Maintenance Modeling for Degrading Systems with Individually Repairable Components using Optimization and Reinforcement Learning

by

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ABSTRACT OF THE DISSERTATION

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There are many different industrial and manufacturing multi-components systems, where each component experiences multiple competing failure processes, such as degradation and environmental shock. In this research there are two competing failure processes for each component, namely soft failure due to the degradation and hard failure due to the random shocks coming to the system. Moreover, each incoming shock results in damage on all the degradation paths of all the components. For some multi-component systems, components can be repaired/replaced individually within the system. For such multi-component systems with individually repairable components and dependent failure processes, it is not economical to replace the whole system if it fails, while in most of the previous studies, the systems are either considered to have independent failure processes or packaged and sealed together, and the whole system was replaced with a new one when any component fails. For systems functioning for a very long time, and each component is repairable within the system, individual components have been replaced several times. Therefore, the starting time or age of all the components within the system are not the same. In this research work, the conditional reliability analysis of such systems is studied considering the initial age of each component, at the beginning of the inspection intervals, as a random variable. For systems, whose costs due to failure are high, it is prudent to avoid the event of failure, i.e., the components should be repaired or replaced before the failure happens.

Condition-based maintenance models recommend a policy to initiate repair or replacement before the failure occurs by detecting the system degradation status at each inspection time interval. Different types of condition-based maintenance models, including both static and dynamic models, are formulated and optimized to find the best maintenance policy. In some of the proposed models, determination of the optimal maintenance thresholds, such as on-condition and opportunistic thresholds for each component, along with optimal inspection time for the whole system, work to prevent failures and minimize cost. For multi-component systems with repairable components, it is also beneficial to have a dynamic maintenance plan based on the current degradation level of all the components in the system. In this study, different dynamic condition-based maintenance models are proposed using optimization and reinforcement learning methods. Moreover, different types of maintenance actions and the uncertainty of the maintenance implementation are also considered in the formulation of the maintenance models. Using a reinforcement learning approach provides a more time-efficient and cost-effective method compared to the traditional maintenance optimization solutions, and it can also provide a dynamic maintenance policy for each specific degradation state of the system. This can be more useful and beneficial compared to the fixed or stationary maintenance plans. The maintenance problems are formulated as a Markov decision process and are solved by using a Q-learning algorithm and deep Q-learning. Overall, the goal of the proposed research is to provide practical and effective maintenance models for a multi-component system with individually repairable components to avoid the failure and high downtime cost, and to minimize the cost.

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NOMENCLATURE

Section 3

$N(t)$	number of shock loads that have arrived by time t;
n	number of components in a series or parallel system;
λ	arrival rate of random shocks;
D_i	threshold for catastrophic/hard failure of i^{th} component;
W_{ij}	size/magnitude of the j^{th} shock load on the i^{th} component;
$F_{Wi}(w)$	cumulative distribution function (cdf) of W_i ;
H_i^1	critical wear degradation failure threshold of the i^{th} component (a fixed

	parameter);
H_i^2	on-condition threshold of the i^{th} component (a decision variable);
$X_i(t)$	wear volume of the i^{th} component due to continuous degradation at t;
$X_{S_i}(t)$	total wear volume of the i^{th} component at t due to both continual wear and instantaneous damage from shocks;
Y_{ij}	damage size contributing to soft failure of the i^{th} component caused by the j^{th} shock load;
$S_i(t)$	cumulative shock damage size of the i^{th} component at t;
$\alpha_i(t)$, β_i	shape and scale parameter for gamma degradation process for component i;
$G_i(x_i,t)$	cumulative distribution function (cdf) of $X_i(t)$;
$F_{X_i}(x_i,t)$	cdf of $X_{S_i}(t)$;
$f_{Y_i}(y)$	probability density function (pdf) of Y_i ;
$f_{Y_i}^{<k>}(y)$	pdf of the sum of k independent and identically distributed (*i.i.d.*) Y_i variables
$f_T(t)$, $F_T(t)$	pdf and cdf of the failure time, T;
$F_T^{\mathbf{H}^1}(t)$	cdf of the failure time T for the whole system considering critical failure threshold;
$F_T^{\mathbf{H}^2}(t)$	cdf of the time when an on-condition threshold is reached;
$C(t)$	cumulative maintenance cost by time t;
$E[TC]$	expected value of the total maintenance cost of the renewal cycle, TC;
τ	periodic inspection interval;
$CR(\tau)$	average long-run maintenance cost rate of the maintenance policy;
$E[K]$	expected renewal cycle length, K of the maintenance policy;
$E[N_I]$	expected number of inspections N_I;
$E[\rho]$	expected system downtime (the expected time from a system failure to the next inspection when the failure is detected);
C_R	replacement cost per unit;
C_I	cost associated with each inspection;
C_ρ	penalty cost rate during downtime per unit of time;
k	number of clusters of degrading components, with dependent component degradation paths;
H_i	critical wear degradation failure threshold of the i^{th} component;

