} \end{aligned}$$

$$\text{i.e., } g^{-1}\sigma(g) \in Cent(T(\mathbb{R}), G(\mathbb{C})) = Cent(T(\mathbb{C}), G(\mathbb{C}))$$

As such, write g_σ for $g^{-1}\sigma(g)$. Then

$$\begin{aligned} g_\sigma \sigma(g_\sigma) &= g^{-1}\sigma(g)\sigma(g^{-1}\sigma(g)) \\ &= g^{-1}\sigma(g)\sigma(g)^{-1}\sigma(\sigma(g)) \\ &= g^{-1}\sigma(g)\sigma(g)^{-1}g \\ &= 1 \ , \end{aligned}$$

indicates that

$$\begin{cases} 1 \to 1 \\ \sigma \to g_\sigma \end{cases}$$

is a 1-cocycle of $Gal(\mathbb{C}/\mathbb{R})$ with values in $T(\mathbb{C})$. As a 1-cocycle of $Gal(\mathbb{C}/\mathbb{R})$ with values in $G(\mathbb{C})$, this is a 1-coboundary. Via Shelstad, $gT(\mathbb{C})$ normalizes $T(\mathbb{R})$ if and only if

$$g \in Norm(T(\mathbb{R}), G(\mathbb{R})) \cdot Norm(T(\mathbb{C}), M(\mathbb{C})),$$

where M is the centralizer of the maximal $\mathbb{R}$-split torus S_T in T, a connected reductive subgroup of G. The roots of T in M are precisely the roots α of T of G for which $\sigma\alpha = -\alpha$, i.e., the imaginary roots. Thus, $\Omega_M(T)$ is generated by the Weyl relfections w_α for the imaginary roots α of T. Notice that

$$\{\pm\alpha\} = \{\alpha, \sigma\alpha\}$$

determines a single Weyl reflection.

We can also consider the orbit of an imaginary root α under $\Omega_M(T)$. In an irreducible root system, all roots of same length lie in a single Weyl group orbit.

Now we develop our algebraic notion of Cayley transform. Recall, via Shelstad, stable conjugacy for maximal tori over $\mathbb{R}$ is the same as $G(\mathbb{R})$-conjugacy. Consider the partially ordered set $t_{st}(G)$ of stable conjugacy classes of maximal tori T over $\mathbb{R}$

in G. Write $\{T\}$ for the stable conjugacy class of T. We have $\{T\} \preceq \{T'\}$ if and only if S_T is $G(\mathbb{R})$-conjugate to a subtorus of $S_{T'}$. We say $\{T\}$ is **adjacent** to $\{T'\}$, and T is **adjacent** to T', if $\{T\} \preceq \{T'\}$ and $dim(S_{T'}) = 1 + dim(S_T)$. Furthermore,

Proposition: T is adjacent to T' if and only if there is $g \in G(\mathbb{C})$ such that $Int(g)$ carries T to T' and $Int(g^{-1}\sigma(g))$ normalizes $T(\mathbb{C})$ and acts on $T(\mathbb{C})$ as the Weyl reflection w_α for an imaginary root α that is not totally compact.

Proof: see [She83]

If α is an imaginary root of T, then α determines a certain 3-dimensional subgroup G_α isogenous over $\mathbb{R}$ to either $SL(2)$ or $SU(2)$. We call α **noncompact** if $G_\alpha \sim SL(2)$ or **compact** if $G_\alpha \sim SU(2)$. We call α **totally compact** if and only if every root in the its imaginary Weyl group orbit is compact. Assume T is anisotropic modulo the center of G. Then,

Proposition: totally compact roots exist for T if and only if G is not quasi-split over $\mathbb{R}$

Proof: see [She]

Also,

Proposition: two noncompact roots in the same Weyl group orbit must be conjugate under $G(\mathbb{R})$

Proof: see [She12]

We return to general maximal torus T defined over $\mathbb{R}$. Assume $\{T\} \preceq \{T'\}$ and $dim(S_{T'}) = 1 + dim(S_T)$. Suppose $g \in Norm(T(\mathbb{C}), G(\mathbb{C}))$ carries $T(\mathbb{R})$ to $T'(\mathbb{R})$, and that the attached cocycle

$$\begin{cases} 1 \to 1 \\ \sigma \to g_\sigma \end{cases}$$

is such that g_σ acts on $T(\mathbb{C})$ as the Weyl reflection w_α for the imaginary root α that is not totally compact. Write s_α for the restriction of $Int(g)$ to $T(\mathbb{C})$ so that $s_\alpha^{-1}\sigma(s_\alpha)$

preserves $T(\mathbb{C})$ and acts on $T(\mathbb{C})$ as w_α. Often we write s in place of s_α. This s is our Cayley transform. We emphasize that our definition is based purely on algebraic group considerations, and notably we have not fixed a Cartan involution on $G(\mathbb{C})$. Notice that if we replace α by w_α, $w \in \Omega_M(T)$, and s_α by s_{w_α}, then we get $s_{w\alpha}(T)$ stably (or $G(\mathbb{R})$-) conjugate to $s_\alpha(T)$.

5 Lattices of Conjugacy Classes of Maximal Tori for Groups of Type C_r

Leveraging the results of the previous two sections, here we construct the lattices of conjugacy classes of maximal tori for the various real forms of interest.

As a starting point, we view the results of section 3.6 to establish the nodes of our lattice, where each node represents one of our conjugacy classes. We will always fix the top node as the element of split rank zero, i.e., the anisotropic torus, and the bottom node as the element of maximal split rank. It is here that we realize our aforementioned notions of these as minimal and maximal elements, respectively. Given a particular level of our lattice, every node in the next level down indicates a conjugacy class of split rank plus one. In rough generality, the nodes of our lattice will appear as

$$\begin{array}{ccccccl}
 & & & \circ & & & \text{split rank 0} \\
 & & \circ & & \circ & & \text{split rank 1} \\
 & \circ & & \circ & & \circ & \text{split rank 2} \\
 & & & \vdots & & & \\
 & \circ & & \circ & & \circ & \text{max. split rank r-2} \\
 & & \circ & & \circ & & \text{max. split rank r-1} \\
 & & & \circ & & & \text{max. split rank r}
\end{array}$$

Note, the lattices of the real forms that are not split are significantly simpler. With the nodes established, our primary task is to populate the edges of our lattice, where each edge represents a Cayley transform between adjacent tori. It is important to note that not every torus represented in a given level is adjacent to every torus in the next level, and contrariwise. Ultimately, within the scope of our work, there are three "types" of Cayley transforms. These allow for descending the lattice, which we will develop in the rank one and rank two cases. There are cases where the root system is such that there is no associated Cayley transform, which we will see in each rank for real forms that are not split and in the rank four case, for the first time, in the split case.

We realize our root system $C_r = C_{\frac{n}{2}}$ as

$$\Delta = \{\pm(e_i \pm e_j)\ ;\ 1 \leq i < j \leq \frac{n}{2}\} \cup \{\pm 2e_k\ ;\ 1 \leq k \leq \frac{n}{2}\},$$

for which we refer to roots of the form $e_i \pm e_j$ as **short** and roots of the form $2e_k$ as **long**. Specifically, we make use of the positive imaginary roots and their associated Cayley transforms.

Each subsequent level of our lattice indicates that the split rank increases by one, which Cayley transforms achieve, by definition. Thus, in order to fully descend the lattice along some path, successive Cayley transforms must correspond to maximal orthogonal sets of roots, which we will leverage heavily as we develop the details.

In each subsequent section, we will begin our work with establishing the maximal set of orthogonal positive roots relevant to the given rank and with providing the general framework of our lattice. For the latter, we will begin by using the representatives of each conjugacy classes established in section 3 for each node in place of the circles utilized above. In establishing the appropriate Cayley transforms, we will initially make specific choices to our sets of roots. But, we will end each discussion with a generalized lattice that shows the equivalence of all such choices, in a sense to be made precise in the course of our discussion.

Finally, we note that our work focuses on imaginary roots to establish the notion of descending the lattice. It is possible to perform the same work, mutatis mutandis, to ascend the lattice using the real roots.

5.1 Rank 1 Groups

The root system for the group rank 1 case is given by A_1, for which we express the positive root as

$$\Delta^+ = \{\alpha\}$$

Naturally, leveraging an orthogonal set of roots is not relevant here.

We begin with the case of the split form, where the basic structure of our lattice is given by

$$S^1$$

$$\mathbb{R}^\times$$

S^1 has a set of positive imaginary roots given by $\Delta^{im} = A_1 = \{\alpha\}$, which is noncompact. As such, to α is associated a Cayley transform that maps $S^1 \to \mathbb{R}^\times$. In doing so, the corresponding action of the Weyl element makes α a real root. Hence, we have that $\Delta^{im} = \emptyset$ for $\mathbb{R}^\times$. It is in this sense that we meant our claim in section 3.6 that the exponent c_1 of $\mathbb{R}^\times$ indicates a contribution of C_{c_1} to Δ^{re} and that the exponent c_3 of S^1 indicates a contribution of C_{c_3} to Δ^{im}. Furthermore, these results indicate that there can not exist a Cayley transform with which we can descend further and, along with the fact that $\mathbb{R}^\times$ achieves the full split rank, that our lattice must terminate here.

This accomplishes two of our goals. First, we have realized the first of our Cayley transforms, $S^1 \to \mathbb{R}^\times$, which in all subsequent sections will be identified with those Cayley transforms that are associated to long roots.

Secondly, we have completed the lattice for $Sp(2, \mathbb{R})$, which is given in figure 5.1.

$$\begin{array}{c} S^1 \\ | \\ \mathbb{R}^\times \end{array}$$

Figure 1: Lattice of Conjugacy Classes of Maximal Tori, $Sp(2,\mathbb{R})$

And, while we were restricted from making any specific choices in this case, the general lattice is given by

$$\begin{array}{c} \circ \\ | \\ \circ \end{array}$$

Next, the real form $Sp(2,0)(\mathbb{R})$ is compact and, hence, only has one conjugacy class of maximal tori. The imaginary root system, again, consists of a singleton, i.e., $\Delta^{im} = A_1$. In this context, this imaginary root is compact and totally compact. Hence, their is no associated Cayley transform and the lattice for $Sp(2,0)(\mathbb{R})$ is simply given by the single node:

$$S^1$$

5.2 Rank 2 Groups

The root system for the group rank 2 case is given by C_2, for which we express the positive roots as

$$\Delta^+ = \{e_1 + e_2, e_1 - e_2, 2e_1, 2e_2\},$$

and for which all maximal orthogonal sets of positive roots are given by

$$\{e_1 \pm e_2\} \ ; \ \{2e_1, 2e_2\}.$$

We begin with the case of the split form, where the basic structure of our lattice is given by

$$\begin{tikzcd}[column sep=small]
 & (S^1)^2 & \\
\mathbb{R}^\times \cdot S^1 & & \mathbb{C}^\times \\
 & (\mathbb{R}^\times)^2 &
\end{tikzcd}$$

$(S^1)^2$ has a set of imaginary roots given by $\Delta^{im} = C_2$, all of which are noncompact. Associated to each of these is a Cayley transform that map to a conjugacy class of split rank one. Guided by the notion that our orthogonal sets provide two potential paths of descent, we will choose first to consider those Cayley transforms associated with the roots $2e_1$ and $e_1 - e_2$. The Cayley transform associated to the long root $2e_1$ will, as indicated in the rank one case, map one copy of S^1 to $\mathbb{R}^\times$, and will provide the edge of our lattice corresponding to a mapping from $(S^1)^2 \to \mathbb{R}^\times \cdot S^1$. The associated Weyl element acting on the set of roots will leave the root $2e_2$ imaginary, make the root $2e_1$ real, and make the two short roots $e_1 \pm e_2$ complex. Hence, we have that $\Delta^{im} = A_1 = \{2e_2\}$ for $\mathbb{R}^\times \cdot S^1$. Next, the Cayley transform associated to the short root $e_1 - e_2$ will map $(S^1)^2 \to \mathbb{C}^\times$. Here, the associated Weyl element will leave the root $e_1 + e_2$ imaginary, make the root $e_1 - e_2$ real, and make the two long roots $2e_1$ and $2e_2$ complex. Hence, we have that $\Delta^{im} = A_1 = \{e_1 + e_2\}$ for $\mathbb{C}^\times$. Furthermore, this latter idea is the second of our Cayley transforms, $(S^1)^2 \to \mathbb{C}^\times$, which in all subsequent sections will be identified with those Cayley transforms that are associated to short roots that are the difference of the basis elements. Furthermore, it is in this sense that we meant our claim in section 3.6 that the exponent c_2 of $\mathbb{C}^\times$ indicates a contribution of c_2 copies of A_1 to each of Δ^{im} and Δ^{re}.

Thus, so far we have accomplished the following with our lattice:

$$\begin{tikzcd}[column sep=small]
 & (S^1)^2 \arrow[dl, dash, "2e_1"'] \arrow[dr, dash, "e_1 - e_2"] & \\
\mathbb{R}^\times \cdot S^1 & & \mathbb{C}^\times \\
 & (\mathbb{R}^\times)^2 &
\end{tikzcd}$$

The result that $\mathbb{R}^\times \cdot S^1$ has $\Delta^{im} = A_1 = \{2e_2\}$, which is noncompact, indicates that there can be only one Cayley transform by which we may map to $(\mathbb{R}^\times)^2$. By doing so, the associated Weyl element will make the root $2e_2$ real, yielding that $(\mathbb{R}^\times)^2$ has $\Delta^{im} = \emptyset$. Similarly, the result that $\mathbb{C}^\times$ has $\Delta^{im} = A_1 = \{e_1 + e_2\}$ indicates that there can be only one Cayley transform by which we may map to $(\mathbb{R}^\times)^2$. Here, the associated Weyl element will make the root $e_1 + e_2$ real, again yielding that $(\mathbb{R}^\times)^2$ has $\Delta^{im} = \emptyset$. Note, that both of these results are consistent with our insistence that, to fully descend the lattice, successive Cayley transforms must be associated with a set of orthogonal roots. Furthermore, this latter transform idea is the third and final of our Cayley transforms, $\mathbb{C}^\times \to (\mathbb{R}^\times)^2$, which in all subsequent sections will be identified with those Cayley transforms that are associated to short roots that are the sum of the basis elements. Recall the perspective we established that building our conjugacy classes amounted to repeatedly leveraging our three basic building blocks, S^1, $\mathbb{R}^\times$, and $\mathbb{C}^\times$, of real points of maximal tori. In very much the same way, our three "types" of Cayley transforms that correspond to descending the lattice will be applied repeatedly to establish the manner in which are conjugacy classes may relate and, consequently, establishing the edges of our lattices.

Thus, the lattice for $Sp(4, \mathbb{R})$ is given in figure 5.2.

$$\begin{tikzcd}
& (S^1)^2 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{R}^\times \cdot S^1 \arrow[dr, dash] & & \mathbb{C}^\times \arrow[dl, dash] \\
& (\mathbb{R}^\times)^2 &
\end{tikzcd}$$

Figure 2: Lattice of Conjugacy Classes of Maximal Tori, $Sp(4, \mathbb{R})$

We have indicated that an orthogonal set of positive imaginary noncompact roots provides guidance on which Cayley transforms may be made in succession. However, this does not specify in which order they must be applied. If we were so inclined, we could choose to establish that the Cayley transform associated to $2e_1$ would map $(S^1)^2 \to \mathbb{R}^\times \cdot S^1$ and the one associated to $2e_2$ would map $(S^1)^2 \to S^1 \cdot \mathbb{R}^\times$. But, because we are concerned only with the conjugacy classes of our tori, these are equivalent choices, and the order in which they are applied has no effect in our setting. In the next section, where our set of orthogonal roots will be slightly more complicated, we will return to this point to make the clarification that some choices do matter. For now, we will simply establish the necessary decorations, l and s for a Cayley transform associated to a long or short root, respectively, in our general lattice.

As such, the general lattice is given by

$$\begin{tikzcd} & \circ \arrow[dl, dash, "l"'] \arrow[dr, dash, "s"] & \\ \circ \arrow[dr, dash, "l"'] & & \circ \arrow[dl, dash, "s"] \\ & \circ & \end{tikzcd}$$

Next, for the real form, $Sp(2,2)(\mathbb{R})$, the two conjugacy classes indicate that the nodes of our lattice are given by

$$(S^1)^2$$

$$\mathbb{C}^\times$$

As in the split case, $(S^1)^2$ has an imaginary set of roots given by $\Delta^{im} = C_2$. However, unlike the split case, not all of the imaginary roots are noncompact. More specifically, here, the long roots are imaginary compact roots, are totally compact, and do not have an associated Cayley transform. This has the effect of reducing the available choices for Cayley transforms with which to descend the lattice and will give these lattices a significantly simpler structure.

To establish the details, we consider the results under the Cayley transform associated to the short root $e_1 - e_2$, This will map $(S^1)^2 \to \mathbb{C}^\times$, which has $\Delta^{im} = A_1 = \{e_1 + e_2\}$. This imaginary root is compact and totally compact, and our lattice terminates here.

Thus, the lattice for $Sp(2,2)(\mathbb{R})$ is given by:

$$\begin{array}{c} (S^1)^2 \\ | \\ \mathbb{C}^\times \end{array}$$

Finally, because the real form $Sp(4,0)(\mathbb{R})$ is compact and only has one conjugacy class of maximal tori, its lattice is given by:

$$(S^1)^2$$

5.3 Rank 3 Groups

The root system for the group rank 3 case is given by C_3, for which we express the positive roots as

$$\Delta^+ = \{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, 2e_1, 2e_2, 2e_3\},$$

and for which all maximal orthogonal sets of positive roots are given by

$$\{e_1 \pm e_2, 2e_3\} \ ; \ \{e_2 \pm e_3, 2e_1\} \ ; \ \{e_1 \pm e_3, 2e_2\} \ ; \ \{2e_1, 2e_2, 2e_3\}$$

We begin with the case of the split form, where the basic structure of our lattice is given by

$$\begin{array}{ccc} & (S^1)^3 & \\ \mathbb{R}^\times \cdot (S^1)^2 & & \mathbb{C}^\times \cdot S^1 \\ (\mathbb{R}^\times)^2 \cdot S^1 & & \mathbb{R}^\times \cdot \mathbb{C}^\times \\ & (\mathbb{R}^\times)^3 & \end{array}$$

To establish the first version of our lattice, we will work with the specific choices for the orthogonal sets of

$$\{e_2 \pm e_3, 2e_1\}\ ;\ \{2e_1, 2e_2, 2e_3\}$$

$(S^1)^2$ has a set of imaginary roots given by $\Delta^{im} = C_3$, all of which are noncompact. We choose to consider the Cayley transforms associated to the roots $2e_1$ and $e_2 - e_3$, which will map to conjugacy classes of split rank one. First, the one associated with $2e_1$ maps $(S^1)^3 \to \mathbb{R}^\times \cdot (S^1)^2$, and the corresponding Weyl element acts on the set of imaginary roots to make $2e_1$ real and leaves $e_2 \pm e_3$, $2e_2$, and $2e_3$ as noncompact imaginary. Hence, we have that $\Delta^{im} = C_2$ for $\mathbb{R}^\times \cdot (S^1)^2$, indicating for future work that we will have multiple potential Cayley transforms with which we may map to the conjugacy classes of split rank two. Next, the Cayley transform associated with $e_2 - e_3$ maps $(S^1)^3 \to \mathbb{C}^\times \cdot S^1$, and the corresponding Weyl element makes $e_2 - e_3$ real, makes $2e_2$ and $2e_3$ complex, and leaves $e_2 + e_3$ and $2e_1$ noncompact imaginary. Hence, we have that $\Delta^{im} = A_1 \times A_1$ for $\mathbb{C}^\times \cdot S^1$, which is our first example of a resulting imaginary root system that is not nonsingular. This indicates that the roots $e_2 + e_3$ and $2e_1$ have distinct Galois orbits within the root system, thus limiting the potential choices of applicable Cayley transforms. With these results, our lattice is now

$$\begin{tikzcd}
 & (S^1)^3 \arrow[dl, dash, "2e_1"'] \arrow[dr, dash, "e_2-e_3"] & \\
\mathbb{R}^\times \cdot (S^1)^2 & & \mathbb{C}^\times \cdot S^1 \\
(\mathbb{R}^\times)^2 \cdot S^1 & & \mathbb{R}^\times \cdot \mathbb{C}^\times \\
 & (\mathbb{R}^\times)^3 &
\end{tikzcd}$$

Then, again, $\mathbb{R}^\times \cdot (S^1)^2$ has $\Delta^{im} = C_2 = \{e_2 \pm e_3, 2e_2, 2e_3\}$, and we may choose Cayley transforms associated to either of the short or long roots. We will choose those associated with $2e_2$ and $e_2 - e_3$. The one associated with $2e_2$, as a long root, maps

$\mathbb{R}^\times \cdot (S^1)^2 \to (\mathbb{R}^\times)^2 \cdot S^1$, and the associated Weyl element leaves only $2e_3$ imaginary. Hence, we have $\Delta^{im} = A_1$ for $(\mathbb{R}^\times)^2 \cdot S^1$. Next, the Cayley transform associated with $e_2 - e_3$, as a short root including a difference, maps $\mathbb{R}^\times \cdot (S^1)^2 \to \mathbb{R}^\times \cdot \mathbb{C}^\times$, and the associated Weyl element leaves only $e_2 + e_3$ imaginary. Before we turn to the other split rank one conjugacy class, our lattice is

$$\begin{tikzcd}
& (S^1)^3 \arrow[dl, no head, "2e_1"'] \arrow[dr, no head, "e_2-e_3"] & \\
\mathbb{R}^\times \cdot (S^1)^2 \arrow[d, no head, "2e_2"'] \arrow[drr, no head, "e_2-e_3"] & & \mathbb{C}^\times \cdot S^1 \\
(\mathbb{R}^\times)^2 \cdot S^1 & & \mathbb{R}^\times \cdot \mathbb{C}^\times \\
& (\mathbb{R}^\times)^3 &
\end{tikzcd}$$

Next, the other split rank 1 conjugacy class, $\mathbb{C}^\times \cdot S^1$, has $\Delta^{im} = A_1 \times A_1 = \{e_2 + e_3, 2e_1\}$. Based on their separate Galois orbits, each of these will correspond to a mapping to a different adjacent torus of split rank 2. The Cayley transform associated with $2e_1$, as a long root, maps $\mathbb{C}^\times \cdot S^1 \to (\mathbb{R}^\times)^2 \cdot S^1$ and the Cayley transform associated with $e_2 + e_3$, as a short including a sum, maps $\mathbb{C}^\times \cdot S^1 \to \mathbb{R}^\times \cdot \mathbb{C}^\times$, both with imaginary root systems as just discussed. As such, our lattice is now,

$$\begin{tikzcd}
& (S^1)^3 \arrow[dl, no head, "2e_1"'] \arrow[dr, no head, "e_2-e_3"] & \\
\mathbb{R}^\times \cdot (S^1)^2 \arrow[d, no head, "2e_2"'] \arrow[drr, no head, "e_2-e_3"] & & \mathbb{C}^\times \cdot S^1 \arrow[dll, no head, "e_2+e_3"] \arrow[d, no head, "2e_1"] \\
(\mathbb{R}^\times)^2 \cdot S^1 & & \mathbb{R}^\times \cdot \mathbb{C}^\times \\
& (\mathbb{R}^\times)^3 &
\end{tikzcd}$$

Both of the conjugacy classes of split rank 2 have singletons for their set of imaginary roots, and we conclude that the lattice for $Sp(6, \mathbb{R})$ is given in figure 5.3.

$$
\begin{tikzcd}[arrows=dash]
 & (S^1)^3 \arrow[dl] \arrow[dr] & \\
\mathbb{R}^\times \cdot (S^1)^2 \arrow[d] \arrow[drr] & & \mathbb{C}^\times \cdot S^1 \arrow[d] \arrow[dll] \\
(\mathbb{R}^\times)^2 \cdot S^1 \arrow[dr] & & \mathbb{R}^\times \cdot \mathbb{C}^\times \arrow[dl] \\
 & (\mathbb{R}^\times)^3 &
\end{tikzcd}
$$

Figure 3: Lattice of Conjugacy Classes of Maximal Tori, $Sp(6, \mathbb{R})$

As discussed in the last section, while we made some specific choices to showcase specific details, we have only done so for the sake of clarity and all such choices of orthogonal sets are ultimately equivalent. However, unlike the last case, we have seen that the order in which successive Cayley transforms are applied does create different paths of descent through the various conjugacy classes. The only exception to this here is in the strongly orthogonal set of long roots. Ultimately, the four paths of descent that we have established correspond to one the following orders of application: lll, lss, sls, and ssl. As such, the general lattice is given by

$$
\begin{tikzcd}[arrows=dash]
 & \circ \arrow[dl, "l"'] \arrow[dr, "s"] & \\
\circ \arrow[d, "l"'] \arrow[drr, "s"] & & \circ \arrow[d, "l"] \arrow[dll, "s"'] \\
\circ \arrow[dr, "l"'] & & \circ \arrow[dl, "s"] \\
 & \circ &
\end{tikzcd}
$$

Next, for the real form $Sp(4, 2)(\mathbb{R})$, the general form of our lattice is given by:

$$(S^1)^3$$

$$\mathbb{C}^\times \cdot S^1$$

$(S^1)^3$ has $\Delta^{im} = C_3$, but this contains a mixture of compact and noncompact roots. Specifically, the compact imaginary roots are $2e_1$, $2e_2$, $2e_3$, and $e_1 \pm e_3$, and the noncompact imaginary roots are $e_1 \pm e_2$ and $e_2 \pm e_3$. Choosing one of these, say $e_1 - e_2$, the associated Cayley transform maps $(S^1)^3 \to \mathbb{C}^\times \cdot S^1$, which has $\Delta^{im} = A_1 \times A_1 = \{e_1 + e_2\} \times \{e_2 - e_3\}$. But, because these both become totally compact roots in this context, our lattice must terminate. Hence, the lattice for $Sp(4,2)(\mathbb{R})$ is given by:

$$\begin{array}{c} (S^1)^3 \\ | \\ \mathbb{C}^\times \cdot S^1 \end{array}$$

Finally, as in previous cases, the lattice for the compact real form $Sp(6,0)(\mathbb{R})$ is given by:

$$(S^1)^3$$

5.4 Rank 4 Groups

The root system for the group rank 4 case is given by C_4, for which we express the positive roots as

$$\Delta^+ = \{e_1 \pm e_2, e_1 \pm e_3, e_1 \pm e_4, e_2 \pm e_3, e_2 \pm e_4, e_3 \pm e_4, 2e_1, 2e_2, 2e_3, 2e_4\},$$

and for which all maximal orthogonal sets of positive roots are given by

$$\{e_1 \pm e_2, 2e_3, 2e_4\}\ ;\ \{e_1 \pm e_3, 2e_2, 2e_4\}\ ;\ \{e_1 \pm e_4, 2e_2, 2e_3\}\ ;\ \{e_2 \pm e_3, 2e_1, 2e_4\}\ ;\ \{e_2 \pm e_4, 2e_1, 2e_3\}\ ;$$

$$\{e_3 \pm e_4, 2e_1, 2e_2\}\ ;\ \{2e_1, 2e_2, 2e_3, 2e_4\}\ ;\ \{e_1 \pm e_2, e_3 \pm e_4\}\ ;\ \{e_1 \pm e_3, e_2 \pm e_4\}\ ;\ \{e_1 \pm e_4, e_2 \pm e_3\}$$

We begin with the case of the split form, where the basic structure of our lattice is given by

$$(S^1)^4$$

$$\mathbb{R}^\times \cdot (S^1)^3 \qquad\qquad \mathbb{C}^\times \cdot (S^1)^2$$

$$(\mathbb{R}^\times)^2 \cdot (S^1)^2 \qquad\qquad \mathbb{R}^\times \cdot \mathbb{C}^\times \cdot S^1 \qquad\qquad (\mathbb{C}^\times)^2$$

$$(\mathbb{R}^\times)^3 \cdot S^1 \qquad\qquad (\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times$$

$$(\mathbb{R}^\times)^4$$

In light of the results of the previous section, and in an effort to vary our presentation, in this case we will follow each path of descent, instead of building our work by increasing the split rank. To accomplish this, we will choose to work with the orthogonal sets

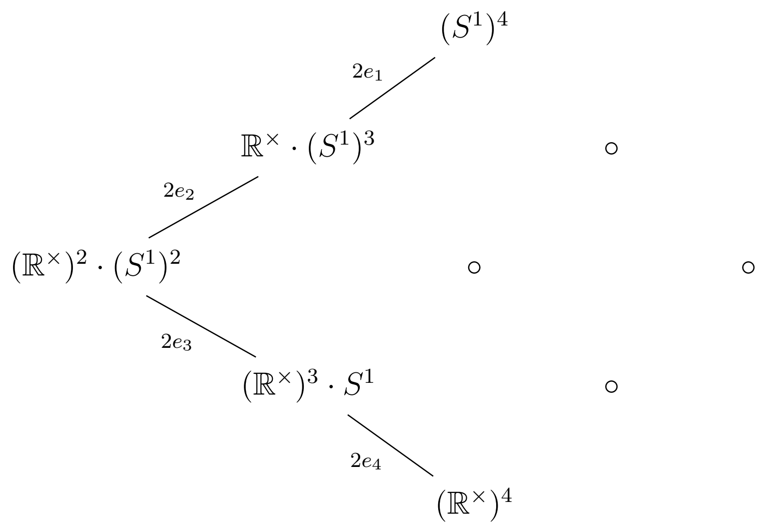
$$\{e_3 \pm e_4, 2e_1, 2e_2\} \ ; \ \{e_1 \pm e_2, e_3 \pm e_4\} \ ; \ \{2e_1, 2e_2, 2e_3, 2e_4\}$$

There are nine paths of descent, corresponding to

$$llll \ ; \ llss \ ; \ lsls \ ; \ lssl \ ; \ ssll \ ; \ ssss \ ; \ slsl \ ; \ slls \ ; \ ssss,$$

and noting that the two instances of $ssss$ are distinct paths that dependent on whether the short root includes a sum or difference. Finally, because the discussions of the last few sections have established almost all of the ideas applicable to the scope of our work, we will highlight the working components of each of the nine cases within the context of the general lattice, and limit our discussion.

Case *llll*:



for which the imaginary root systems are described in table 9.

$\mathbb{R}^\times \cdot (S^1)^3$:	$\Delta^{im} = C_3 = \{2e_2, 2e_3, 2e_4, e_2 \pm e_3, e_3 \pm e_4, e_2 \pm e_4\}$
$(\mathbb{R}^\times)^2 \cdot (S^1)^2$:	$\Delta^{im} = C_2 = \{2e_3, 2e_4, e_3 \pm e_4\}$
$(\mathbb{R}^\times)^3 \cdot S^1$:	$\Delta^{im} = A_1 = \{2e_4\}$

Table 9: Δ^{im}: Case *llll*

Note, the Cayley transform associated with the root $2e_1$ maps $(S^1)^4 \to \mathbb{R}^\times \cdot (S^1)^3$. And, the corresponding Weyl element yields compact roots $e_2 \pm e_3$, $e_2 \pm e_4$ and noncompact roots $2e_2$, $2e_3$, $2e_4$, $e_3 \pm e_4$, all of which are imaginary. Because the Weyl group orbits of these compact imaginary roots include noncompact imaginary roots, the Cayley transforms associated to these compact imaginary roots will have the same effect as those associated to the noncompact imaginary roots. And, as we have seen, those associated to the noncompact imaginary roots map to a torus to which this one is adjacent. Note, in the event that all of the roots in its Weyl group orbit are compact, we call them totally compact, and there is no associated Cayley transform. But, this latter point does not occur in quasi-split groups, i.e., for a quasi-split group every imaginary root for every T has a Cayley transform [She83].

Case $llss$:

$$
\begin{tikzcd}
 & & (S^1)^4 \arrow[dl, dash, "2e_1"'] & & \\
 & \mathbb{R}^\times \cdot (S^1)^3 \arrow[dl, dash, "2e_2"'] & & \circ & \\
 (\mathbb{R}^\times)^2 \cdot (S^1)^2 \arrow[drrr, dash, "e_3-e_4"] & & \circ & & \circ \\
 & \circ & & (\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times \arrow[dl, dash, "e_3+e_4"] & \\
 & & (\mathbb{R}^\times)^4 & &
\end{tikzcd}
$$

for which the imaginary root systems are described in table 10.

$\mathbb{R}^\times \cdot (S^1)^3 :$	$\Delta^{im} = C_3 = \{2e_2, 2e_3, 2e_4, e_2 \pm e_3, e_3 \pm e_4, e_2 \pm e_4\}$
$(\mathbb{R}^\times)^2 \cdot (S^1)^2 :$	$\Delta^{im} = C_2 = \{2e_3, 2e_4, e_3 \pm e_4\}$
$(\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times :$	$\Delta^{im} = A_1 = \{e_3 + e_4\}$

Table 10: Δ^{im}: Case $llss$

Case *lsls*:

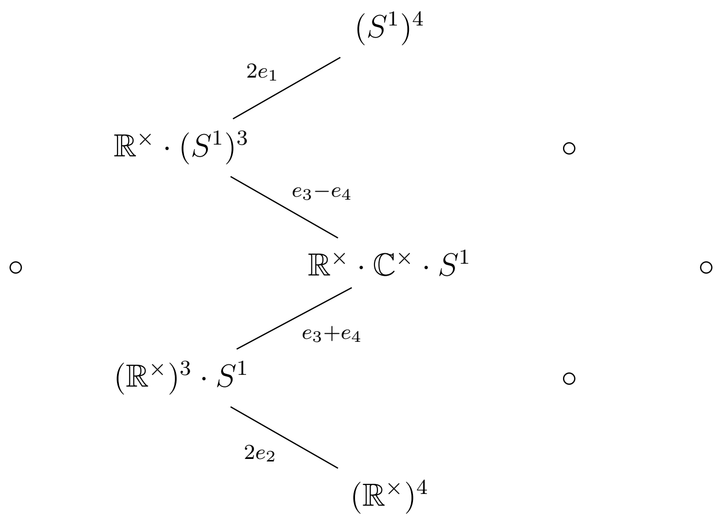
$$
\begin{tikzcd}
 & & (S^1)^4 \arrow[dl, dash, "2e_1"'] & & \\
 & \mathbb{R}^\times \cdot (S^1)^3 \arrow[dr, dash, "e_3-e_4"] & & \circ & \\
\circ & & \mathbb{R}^\times \cdot \mathbb{C}^\times \cdot S^1 \arrow[dr, dash, "2e_2"] & & \circ \\
 & \circ & & (\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times \arrow[dl, dash, "e_3+e_4"] & \\
 & & (\mathbb{R}^\times)^4 & &
\end{tikzcd}
$$

for which the imaginary root systems are described in table 11.

$\mathbb{R}^\times \cdot (S^1)^3$:	$\Delta^{im} = C_3 = \{2e_2, 2e_3, 2e_4, e_2 \pm e_3, e_3 \pm e_4, e_2 \pm e_4\}$
$\mathbb{R}^\times \cdot \mathbb{C}^\times \cdot S^1$:	$\Delta^{im} = A_1 \times A_1 = \{2e_2\} \times \{e_3 + e_4\}$
$(\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times$:	$\Delta^{im} = A_1 = \{e_3 + e_4\}$

Table 11: Δ^{im}: Case *lsls*

Case *lssl*:

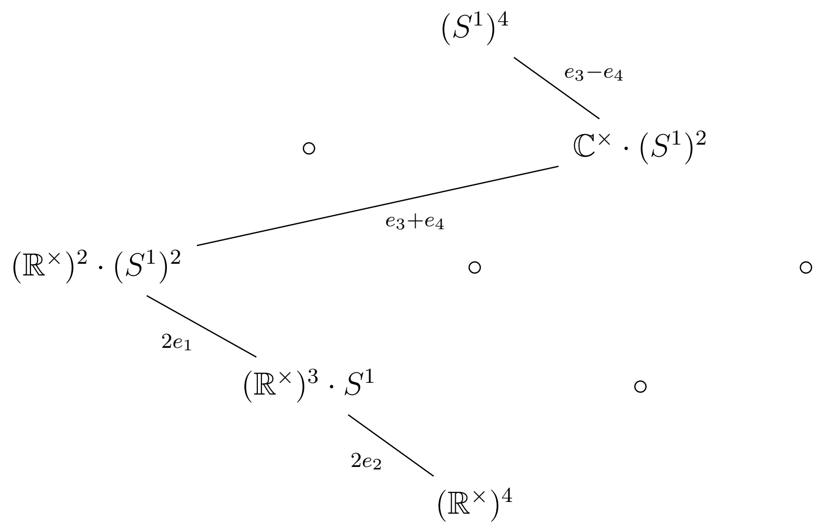


for which the imaginary root systems are described in table 12.

$\mathbb{R}^\times \cdot (S^1)^3$:	$\Delta^{im} = C_3 = \{2e_2, 2e_3, 2e_4, e_2 \pm e_3, e_3 \pm e_4, e_2 \pm e_4\}$
$\mathbb{R}^\times \cdot \mathbb{C}^\times \cdot S^1$:	$\Delta^{im} = A_1 \times A_1 = \{2e_2\} \times \{e_3 + e_4\}$
$(\mathbb{R}^\times)^3 \cdot S^1$:	$\Delta^{im} = A_1 = \{2e_2\}$

Table 12: Δ^{im}: Case *lssl*

Case $ssll$:



for which the imaginary root systems are described in table 13.

$\mathbb{C}^\times \cdot (S^1)^2$:	$\Delta^{im} = A_1 \times C_2 = \{e_3 + e_4\} \times \{2e_1, 2e_2, e_1 \pm e_2\}$
$(\mathbb{R}^\times)^2 \cdot (S^1)^2$:	$\Delta^{im} = C_2 = \{2e_1, 2e_2, e_1 \pm e_2\}$
$(\mathbb{R}^\times)^3 \cdot S^1$:	$\Delta^{im} = A_1 = \{2e_2\}$

Table 13: Δ^{im}: Case $ssll$

Case $ssss$:

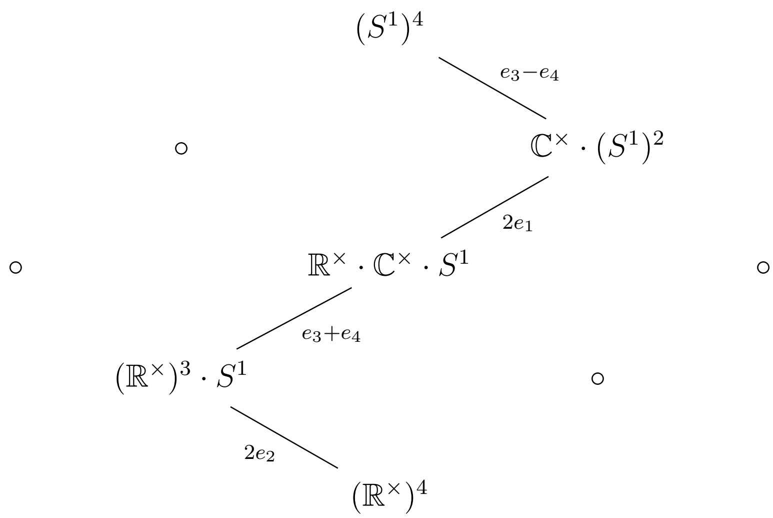
$$\begin{tikzcd}
 & & (S^1)^4 \arrow[dr, dash, "e_3-e_4"] & & \\
 & \circ & & \mathbb{C}^\times \cdot (S^1)^2 \arrow[dlll, dash, "e_3+e_4"] & \\
(\mathbb{R}^\times)^2 \cdot (S^1)^2 \arrow[drrr, dash, "e_1-e_2"'] & & \circ & & \circ \\
 & \circ & & (\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times \arrow[dl, dash, "e_1+e_2"] & \\
 & & (\mathbb{R}^\times)^4 & &
\end{tikzcd}$$

for which the imaginary root systems are described in table 14.

$\mathbb{C}^\times \cdot (S^1)^2$:	$\Delta^{im} = A_1 \times C_2 = \{e_3 + e_4\} \times \{2e_1, 2e_2, e_1 \pm e_2\}$
$(\mathbb{R}^\times)^2 \cdot (S^1)^2$:	$\Delta^{im} = C_2 = \{2e_1, 2e_2, e_1 \pm e_2\}$
$(\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times$:	$\Delta^{im} = A_1 = \{e_1 + e_2\}$

Table 14: Δ^{im}: Case $ssss$

Case *slsl*:



for which the imaginary root systems are described in table 15.

$\mathbb{C}^\times \cdot (S^1)^2$:	$\Delta^{im} = A_1 \times C_2 = \{e_3 + e_4\} \times \{2e_1, 2e_2, e_1 \pm e_2\}$
$\mathbb{R}^\times \cdot \mathbb{C}^\times \cdot S^1$:	$\Delta^{im} = A_1 \times A_1 = \{2e_2\} \times \{e_3 + e_4\}$
$(\mathbb{R}^\times)^3 \cdot S^1$:	$\Delta^{im} = A_1 = \{2e_2\}$

Table 15: Δ^{im}: Case *slsl*

Case *slls*:

$$
\begin{tikzcd}
 & & (S^1)^4 \arrow[dr, dash, "e_3-e_4"] & & \\
 & \circ & & \mathbb{C}^\times \cdot (S^1)^2 \arrow[dl, dash, "2e_1"] & \\
\circ & & \mathbb{R}^\times \cdot \mathbb{C}^\times \cdot S^1 \arrow[dr, dash, "2e_2"'] & & \circ \\
 & \circ & & (\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times \arrow[dl, dash, "e_3+e_4"] & \\
 & & (\mathbb{R}^\times)^4 & &
\end{tikzcd}
$$

for which the imaginary root systems are described in table 16.

$\mathbb{C}^\times \cdot (S^1)^2$:	$\Delta^{im} = A_1 \times C_2 = \{e_3 + e_4\} \times \{2e_1, 2e_2, e_1 \pm e_2\}$
$\mathbb{R}^\times \cdot \mathbb{C}^\times \cdot S^1$:	$\Delta^{im} = A_1 \times A_1 = \{2e_2\} \times \{e_3 + e_4\}$
$(\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times$:	$\Delta^{im} = A_1 = \{e_3 + e_4\}$

Table 16: Δ^{im}: Case *slls*

Case $ssss$:

$$
\begin{tikzcd}
 & & (S^1)^4 \arrow[dr, dash, "e_3-e_4"] & & \\
 & \circ & & \mathbb{C}^\times \cdot (S^1)^2 \arrow[dr, dash, "e_1-e_2"] & \\
\circ & & \circ & & (\mathbb{C}^\times)^2 \arrow[dl, dash, "e_1+e_2"] \\
 & \circ & & (\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times \arrow[dl, dash, "e_3+e_4"] & \\
 & & (\mathbb{R}^\times)^4 & &
\end{tikzcd}
$$

for which the imaginary root systems are described in table 17.

$\mathbb{C}^\times \cdot (S^1)^2$:	$\Delta^{im} = A_1 \times C_2 = \{e_3 + e_4\} \times \{2e_1, 2e_2, e_1 \pm e_2\}$
$(\mathbb{C}^\times)^2$:	$\Delta^{im} = A_1 \times A_1 = \{e_3 + e_4\} \times \{e_1 + e_2\}$
$(\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times$:	$\Delta^{im} = A_1 = \{e_3 + e_4\}$

Table 17: Δ^{im}: Case $ssss$

Finally, the complete lattice for $Sp(8,\mathbb{R})$ is given in figure 5.4.

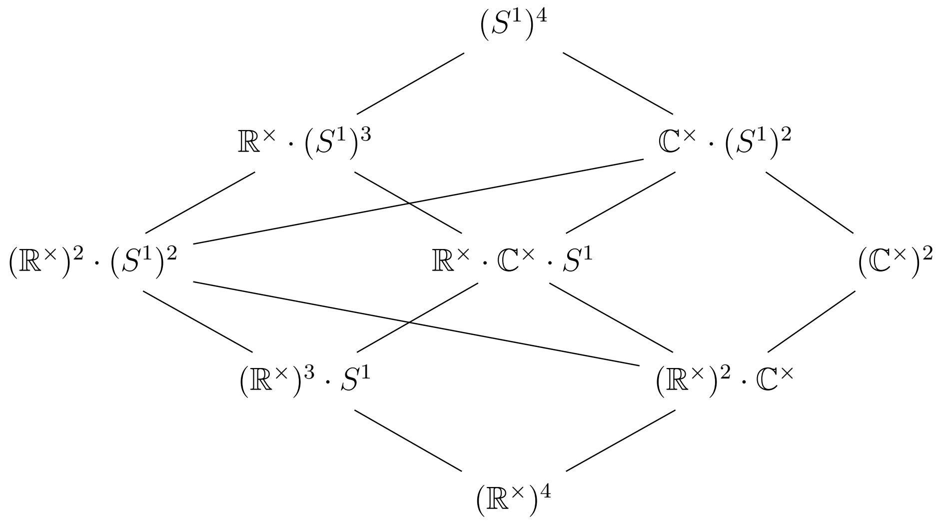
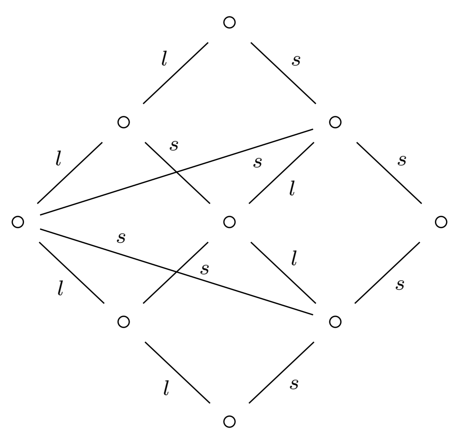


Figure 4: Lattice of Conjugacy Classes of Maximal Tori, $Sp(8,\mathbb{R})$

And the general lattice is given by:



Next, for the real form $Sp(4,4)(\mathbb{R})$, the general form of our lattice is given by

$$(S^1)^4$$

$$\mathbb{C}^\times \cdot (S^1)^2$$

$$(\mathbb{C}^\times)^2$$

$(S^1)^4$ has $\Delta^{im} = C_4$, of which $e_1 \pm e_3,\ e_2 \pm e_4,\ 2e_1,\ 2e_2,\ 2e_3,\ 2e_4$ are compact and of which $e_1 \pm e_2,\ e_1 \pm e_4,\ e_2 \pm e_3,\ e_3 \pm e_4$ are noncompact. Taking the Cayley transform associated with one of the noncompact imaginary roots, say e_1-e_2, will map $(S^1)^4 \to \mathbb{C}^\times \cdot (S^1)^2$, which has $\Delta^{im} = A_1 \times C_2 = \{e_1+e_2\} \times \{e_3 \pm e_4, 2e_3, 2e_4\}$. Of these, we now choose the Cayley transform associated to the noncompact imaginary root e_3-e_4, which maps $\mathbb{C}^\times \cdot (S^1)^2 \to (\mathbb{C}^\times)^2$. This has $\Delta^{im} = A_1 \times A_1 = \{e_1+e_2\} \times \{e_3+e_4\}$, for which both singletons are totally compact and our descent terminates. Thus, the lattice for $Sp(4,4)(\mathbb{R})$ is given by:

$$\begin{tikzcd} (S^1)^4 \arrow[d, dash] \\ \mathbb{C}^\times \cdot (S^1)^2 \arrow[d, dash] \\ (\mathbb{C}^\times)^2 \end{tikzcd}$$

Next, for the real form $Sp(6,2)(\mathbb{R})$, the Weyl element corresponding to any of the Cayley transforms associated to the noncompact imaginary roots of $(S^1)^4$, will yield all imaginary roots of $\mathbb{C}^\times \cdot (S^1)^2$ as compact. Hence, the lattice for $Sp(6,2)(\mathbb{R})$ is given by:

$$\begin{tikzcd} (S^1)^4 \arrow[d, dash] \\ \mathbb{C}^\times \cdot (S^1)^2 \end{tikzcd}$$

Finally, the lattice for the compact real form $Sp(8,0)(\mathbb{R})$ is given by:

$$(S^1)^2$$

5.5 Rank 5 Groups

In this case, we list our results directly.

The lattice for $Sp(10, \mathbb{R})$ is given in figure 5.5.

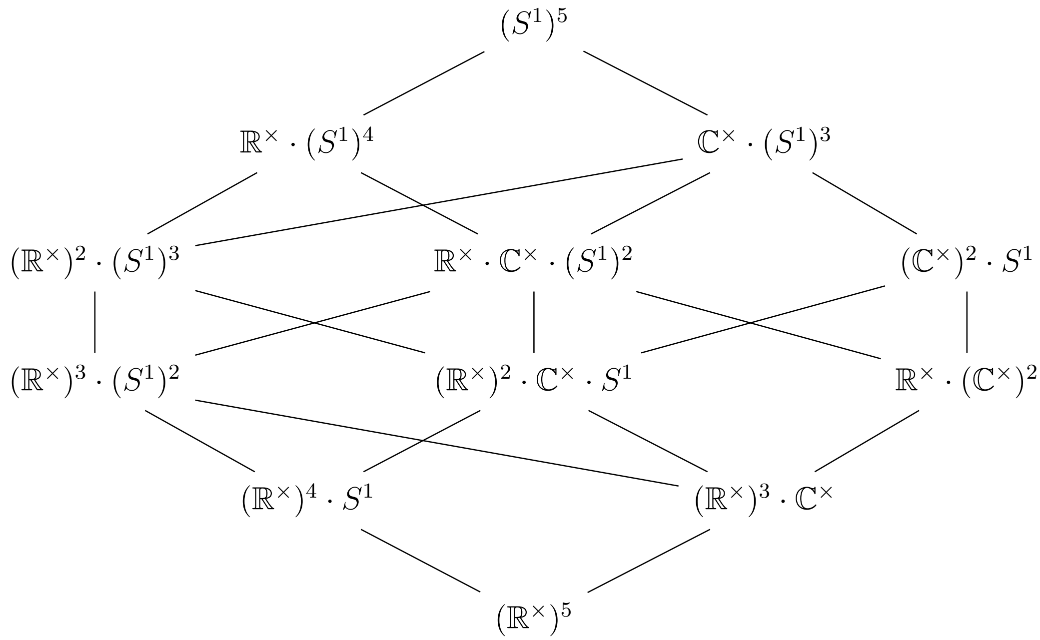


Figure 5: Lattice of Conjugacy Classes of Maximal Tori, $Sp(10, \mathbb{R})$

Next, the lattice for $Sp(6, 4)(\mathbb{R})$ is given by:

$$\begin{array}{c} (S^1)^5 \\ | \\ \mathbb{C}^\times \cdot (S^1)^3 \\ | \\ (\mathbb{C}^\times)^2 \cdot S^1 \end{array}$$

Next, the lattice for $Sp(8, 2)(\mathbb{R})$ is given by:

$$\begin{array}{c} (S^1)^5 \\ | \\ \mathbb{C}^\times \cdot (S^1)^3 \end{array}$$

Finally, the lattice for $Sp(10, 0)(\mathbb{R})$ is given by:

$$(S^1)^5$$

6 A Complete Example: $Sp(12, \mathbb{C})$

So that the interested reader may have a more direct workflow of constructing the lattice for their case of interest, we combine the ideas of the previous three sections into a simplified discussion in order to construct the lattices corresponding to the real forms of $Sp(12, \mathbb{C})$. We will discuss first the more involved lattice of the split real form and then use this result to establish those of the other real forms. We will do so by first setting up some basic ideas and, then, by establishing the representatives of the conjugacy classes and the appropriate Cayley transforms for each sequential split rank.

The relevant root system is C_6, for which the set of positive roots may be realized as

$$\Delta^+ = \{e_i \pm e_j \ ; \ 1 \leq i < j \leq 6\} \cup \{2e_k \ ; \ 1 \leq k \leq 6\}.$$

It is an exercise in linear algebra to establish the sets of orthogonal roots. The reader may find it helpful to follow the subsequent discussion by having these at hand, but we will not utilize them explicitly.

To begin, for the split real form $Sp(12, \mathbb{R})$, we use the results of section 3.6 to observe that the orders of the sets of conjugacy classes of each split rank are as described in table 18.

	Number of Conj. Classes of Max. Tori of Split Rank							
	0	1	2	3	4	5	6	N
$Sp(12, \mathbb{R})$	1	2	3	4	3	2	1	16

Table 18: Number of Conjugacy Classes in $Sp(12, \mathbb{R})$

From this, and by further recalling the standard representatives of the anisotropic and split tori, we can establish the basic structure of our lattice as

$$
\begin{array}{ccccccc@{\qquad\qquad}l}
 & & & (S^1)^6 & & & & \text{split rank 0} \\
 & & \circ & & \circ & & & \text{split rank 1} \\
 & \circ & & \circ & & \circ & & \text{split rank 2} \\
\circ & & \circ & & \circ & & \circ & \text{split rank 3} \\
 & \circ & & \circ & & \circ & & \text{split rank 4} \\
 & & \circ & & \circ & & & \text{split rank 5} \\
 & & & (\mathbb{R}^\times)^6 & & & & \text{split rank 6}
\end{array}
$$

Furthermore, the Cayley transforms associated to the set of strongly orthogonal long roots provide a path of descent for which each sequential actions maps a copy of S^1 to a copy of $\mathbb{R}^\times$. As such, before beginning with any gritty details, we can establish our lattice as

$$
\begin{tikzcd}[column sep=small]
 & & & (S^1)^6 \arrow[dl, no head] & & & \\
 & & \mathbb{R}^\times \cdot (S^1)^5 \arrow[dl, no head] & & \circ & & \\
 & (\mathbb{R}^\times)^2 \cdot (S^1)^4 \arrow[dl, no head] & & \circ & & \circ & \\
(\mathbb{R}^\times)^3 \cdot (S^1)^3 \arrow[dr, no head] & & \circ & & \circ & & \circ \\
 & (\mathbb{R}^\times)^4 \cdot (S^1)^2 \arrow[dr, no head] & & \circ & & \circ & \\
 & & (\mathbb{R}^\times)^5 \cdot S^1 \arrow[dr, no head] & & \circ & & \\
 & & & (\mathbb{R}^\times)^6 & & &
\end{tikzcd}
$$

Note that each of these representatives both agree with the appropriate split rank and satisfy the necessary 12×12 structure of the associated matrices. While we will repeat some of these details as our discussion progresses, because this path of descent exists for every rank, we include this as a reasonable starting point should the reader need to quickly find representations of each split rank.

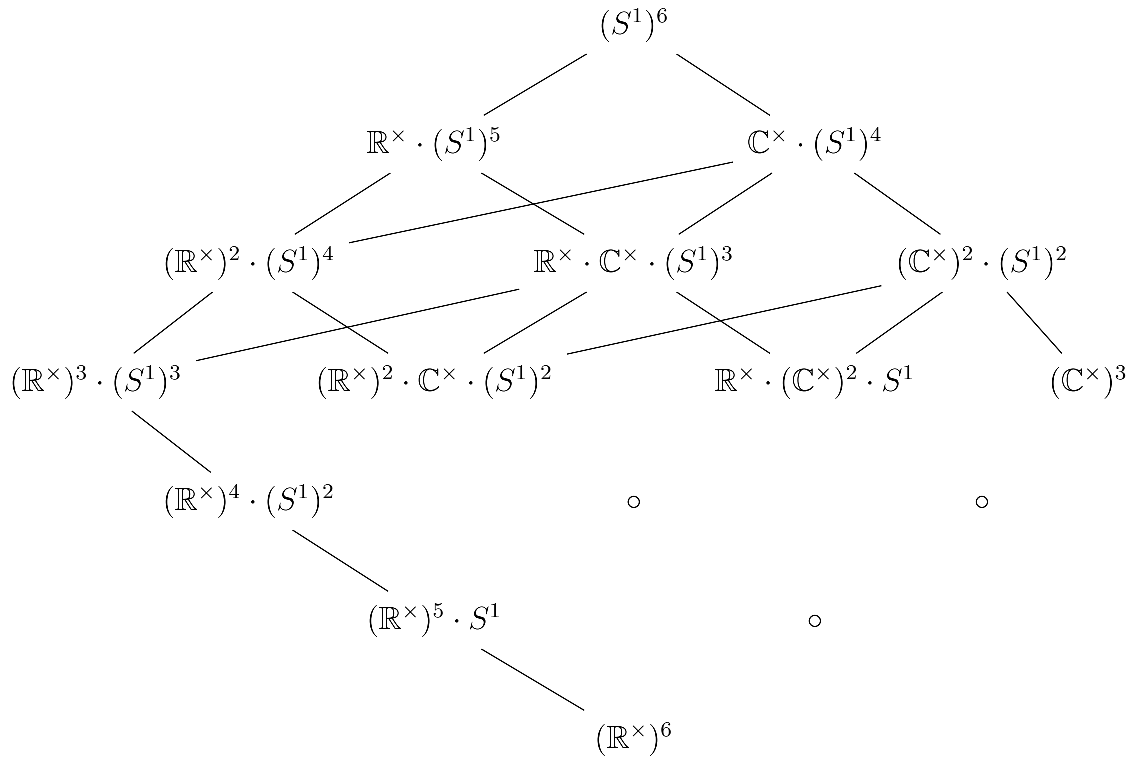
Beginning again at $(S^1)^6$, we may choose to map by a Cayley transform associated to either a long or a short simple root. That associated with the long root will map to $\mathbb{R}^\times \cdot (S^1)^5$ and that associated with the short root based on the difference of basis elements will map to $\mathbb{C}^\times \cdot (S^1)^4$, both of which are split rank one. As such, our lattice is now

$$
\begin{tikzcd}[column sep=small]
 & & & (S^1)^6 \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & \mathbb{R}^\times \cdot (S^1)^5 \arrow[dl, dash] & & \mathbb{C}^\times \cdot (S^1)^4 & & \\
 & (\mathbb{R}^\times)^2 \cdot (S^1)^4 \arrow[dl, dash] & & \circ & & \circ & \\
(\mathbb{R}^\times)^3 \cdot (S^1)^3 \arrow[dr, dash] & & \circ & & \circ & & \circ \\
 & (\mathbb{R}^\times)^4 \cdot (S^1)^2 \arrow[dr, dash] & & \circ & & \circ & \\
 & & (\mathbb{R}^\times)^5 \cdot S^1 \arrow[dr, dash] & & \circ & & \\
 & & & (\mathbb{R}^\times)^6 & & &
\end{tikzcd}
$$

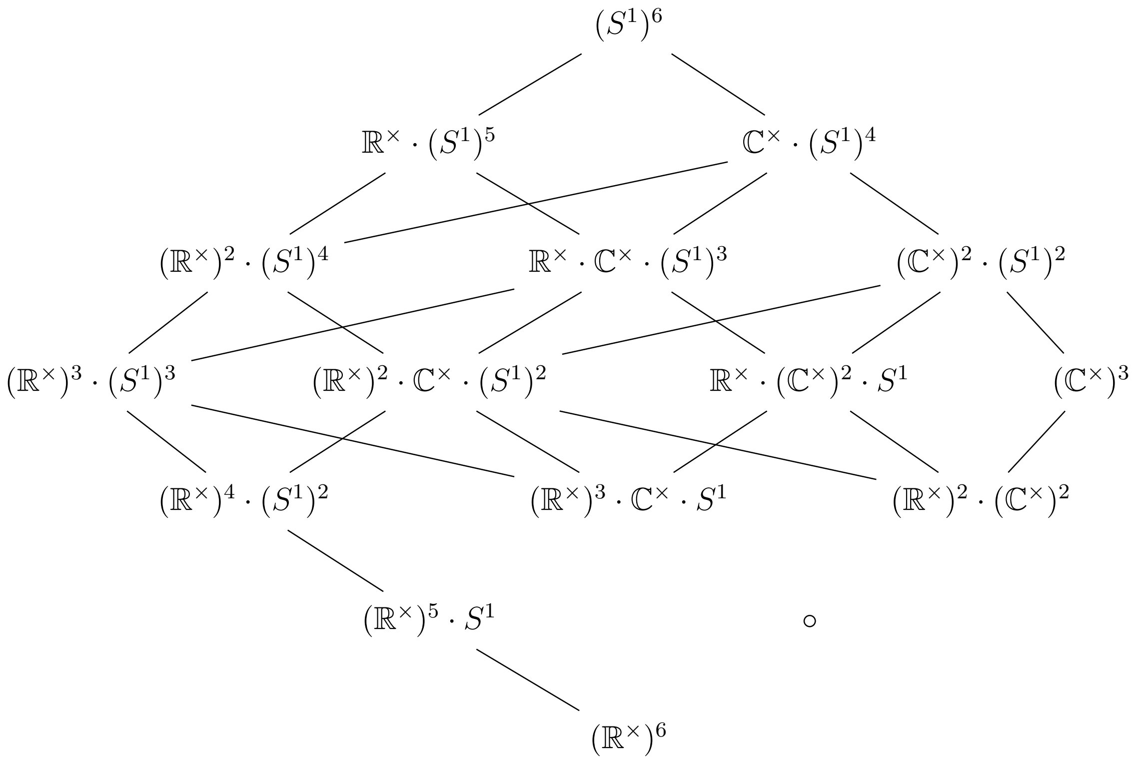
Next, observe that the only possibilities for representatives of conjugacy classes of split rank two are given by $(\mathbb{R}^\times)^2 \cdot (S^1)^4$, $\mathbb{R}^\times \cdot \mathbb{C}^\times \cdot (S^1)^3$, and $(\mathbb{C}^\times)^2 \cdot (S^1)^2$. $\mathbb{R}^\times \cdot (S^1)^5$ may map to the first two of these by Cayley transforms associated to a long and short (of difference of basis elements) root, respectively. $\mathbb{C}^\times \cdot (S^1)^4$ may map to all three of these by Cayley transforms associated to a short (of sum of basis elements), a long, and a short (difference), respectively. As such, our lattice is now

$$\begin{tikzcd}[column sep=small]
 & & & (S^1)^6 \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & \mathbb{R}^\times \cdot (S^1)^5 \arrow[dl, dash] \arrow[dr, dash] & & \mathbb{C}^\times \cdot (S^1)^4 \arrow[dlll, dash] \arrow[dl, dash] \arrow[dr, dash] & & \\
 & (\mathbb{R}^\times)^2 \cdot (S^1)^4 \arrow[dl, dash] & & \mathbb{R}^\times \cdot \mathbb{C}^\times \cdot (S^1)^3 & & (\mathbb{C}^\times)^2 \cdot (S^1)^2 & \\
(\mathbb{R}^\times)^3 \cdot (S^1)^3 \arrow[dr, dash] & & \circ & & \circ & & \circ \\
 & (\mathbb{R}^\times)^4 \cdot (S^1)^2 \arrow[dr, dash] & & \circ & & \circ & \\
 & & (\mathbb{R}^\times)^5 \cdot S^1 \arrow[dr, dash] & & \circ & & \\
 & & & (\mathbb{R}^\times)^6 & & &
\end{tikzcd}$$

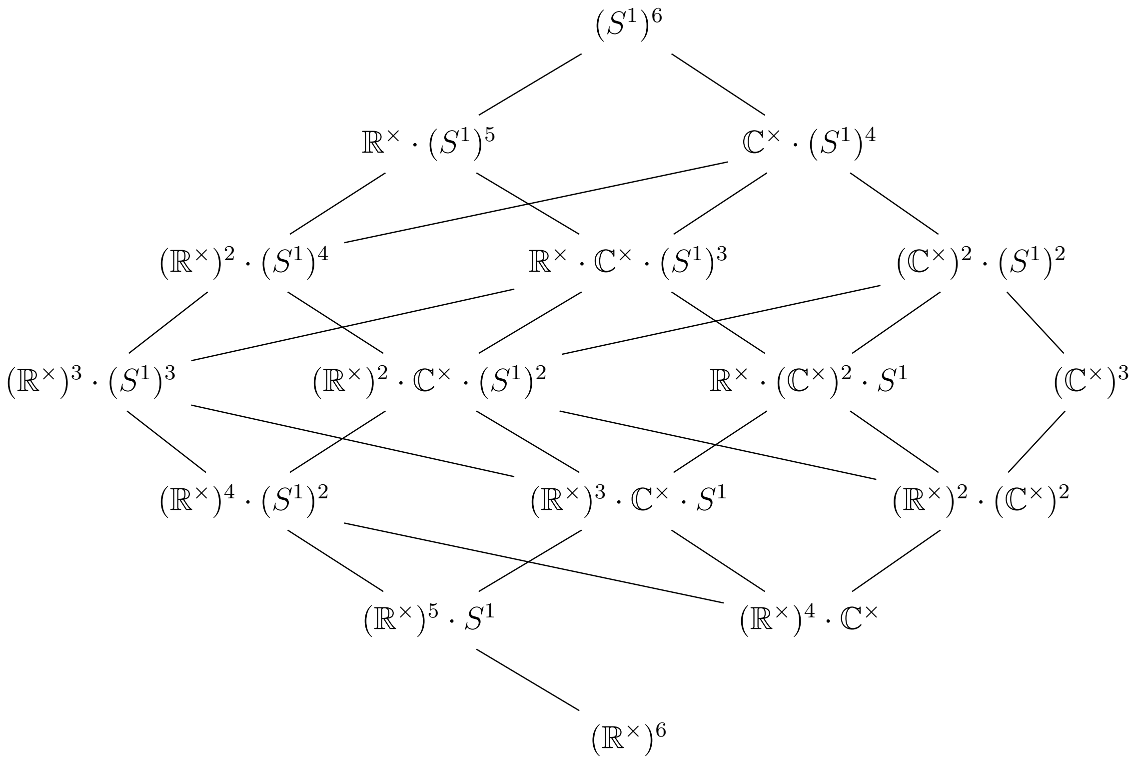
Next, the only possibilities for representatives of split rank 3 are given by $(\mathbb{R}^\times)^3 \cdot (S^1)^3$, $(\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times \cdot (S^1)^2$, $\mathbb{R}^\times \cdot (\mathbb{C}^\times)^2 \cdot S^1$, and $(\mathbb{C}^\times)^3$. $(\mathbb{R}^\times)^2 \cdot (S^1)^4$ may map to the first two of these by Cayley transforms associated to a long and a short (difference) root, respectively. $\mathbb{R}^\times \cdot \mathbb{C}^\times \cdot (S^1)^3$ may map the first three of these by Cayley transforms associated to a short (sum), a long, and a short (difference) root, respectively. And, $(\mathbb{C}^\times)^2 \cdot (S^1)^2$ may map to the last three of these by Cayley transforms associated to a short (sum), a long, and a short (difference) root, respectively. As such, our lattice is now



Next, the representatives of split rank 4 are given by $(\mathbb{R}^\times)^4 \cdot (S^1)^2$, $(\mathbb{R}^\times)^3 \cdot \mathbb{C}^\times \cdot S^1$, and $(\mathbb{R}^\times)^2 \cdot (\mathbb{C}^\times)^2$. $(\mathbb{R}^\times)^3 \cdot (S^1)^3$ may map to the first two of these by Cayley transforms associated to a long and a short (difference) root, respectively. $(\mathbb{R}^\times)^2 \cdot \mathbb{C}^\times \cdot (S^1)^2$ may map to all three of these by Cayley transforms associated to a short (sum), a long, and a short (difference) root, respectively. $\mathbb{R}^\times \cdot (\mathbb{C}^\times)^2 \cdot S^1$ may map to the last two of these by Cayley transforms associated to a short (sum) and a long root, respectively. And, $(\mathbb{C}^\times)^3$ may map to the last of these by Cayley transforms associated to a short (sum) root. As such, our lattice is now



Next, the representatives of split rank 5 are given by $(\mathbb{R}^\times)^5 \cdot S^1$ and $(\mathbb{R}^\times)^4 \cdot \mathbb{C}^\times$. $(\mathbb{R}^\times)^4 \cdot (S^1)^2$ may map to both of these by Cayley transforms associated a long and a short (difference) root, respectively. $(\mathbb{R}^\times)^3 \cdot \mathbb{C}^\times \cdot S^1$ may map to both of these by Cayley transforms associated a short (sum) and a long root, respectively. And, $(\mathbb{R}^\times)^2 \cdot (\mathbb{C}^\times)^2$ may map to the last of these by Cayley transforms associated a short (sum) root. As such, our lattice is now



Finally, the complete lattice for the split real form $Sp(12, \mathbb{R})$ is given in figure 6.

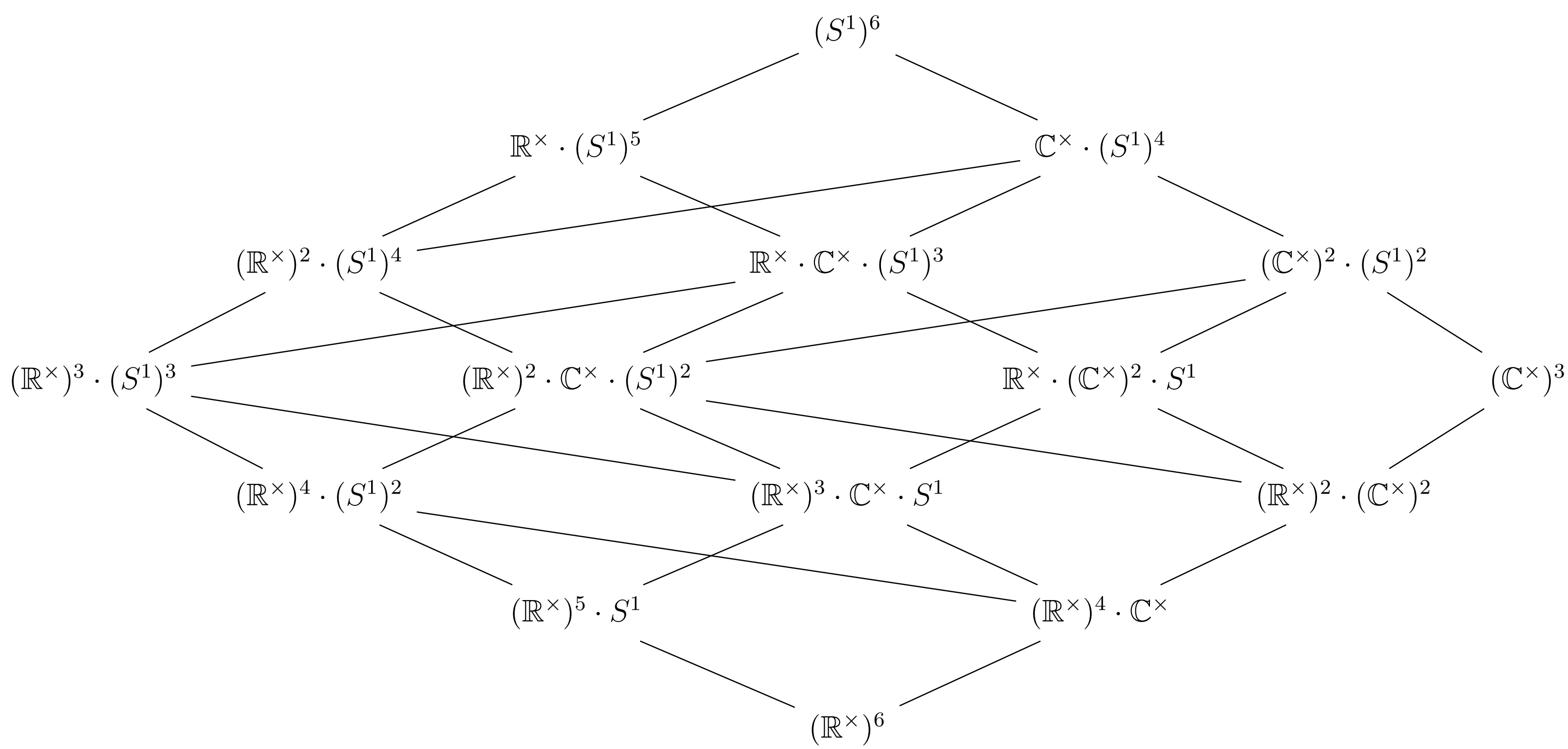


Figure 6: Lattice of Conjugacy Classes of Maximal Tori, $Sp(12, \mathbb{R})$

With the split form lattice established, we turn our attention to the lattices of the other real forms. Based on the results of section 3.6, the four real forms that are not split are $Sp(6,6)(\mathbb{R})$, $Sp(8,4)(\mathbb{R})$, $Sp(10,2)(\mathbb{R})$, and $Sp(12,0)(\mathbb{R})$. As we have done in previous discussions, by choosing the representatives of conjugacy classes that do not include any copies of $\mathbb{R}^\times$ and by identifying the various paths of descent, the associated lattices are given by:

$$
\begin{array}{cccc}
\underline{Sp(6,6)(\mathbb{R})} & \underline{Sp(8,4)(\mathbb{R})} & \underline{Sp(10,2)(\mathbb{R})} & \underline{Sp(12,0)(\mathbb{R})} \\
 & & & \\
(S^1)^6 & (S^1)^6 & (S^1)^6 & (S^1)^6 \\
\Big| & \Big| & \Big| & \\
\mathbb{C}^\times \cdot (S^1)^4 & \mathbb{C}^\times \cdot (S^1)^4 & \mathbb{C}^\times \cdot (S^1)^4 & \\
\Big| & \Big| & & \\
(\mathbb{C}^\times)^2 \cdot (S^1)^2 & (\mathbb{C}^\times)^2 \cdot (S^1)^2 & & \\
\Big| & & & \\
(\mathbb{C}^\times)^3 & & &
\end{array}
$$

7 $\mathcal{D}(T)$, $\mathcal{E}(T)$, and Extended Groups

7.1 General Notions in the Unitary Case

We begin by summarizing the ideas of Shelstad in [She08a] to establish the general notions of these objects in the unitary case. Let $\mathbb{U}(p,q)$, with $p+q=n$, be the real unitary groups of $n \times n$ matrices. Set $G^j = \mathbb{U}(n-j,j)$, so that G^0 is the compact form and G^m is the quasi-split form, with $m = \lfloor \frac{n}{2} \rfloor$.

Fix a group G^j. Given a regular elliptic stable conjugacy class in $G^j(\mathbb{R})$, fix an element δ of $T(\mathbb{R})$ in this class. Then, $g^{-1}\delta g$, for $g \in GL(n,\mathbb{C})$, is also in this class if and only if $g\sigma(g)^{-1}$ is in $T(\mathbb{C})$. As such, the $G^j(\mathbb{R})$-conjugacy classes are parametrized by the set $\mathcal{D}_j(T)$ of elements in $H^1(\Gamma, T)$ that become trivial in $H^1(\Gamma, G^j)$ under the map determined by inclusion. Note that this is a set and not a group, and its elements vary depending on the chosen form. The set is trivial for the compact form and is largest for the quasi-split form. Furthermore, each $\mathcal{D}_j(T)$ is contained in the group

$$\mathcal{E}(T) = Im(H^1(\Gamma, T_{sc}) \to H^1(\Gamma, T)),$$

where T_{sc} is the maximal torus anisotropic over $\mathbb{R}$ in the simply-connected covering of the derived group of G. There is a partition for this group, $\mathcal{E}(T) = \sqcup_j \mathcal{D}_j(T)$. To describe this partition, identify $\mathcal{D}_j(T)$ with its image in $\mathcal{E}(T)$ under the twist from G^j to G^m. Then, $\mathcal{D}_j^{'}(T)$ is the translate of $\mathcal{D}_j(T)$ by the class of the twisting cocycle x_σ^j or $x_\sigma^{j,1}$, and $\mathcal{D}_j^{''}(T)$ is defined relative to the second twist and cocycle, both of which are explained in more detail below. And, the size of this group and this set are given by $|\mathcal{E}(T)| = 2^{n-1}$ and $|\mathcal{D}_j(T)| = \binom{n}{j}$, respectively.

Then, Tate-Nakayama duality is used to identify $\mathcal{E}(T)$ with

$$Im(H^{-1}(\Gamma, X_*(T_{sc})) \to H^{-1}(\Gamma, X_*(T))).$$

The roots of T are given by

$$t_k - t_l : diag(t_1, ..., t_n) \to \frac{t_k}{t_l},$$

and corresponding coroots are given by

$$e_k - e_l : t \to diag(t_1, ..., t_n),$$

where $t_k = t = t_l^{-1}$ and all other entries are 1. Each element of $\mathcal{E}(T)$ can be written as a sum of coroots

$$(e_{k_1} - e_{l_1}) + (e_{k_2} - e_{l_2}) + \cdots + (e_{k_r} - e_{l_r})$$

modulo $2X_*(T_{sc})$. The root $t_k - t_l$ is compact in G^j if the k^{th} and l^{th} diagonal entries in the matrix $I_{n-j,j}$ are the same and noncompact otherwise. If $t_k - t_l$ is compact, then the Weyl reflection $\omega_{k,l}$ relative to $t_k - t_l$ determines the trivial element of $\mathcal{D}_j(T)$. If $t_k - t_l$ is noncompact, then $\omega_{k,l}$ determines the nontrivial element $e_k - e_l$ of $\mathcal{D}_j(T)$.

These ideas are a consequence of her construction of the extended groups, which we now summarize. In the unitary case, there are two related constructions for n odd or even. First, take n to be odd. Then $\mathbf{G}$ is given by $G^0 \sqcup G^1 \sqcup \cdots \sqcup G^m$ with twists $\psi_{i,j} : G^i \to G^j$ each equal to the identity map. Additionally, take $\psi : G^m \to G^*$, then there is the twist $\psi \circ \psi_{j,m} = \psi_j : G^j \to G^*$. Specify a 1-cocycle of Γ in a group X by an element $x_\sigma \in X$ such that $x_\sigma \sigma(x_\sigma) = 1$. Then $\psi_j \sigma(\psi_j)^{-1}$ is equivalent to conjugation by $\psi(x_\sigma^j)$, where $x_\sigma^m = I$ and x_σ^j is the cocycle $diag(1, \cdots, 1, -1, \cdots, -1, 1, \cdots, 1)$, for which the change of terms occur at the $m+1$ and $n-j$ positions, in G^m.

Next, take n even. As in the odd case, there are $m+1$ cocycles $\psi(x_\sigma^j)$ in G^*, for which the negative entries in x_σ^j begin at the $m+1$ position. If $m-j$ is even and $j < m$, then there are two distinct classes, $\pm\psi(x_\sigma^j)$. When $j = m$, these two classes are trivial. Hence, we have $\mathbf{G}$ is given by $G^m \sqcup G^{m-2} \sqcup G^{m-2} \sqcup \cdots$. The remaining inner forms establish $\mathbf{G}' = G^{m-1} \sqcup G^{m-1} \sqcup G^{m-3} \sqcup G^{m-3} \sqcup \cdots$.

7.2 Details of Relevant Unitary Cases

Here we establish the details of the previous section for those cases that are relevant to our work.

rank 1:

Corresponding to $\mathbb{U}(2,0)$, we have

$$\mathcal{D}_0(T) = \{0\}.$$

Adjusting this, we take

$$\mathcal{D}_0'(T) = e_1 - e_2 + \mathcal{D}_0(T).$$

As such, for the rank 1 case

$$\mathcal{E}(T) = \mathcal{D}_0(T) \sqcup \mathcal{D}_0'(T) \qquad (2 = 1 + 1)$$

And, the extended group is given by

$$G' = \mathbb{U}(2,0) \sqcup \mathbb{U}(2,0)$$

rank 2:

Corresponding to $\mathbb{U}(4,0)$, we have

$$\mathcal{D}_0(T) = \{0\}.$$

Corresponding to $\mathbb{U}(2,2)$, we have

$$\mathcal{D}_2(T) = \{0,\ e_1 - e_4,\ e_1 - e_3,\ e_2 - e_4,\ e_2 - e_3,\ e_1 - e_4 + e_2 - e_3\}$$

Adjusting the above, we take

$$\mathcal{D}_0'(T) = e_1 - e_2 + \mathcal{D}_0(T).$$

$$\mathcal{D}_0''(T) = e_3 - e_4 + \mathcal{D}_0(T).$$

As such, for the rank 2 case

$$\mathcal{E}(T) = \mathcal{D}_0'(T) \sqcup \mathcal{D}_0''(T) \sqcup \mathcal{D}_2(T) \qquad (8 = 1 + 1 + 6)$$

And, the extended group is given by

$$G = \mathbb{U}(4,0) \sqcup \mathbb{U}(4,0) \sqcup \mathbb{U}(2,2)$$

rank 3:

Corresponding to $\mathbb{U}(6,0)$, we have

$$\mathcal{D}_0(T) = \{0\}.$$

Corresponding to $\mathbb{U}(4,2)$, we have

$$\begin{aligned}\mathcal{D}_2(T) = \{0,\ & e_1 - e_5,\ e_1 - e_6,\ e_2 - e_5,\ e_2 - e_6,\ e_3 - e_5,\ e_3 - e_6,\ e_4 - e_5,\ e_4 - e_6,\\ & e_1 - e_5 + e_3 - e_6,\ e_1 - e_5 + e_4 - e_6,\ e_1 - e_5 + e_2 - e_6,\\ & e_2 - e_5 + e_3 - e_6,\ e_2 - e_5 + e_4 - e_6,\ e_3 - e_5 + e_4 - e_6,\ \}\end{aligned}$$

Adjusting the above, we take

$$\mathcal{D}_0'(T) = e_1 - e_2 + \mathcal{D}_0(T).$$

$$\mathcal{D}_0''(T) = e_5 - e_6 + \mathcal{D}_0(T).$$

$$\begin{aligned}\mathcal{D}_2'(T) = e_3 - e_4 + \{0,\ & e_1 - e_3,\ e_1 - e_4,\ e_1 - e_5,\ e_1 - e_6,\\ & e_2 - e_3,\ e_2 - e_4,\ e_2 - e_5,\ e_2 - e_6,\\ & e_1 - e_3 + e_2 - e_4,\ e_1 - e_3 + e_2 - e_5,\ e_1 - e_3 + e_2 - e_6,\\ & e_1 - e_4 + e_2 - e_5,\ e_1 - e_4 + e_2 - e_6,\ e_1 - e_5 + e_2 - e_6,\ \}\end{aligned}$$

As such, for the rank 3 case

$$\mathcal{E}(T) = \mathcal{D}_0'(T) \sqcup \mathcal{D}_0''(T) \sqcup \mathcal{D}_2'(T) \sqcup \mathcal{D}_2(T)$$
$$(32 = 1 + 1 + 15 + 15)$$

And, the extended group is given by

$$G' = \mathbb{U}(6,0) \sqcup \mathbb{U}(6,0) \sqcup \mathbb{U}(4,2) \sqcup \mathbb{U}(4,2)$$

rank 4:

Corresponding to $\mathbb{U}(8,0)$, we have

$$\mathcal{D}_0(T) = \{0\}.$$

Corresponding to $\mathbb{U}(6,2)$, we have

$$\begin{aligned}\mathcal{D}_2(T) = \{0,\ & e_1-e_7,\ e_2-e_7,\ e_3-e_7,\ e_4-e_7,\ e_5-e_7,\ e_6-e_7,\\ & e_1-e_8,\ e_2-e_8,\ e_3-e_8,\ e_4-e_8,\ e_5-e_8,\ e_6-e_8,\\ & e_1-e_7+e_2-e_8,\ e_1-e_7+e_3-e_8,\ e_1-e_7+e_4-e_8,\ e_1-e_7+e_5-e_8,\\ & e_1-e_7+e_6-e_8,\\ & e_2-e_7+e_3-e_8,\ e_2-e_7+e_4-e_8,\ e_2-e_7+e_5-e_8,\ e_2-e_7+e_6-e_8,\\ & e_3-e_7+e_4-e_8,\ e_3-e_7+e_5-e_8,\ e_3-e_7+e_6-e_8,\\ & e_4-e_7+e_5-e_8,\ e_4-e_7+e_6-e_8,\\ & e_5-e_7+e_6-e_8,\ \}\end{aligned}$$

Corresponding to $\mathbb{U}(4,4)$, we have

$$
\begin{aligned}
\mathcal{D}_4(T) = \{0,\ & e_1-e_5,\ e_1-e_6,\ e_1-e_7,\ e_1-e_8,\ e_2-e_5,\ e_2-e_6,\ e_2-e_7,\ e_2-e_8,\\
& e_3-e_5,\ e_3-e_6,\ e_3-e_7,\ e_3-e_8,\ e_4-e_5,\ e_4-e_6,\ e_4-e_7,\ e_4-e_8,\\
& e_1-e_5+e_2-e_6,\ e_1-e_5+e_2-e_7,\ e_1-e_5+e_2-e_8,\ e_1-e_5+e_3-e_6,\\
& \quad e_1-e_5+e_3-e_7,\ e_1-e_5+e_3-e_8,\ e_1-e_5+e_4-e_6,\ e_1-e_5+e_4-e_7,\\
& \quad e_1-e_5+e_4-e_8,\\
& e_1-e_6+e_2-e_7,\ e_1-e_6+e_2-e_8,\ e_1-e_6+e_3-e_7,\ e_1-e_6+e_3-e_8,\\
& \quad e_1-e_6+e_4-e_7,\ e_1-e_6+e_4-e_8,\\
& e_1-e_7+e_2-e_8,\ e_1-e_7+e_3-e_8,\ e_1-e_7+e_4-e_8,\\
& e_2-e_5+e_3-e_6,\ e_2-e_5+e_3-e_7,\ e_2-e_5+e_3-e_8,\ e_2-e_5+e_4-e_6,\\
& \quad e_2-e_5+e_4-e_7,\ e_2-e_5+e_4-e_8,\\
& e_2-e_6+e_3-e_7,\ e_2-e_6+e_3-e_8,\ e_2-e_6+e_4-e_7,\ e_2-e_6+e_4-e_8,\\
& e_2-e_7+e_3-e_8,\ e_2-e_7+e_4-e_8,\\
& e_3-e_5+e_4-e_6,\ e_3-e_5+e_4-e_7,\ e_3-e_5+e_4-e_8,\\
& e_3-e_6+e_4-e_7,\ e_3-e_6+e_4-e_8,\ e_3-e_7+e_4-e_8,\\
& e_1-e_5+e_2-e_6+e_3-e_7,\ e_1-e_5+e_2-e_6+e_3-e_8,\\
& e_1-e_5+e_2-e_6+e_4-e_7,\ e_1-e_5+e_2-e_6+e_4-e_8,\\
& e_1-e_5+e_2-e_7+e_3-e_8,\ e_1-e_5+e_2-e_7+e_4-e_8,\\
& e_1-e_5+e_3-e_6+e_4-e_7,\ e_1-e_5+e_3-e_6+e_4-e_8,\\
& e_1-e_5+e_3-e_7+e_4-e_8,\ e_1-e_6+e_2-e_7+e_3-e_8,\\
& e_1-e_6+e_2-e_7+e_4-e_8,\ e_1-e_6+e_3-e_7+e_4-e_8,\\
& e_2-e_5+e_3-e_6+e_4-e_7,\ e_2-e_5+e_3-e_6+e_4-e_8,\\
& e_2-e_5+e_3-e_7+e_4-e_8,\ e_2-e_6+e_3-e_7+e_4-e_8,\\
& e_1-e_5+e_2-e_6+e_3-e_7+e_4-e_8\ \}
\end{aligned}
$$

Adjusting the above, we take

$$\mathcal{D}_0'(T) = e_1 - e_2 + \mathcal{D}_0(T).$$

$$\mathcal{D}_0^{''}(T) = e_7 - e_8 + \mathcal{D}_0(T).$$

$$\begin{aligned}\mathcal{D}_2^{'}(T) = e_3 - e_6 + \{0,\ & e_1 - e_3,\ e_1 - e_4,\ e_1 - e_5,\ e_1 - e_6,\ e_1 - e_7,\ e_1 - e_8, \\ & e_2 - e_3,\ e_2 - e_4,\ e_2 - e_5,\ e_2 - e_6,\ e_2 - e_7,\ e_2 - e_8, \\ & e_1 - e_3 + e_2 - e_4,\ e_1 - e_3 + e_2 - e_5,\ e_1 - e_3 + e_2 - e_6, \\ & e_1 - e_3 + e_2 - e_7,\ e_1 - e_3 + e_2 - e_8,\ e_1 - e_4 + e_2 - e_5, \\ & e_1 - e_4 + e_2 - e_6,\ e_1 - e_4 + e_2 - e_7,\ e_1 - e_4 + e_2 - e_8, \\ & e_1 - e_5 + e_2 - e_6,\ e_1 - e_5 + e_2 - e_7,\ e_1 - e_5 + e_2 - e_8, \\ & e_1 - e_6 + e_2 - e_7,\ e_1 - e_6 + e_2 - e_8,\ e_1 - e_7 + e_2 - e_8\ \}\end{aligned}$$

$$\begin{aligned}\mathcal{D}_2^{''}(T) = e_4 - e_5 + \{0,\ & e_1 - e_3,\ e_1 - e_4,\ e_1 - e_5,\ e_1 - e_6,\ e_1 - e_7,\ e_1 - e_8, \\ & e_2 - e_3,\ e_2 - e_4,\ e_2 - e_5,\ e_2 - e_6,\ e_2 - e_7,\ e_2 - e_8, \\ & e_1 - e_3 + e_2 - e_4,\ e_1 - e_3 + e_2 - e_5,\ e_1 - e_3 + e_2 - e_6, \\ & e_1 - e_3 + e_2 - e_7,\ e_1 - e_3 + e_2 - e_8,\ e_1 - e_4 + e_2 - e_5, \\ & e_1 - e_4 + e_2 - e_6,\ e_1 - e_4 + e_2 - e_7,\ e_1 - e_4 + e_2 - e_8, \\ & e_1 - e_5 + e_2 - e_6,\ e_1 - e_5 + e_2 - e_7,\ e_1 - e_5 + e_2 - e_8, \\ & e_1 - e_6 + e_2 - e_7,\ e_1 - e_6 + e_2 - e_8,\ e_1 - e_7 + e_2 - e_8\ \}\end{aligned}$$

As such, for the rank 4 case

$$\mathcal{E}(T) = \mathcal{D}_0^{'}(T) \sqcup \mathcal{D}_0^{''}(T) \sqcup \mathcal{D}_2^{'}(T) \sqcup \mathcal{D}_2^{''}(T) \sqcup \mathcal{D}_4(T)$$

$$(128 = 1 + 1 + 28 + 28 + 70)$$

And, the extended group is given by

$$G = \mathbb{U}(8,0) \sqcup \mathbb{U}(8,0) \sqcup \mathbb{U}(6,2) \sqcup \mathbb{U}(6,2) \sqcup \mathbb{U}(4,4)$$

rank 5:

Because of the size of the underlying sets, in this case, we directly state the results, which are:

$$\mathcal{E}(T) = \mathcal{D}_0^{'}(T) \sqcup \mathcal{D}_0^{''}(T) \sqcup \mathcal{D}_2^{'}(T) \sqcup \mathcal{D}_2^{''}(T) \sqcup \mathcal{D}_4^{'}(T) \sqcup \mathcal{D}_4(T)$$
$$(512 = 1 + 1 + 45 + 45 + 210 + 210)$$

And, the extended group is given by

$$G' = \mathbb{U}(10,0) \sqcup \mathbb{U}(10,0) \sqcup \mathbb{U}(8,2) \sqcup \mathbb{U}(8,2) \sqcup \mathbb{U}(6,4) \sqcup \mathbb{U}(6,4)$$

7.3 Extended Groups for Groups of Type C_r

The symplectic case slightly differs from the unitary case. First, for groups of type C, the split form realizes $\mathcal{E}(T) = \mathcal{D}(T)$. We see this below in that the extended group may simply be taken to be the split form. Secondly, there are three types of roots:

$$2t_k : diag(t_1, ..., t_n) \to (t_k)^2$$

$$t_k - t_l : diag(t_1, ..., t_n) \to \frac{t_k}{t_l}$$

$$t_k + t_l : diag(t_1, ..., t_n) \to t_k t_l.$$

As we have seen in section 5, for the nonsplit forms, the long roots are always compact and the noncompact roots will always come in the pairs $t_k \pm t_l$. Specifically, this implies that it cannot be the case that $t_k - t_l$ is noncompact and $t_k + t_l$ is compact, and vice versa. In this context, we see this idea realized in that $t_k \pm t_l$ will both depend on the same k^{th} and l^{th} diagonal entries. As such, they will realize the same element of the group $\mathcal{E}(T)$, and the results for the unitary case are the same for the symplectic case. Finally, the extended groups, for nonsplit forms, follow the general pattern that the odd rank cases will have two copies of each real form and the even rank cases will have one copy of G^r and two copies the other real forms. The results for our specific cases are given by:

$$
\begin{aligned}
\text{rank 1,}\quad & G = Sp(2,\mathbb{R}) \qquad \text{or}\\
& G = Sp(2,0)(\mathbb{R}) \sqcup Sp(2,0)(\mathbb{R})\\
\text{rank 2,}\quad & G = Sp(4,\mathbb{R}) \qquad \text{or}\\
& G = Sp(4,0)(\mathbb{R}) \sqcup Sp(4,0)(\mathbb{R}) \sqcup Sp(2,2)(\mathbb{R})\\
\text{rank 3,}\quad & G = Sp(6,\mathbb{R}) \qquad \text{or}\\
& G = Sp(6,0)(\mathbb{R}) \sqcup Sp(6,0)(\mathbb{R}) \sqcup Sp(4,2)(\mathbb{R}) \sqcup Sp(4,2)(\mathbb{R})\\
\text{rank 4,}\quad & G = Sp(8,\mathbb{R}) \qquad \text{or}\\
& G = Sp(8,0)(\mathbb{R}) \sqcup Sp(8,0)(\mathbb{R}) \sqcup Sp(6,2)(\mathbb{R}) \sqcup Sp(6,2)(\mathbb{R}) \sqcup Sp(4,4)(\mathbb{R})\\
\text{rank 5,}\quad & G = Sp(10,\mathbb{R}) \qquad \text{or}\\
& G = Sp(10,0)(\mathbb{R}) \sqcup Sp(10,0)(\mathbb{R}) \sqcup Sp(8,2)(\mathbb{R}) \sqcup Sp(8,2)(\mathbb{R})\\
& \qquad\qquad \sqcup Sp(6,4)(\mathbb{R}) \sqcup Sp(6,4)(\mathbb{R})
\end{aligned}
$$

8 Towards Stable-Stable Transfer

Following notions Shelstad establishes in [Shear], we use this section to discuss some results towards stable-stable transfer involving symplectic groups.

The notion of transfer is at the heart of Langlands functoriality. Broadly speaking, algebraic functoriality is the idea that homomorphisms between dual groups induce a transfer of the irreducible representations of these groups. For reductive groups, the classification of irreducible representations is sufficient to classify all algebraic representations. Langlands functoriality, specifically in the local setting, claims that homorphisms of L-groups induce a transfer of the isomorphism classes of irreducible automorphic representations. Such L-homomorphisms lead to a geometric transfer of orbital integrals, and any geometric transfer uniquely determines a dual transfer of distributions. While it is reasonable to assume that we may instead start with a transfer of characters that would yield a uniquely determined transfer of orbital integrals, to do so in practice would require a deep understanding of the representation theory of the group. The constructive methods on the orbital integrals are more accessible and lead to simplified arguments on the spectral side.

Endoscopic transfer, for a given reductive group G, instantiated the ideas of functoriality by fixing a well-chosen quasi-split group, the endoscopic group H, and considers an injective L-homomorphism ${}^LH \to {}^LG$. It is crucial that, in endoscopy, the lattice of stable conjugacy classes play a dominant role in various constructions and proofs of results. Notably, it is mappings of the lattices, stemming from the L-group construction, that establish the notion of Cartan subgroups in an endoscopic group H originating in G, giving a relation between the regular elements of both groups. Then, the parameters defined by classes of adjacent tori establish the character theory for a transfer of orbital integrals. We also see their utility in establishing various technical details, such as how the notion of adjacency aids in understanding jump behavior and establish the smoothness of suitable variants of Harish Chandra's F_f-transform.

Ultimately, endoscopy comes in two kinds, ordinary and twisted, where the former is a special case of the latter and reached a complete theory more quickly. Because this is not a paper on endoscopy, we provide the reader with some resources for material beyond the basic framework we established in section 1. Details on standard endoscopy may be found in, for example, [She82], [K$^+$84], and [LS87], and on twisted endoscopy in [KS99]. Relevant to our group of interest, an expansive discussion of endoscopy and symplectic groups may be found in [Art13].

While endoscopic transfer allowed for substantial progress, it has at least one aspect that indicates another construction of transfer is needed. That is, for some groups, the set of conjugacy classes in the stable conjugacy classes of a strongly regular element in $G(\mathbb{R})$ has the structure of a finite abelian group, but this group structure is not uniquely determined by the stable conjugacy class. Despite this, for our purposes, endoscopic transfer has one distinguished achievement. Specifically, it indicates that orbital integrals along conjugacy classes in $G(\mathbb{R})$ can be expressed in terms of orbital integrals along stable conjugacy classes from a certain related finite collection of groups $H(\mathbb{R})$. For the attached dual transfer, if we consider the HCS-case then we know that all tempered characters on $G(\mathbb{R})$ are nicely expressed in terms of stable characters on these $H(\mathbb{R})$.

Finally, before actually coming to our case of interest, we discuss some heuristics, in a general setting, of Shelstad that lead to a statement of stable-stable transfer. In order to fix notation, take $\mathbf{\Gamma}$ to be the space of stable orbital integrals and $d\Gamma$ the measure on it. Write $O(\Gamma, f)$ for the stable orbital integral of a suitable function along the stable conjugacy class Γ, and $\widehat{O}(\Gamma, f)$ for the normalized version. Let $\mathbf{\Pi}$ denote the space of tempered packets with measure $d\Pi$, $Tr(\Pi, *)$ the stable trace for the packet Π, and $Ch(\Pi, *)$ the real analytic function on the regular semisimple elements of $G(\mathbb{R})$ that represents the stable trace, for which $\widehat{Ch}(\Pi, *)$ is the normalized version. Label our two groups of interest H and G; we do not assume that H is endoscopic for G, but we do assume that they have same rank and there is a primary datum ξ that embeds ${}^L H$ in ${}^L G$. This determines a map $\mathbf{\Pi_H} \to \mathbf{\Pi_G}$, in which we will denote the image of Π_H as $\Pi_{H\to G}$.

The heuristics are developed as follows. For each appropriate function f_G on $G(\mathbb{R})$ there exists a function $f_H = (f_G)_H$ on $H(\mathbb{R})$ such that

$$\widehat{O}(\Gamma_H, (f_G)_H) = \int_{\mathbf{\Gamma}_G} \Theta(\Gamma_H, \Gamma_G)\widehat{O}(\Gamma_G, f_G) d\Gamma_G$$

and

$$\widehat{Ch}(\Pi_{H\to G}, \Gamma_G) = \int_{\mathbf{\Gamma}_H} \widehat{Ch}(\Pi_H, \Gamma_H)\Theta(\Gamma_H, \Gamma_G) d\Gamma_H$$

for all strongly regular semisimple Γ_H and Γ_G. Then,

$$\begin{aligned}
Tr(\Pi_{H\to G}, f_G) \ \text{ given by } \quad & \int_{\mathbf{\Pi}_G} \widehat{Ch}(\Pi_{H\to G}, \Gamma_G)\widehat{O}(\Gamma_G, f_G) d\Gamma_G \\
&= \int_{\mathbf{\Pi}_G} \int_{\mathbf{\Pi}_G} \widehat{Ch}(\Pi_H, \Gamma_H)\Theta(\Gamma_H, \Gamma_G)\widehat{O}(\Gamma_G, f_G) d\Gamma_H d\Gamma_G \\
&= \int_{\mathbf{\Pi}_H} \widehat{Ch}(\Pi_H, \Gamma_H) \int_{\mathbf{\Pi}_G} \Theta(\Gamma_H, \Gamma_G)\widehat{O}(\Gamma_G, f_G) d\Gamma_G d\Gamma_H \\
&= \int_{\mathbf{\Pi}_H} \widehat{Ch}(\Pi_H, \Gamma_H)\widehat{O}(\Gamma_H, (f_G)_H) d\Gamma_H \\
&= Tr(\Pi_H, (f_G)_H),
\end{aligned}$$

which is the final stable-stable transfer formula at the level of traces. This includes the term

$$\Theta(\Gamma_H, \Gamma_G) = \int_{\mathbf{\Pi}_H} \Delta(\Gamma_H, \Pi_H)\widehat{Ch}(\Pi_{H\to G}, \Gamma_G) d\Pi_H$$

where $\Delta(\Gamma_H, \Pi_H)$ is the coefficient term that appears in the Fourier inversion of the stable orbital integral:

$$\widehat{O}(\Gamma_H, f_H) = \int_{\mathbf{\Pi}_H} \Delta(\Gamma_H, \Pi_H) Tr(\Pi_H, f_H) d\Pi_H$$

It will be helpful below that, in general, stable-stable transfer for orbital integrals may be summarized by the following three statements: $\forall f_G$, $\exists f_H$, $\forall \Gamma_H$

$$\widehat{O}(\Gamma_H, f_H) = \int_{\mathbf{\Gamma}_G} \Theta(\Gamma_H, \Gamma_G)\widehat{O}(\Gamma_G, f_G) d\Gamma_G \tag{1a}$$

where

$$\Theta(\Gamma_H, \Gamma_G) = \int_{\mathbf{\Pi}_H} \Delta(\Gamma_H, \Pi_H)\widehat{Ch}(\Pi_{H\to G}, \Gamma_G) d\Pi_H \tag{1b}$$

and

$$\widehat{O}(\Gamma_H, f_H) = \int_{\mathbf{\Pi}_H} \Delta(\Gamma_H, \Pi_H)\mathrm{Trace}(\Pi_H(f_H)) d\Pi_H \tag{1c}$$

For our context here, we consider a pair of connected reductive algebraic groups, H and G, defined over $\mathbb{R}$, and a morphism between their L-groups. We can reduce this, via arguments involving endoscopic transfer and other technical details, to considering the case in which both H and G are quasi-split over $\mathbb{R}$, where the morphism is injective. Here, it may be that H is an endoscopic group for G, but that has no bearing on the development of stable-stable transfer. A helpful starting place is to consider an H that is endoscopic only in some special cases. As such, we start by taking H to be any maximal torus over $\mathbb{R}$ in G. Specifically, we take H to be the elliptic maximal torus in Sp_2, i.e.,

$$H(\mathbb{R}) = S^1 \quad \text{and} \quad G(\mathbb{R}) = Sp(2, \mathbb{R}).$$

Here, H is abelian, each conjugacy class is a singleton, and, hence, we may write either of the terms $\Gamma_H = r(\theta)$. This implies that the orbital integral with respect to the function f_H is simply the value of the function at the point Γ_H in $H(\mathbb{R})$, i.e., $\widehat{O}(\Gamma_H, f_H) = f_H(\Gamma_H)$. In this context, (1c) is the Fourier inversion for the function f_H on the connected compact abelian group $H(\mathbb{R})$. The integral is, here, the sum over all unitary characters. In light of these observations, we may rephrase the statements in (1) as

$\forall f_G$, $\exists f_H$, $\forall \Gamma_H$

$$f_H(\Gamma_H) = \int_{\mathbf{\Gamma}_G} \Theta(\Gamma_H, \Gamma_G)\ \widehat{O}(\Gamma_G, f_G) d\Gamma_G \tag{2a}$$

where

$$\Theta(\Gamma_H, \Gamma_G) = \sum_{n \in \mathbb{Z}} e^{in\theta}\ \widehat{Ch}(\Pi_{H \to G}, \Gamma_G) \tag{2b}$$

and

$$f_H(\Gamma_H) = \sum_{\substack{\text{unitary} \\ \text{char.}}} e^{in\theta}\ \Pi_H(f_H) \tag{2c}$$

The first statement, (2a), establishes that there is way to attach to a fixed and suitable function f_G on $G(\mathbb{R})$ a smooth function f_H on $H(\mathbb{R})$. Thus, it needs to be verified that the right-hand side of (2a) is smooth at any particular point Γ_H. To do so, it is useful to explicitly inspect the elements involved.

Theorem: Let $H = S^1$ and $G = Sp(2, \mathbb{R})$. Then, for all suitable choices of f_G, there exists a function f_H such that, for all points Γ_H

$$f_H(\Gamma_H) = \int_{\mathbf{\Gamma}_G} \Theta(\Gamma_H, \Gamma_G)\ \widehat{O}(\Gamma_G, f_G) d\Gamma_G$$

and f_H is smooth.

Proof: The initial hurdle is the divergence of the Θ term. For each pair (Γ_H, Γ_G), the expression $\Theta(\Gamma_H, \Gamma_G)$ is given by the series

$$\Theta(\Gamma_H, \Gamma_G) = \sum_{n \in \mathbb{Z}} e^{in\theta}\ \widehat{Ch}(\Pi_{H \to G}, \Gamma_G).$$

Recalling that the Weyl group for T_{an} and T_{sp} have orders one and two, respectively, the normalized characters in this series are given by

$$\widehat{Ch}(\Pi_{H\to G},\Gamma_G) = \begin{cases} \dfrac{\mp e^{in\theta}}{|e^{i\theta}-e^{-i\theta}|} & \text{if } \Gamma_G \in T_{an} \\ \dfrac{x^n+x^{-n}}{|x-x^{-1}|} & \text{if } \Gamma_G \in T_{sp} \end{cases}$$

As such, $\Theta(\Gamma_H,\Gamma_G)$ is an infinite divergent series.

We analyze this divergence in more detail. We inspect both conjugacy classes of maximal tori in G, and we may break our integral according to Γ_G being regular elliptic or regular hyperbolic:

$$\begin{aligned} f_H(\Gamma_H) &= \int_{\mathbf{\Gamma}_G} \Theta(\Gamma_H,\Gamma_G)\ \widehat{O}(\Gamma_G,f_G)d\Gamma_G \\ &= \int_{\Gamma_G\in T_{an}} \Theta(\Gamma_H,\Gamma_G)\ \widehat{O}(\Gamma_G,f_G)d\Gamma_G + \int_{\Gamma_G\in T_{sp}} \Theta(\Gamma_H,\Gamma_G)\ \widehat{O}(\Gamma_G,f_G)d\Gamma_G. \end{aligned}$$

We may inspect these terms separately now. In T_{sp}, because it is abelian, each conjugacy class is comprised of a single element, i.e., $\{X\}$ for $X = diag(x,x^{-1})$. So, take $\Gamma_G = X$ for $x \in \mathbb{R}_{>0}$. Then,

$$\begin{aligned} \Theta(\Gamma_H,\Gamma_G) &= \sum_{n\in\mathbb{Z}} e^{-in\theta}\ \widehat{Ch}(\Pi_{H\to G},\Gamma_G) \\ &= \sum_{n\in\mathbb{Z}_{\geq 0}} e^{-in\theta}\ (x^n+x^{-n}) \\ &= \lim_{n\to\infty} \sum_{n}^{N} e^{-in\theta}\ (x^n+x^{-n}), \end{aligned}$$

in which we will refer to the partial sum as $\Theta_N(\Gamma_H,\Gamma_G)$. We may further break up this integral as

$$\int_{\Gamma_G \in T_{sp}} \Theta(\Gamma_H, \Gamma_G)\ \widehat{O}(\Gamma_G, f_G) d\Gamma_G = \int_{0<x<1} \lim_{n\to\infty} \sum_n^N e^{-in\theta}\ (x^n + x^{-n})\ \widehat{O}(X, f_G) d^\times x$$
$$+ \int_{x>1} \lim_{n\to\infty} \sum_n^N e^{-in\theta}\ (x^n + x^{-n})\ \widehat{O}(X, f_G) d^\times x,$$

where $d^\times x$ denotes the Haar measure $\frac{dx}{x}$. Observe that the x^{-n} terms on $0 < x < 1$ and the x^n terms on $x > 1$ will both grow with N. As such, on $x > 1$ consider the change of variable $u = x^{-1}$, for which some small manipulations yield $\frac{-du}{u} = \frac{dx}{x}$, and

$$\int_{x>1} \lim_{n\to\infty} \sum_n^N e^{-in\theta}\ (x^n + x^{-n})\ \widehat{O}(X, f_G) d^\times x$$
$$= \int_{0<u<1} \lim_{n\to\infty} \sum_n^N e^{-in\theta}\ (u^{-n} + u^n)\ \widehat{O}(X, f_G)(-d^\times u)$$
$$= \int_{0<u<1} \lim_{n\to\infty} \sum_n^N e^{-in\theta}\ (-u^{-n} - u^n)\ \widehat{O}(X, f_G) d^\times u.$$

Then, placed together

$$\int_{\Gamma_G \in T_{sp}} \Theta(\Gamma_H, \Gamma_G)\ \widehat{O}(\Gamma_G, f_G) d\Gamma_G = \int_{0<x<1} \lim_{n\to\infty} \sum_n^N e^{-in\theta}\ (x^n + x^{-n})\ \widehat{O}(X, f_G) d^\times x$$
$$+ \int_{0<u<1} \lim_{n\to\infty} \sum_n^N e^{-in\theta}\ (-u^{-n} - u^n)\ \widehat{O}(X, f_G) d^\times u,$$

achieving a cancellation of the problematic terms in light of their reciprocal nature over the two intervals of integration.

Finally, we realize

$$\int_{\Gamma_G \in T_{sp}} \Theta(\Gamma_H, \Gamma_G)\ \widehat{O}(\Gamma_G, f_G) d\Gamma_G = \lim_{n\to\infty} \int_{x>0} \sum_n^N 2e^{-in\theta}\ \widehat{O}(X, f_G)\ d^\times x,$$

which is smooth.

Then, for the term corresponding to the regular elliptic elements, we observe

$$\begin{aligned}\int_{\Gamma_G \in T_{an}} \Theta(\Gamma_H, \Gamma_G)\ \widehat{O}(\Gamma_G, f_G) d\Gamma_G &= \int_{\Gamma_G \in T_{an}} \sum_{n \in \mathbb{Z}} e^{-in\theta}(\mp e^{in\theta}) f_G(\Gamma_G)\ d\Gamma_G \\ &= \int_{r(\theta) \in T_{an}} f_G(r(\theta))\ d\theta,\end{aligned}$$

which is also smooth, yielding the theorem.

Thus, we have achieved the requisite smoothness and successfully established stable-stable transfer for orbital integrals for our case of interest. With these principal ideas resolved, additional adjustments will be made to extend our explicit formulations to any maximal torus and, subsequently, to symplectic groups of any rank. We will be able to then extend notions of stable-stable transfer to cases involving other algebraic groups.

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