$\theta(i)$	gamma process scale parameter for component i;
$\alpha_{q,i}$	scale parameter transmission constant for component i in cluster q;
θ_q	scale parameter clustering variable for cluster q, q=1,2,…,k;
$v_i(t)$	gamma process shape parameter for component i, a nondecreasing right-continuous function for $t > 0$;

Section 4

$X(t)$	Degradation at time t
$\alpha_i(t)$	Shape parameter of gamma distribution for component i at time t
β_i	Scale parameter of gamma distribution for component i
H_i^1	Failure threshold of component i
H_i^2	On-condition threshold for component i
U_i	Initial degradation of component i at steady state (random variable)
τ	Inspection interval
$g(\cdot)$	Gamma probability density function
$G(\cdot)$	Cumulative distribution function of gamma distribution
$f_{U_n}(u)$	Probability density function (pdf) of initial degradation of component n
$R(t)$	Reliability for time 0 to t
C_R	Cost of replacement for each component
C_s	Setup cost for replacement
C_I	Cost of inspection
C_ρ	Penalty cost due to system downtime
$F_T(v)$	Probability that the component fails by time v
$F_{X(t)}(H)$	Probability that the degradation passes the threshold H by time t
λ	Shock arrival rate
$S_i(t)$	Total of m shock damages by time t for component i
$N(t)$	Number of shocks arrived to the system by time t
$f_Y^{<m>}(y)$	pdf of m shock damage sizes
$P_{NH_i}(t)$	Probability of having no hard failure by time t
$P_{NS_i}(t)$	Probability of having no soft failure by time t
$P_{NR_i}(t)$	Probability of having no replacement by time t

Section 5

$N(t)$	number of shock loads that have arrived by time t;
n	number of components in a series or parallel system;
λ	arrival rate of random shocks;
D_i	threshold for catastrophic/hard failure of i^{th} component;
W_{ij}	size/magnitude of the j^{th} shock load on the i^{th} component;
$F_{W_i}(w)$	cumulative distribution function (cdf) of W_i ;
H_i^1	critical wear degradation failure threshold of the i^{th} component (a fixed parameter);
H_i^2	Preventive maintenance threshold of the i^{th} component (a decision variable);
$X_i(t)$	wear volume of the i^{th} component due to continuous degradation at t;
$X_{S_i}(t)$	total wear volume of the i^{th} component at t due to both continual wear and instantaneous damage from shocks;
Y_{ij}	damage size contributing to soft failure of the i^{th} component caused by the j^{th} shock load;
$S_i(t)$	cumulative shock damage size of the i^{th} component at t;
$\alpha_i(t)$, β_i	shape and scale parameter for gamma degradation process for component *i;*
$G_i(x_i,t)$	cumulative distribution function (cdf) of $X_i(t)$;
$F_{X_i}(x_i,t)$	cdf of $X_{S_i}(t)$;
$f_{Y_i}(y)$	probability density function (pdf) of Y_i ;
$f_{Y_i}^{<k>}(y)$	pdf of the sum of k independent and identically distributed (*i.i.d.*) Y_i variables
$f_T(t;\mathbf{H}^1-\mathbf{u})$	The probability density function of residual failure time, given observed degradation **u** at the beginning of the interval
$F_T(t;\mathbf{H}^1-\mathbf{u})$	The cumulative distribution function of residual failure time, given observed degradation **u** at the beginning of the interval
C_I	The cost of inspection
C_R	The cost of replacement
C_ρ	The penalty cost due to system shut down

C_S	The setup cost for maintenance implementation
θ_i	The weight of input i in neural network
b	The bias for the neural network
o_e	The actual value of a decision variable i.e the inspection interval
$\hat{o}_e$	The predicted value of a decision variable i.e the inspection interval

Section 6

$X(t)$	Degradation at time t
$\alpha_i(t)$	Shape parameter of gamma distribution for component i at time t
β_i	Scale parameter of gamma distribution for component i
$G(\cdot)$	Cumulative distribution function of gamma distribution
λ	Shock arrival rate
$S_i(t)$	Total of m shock damages by time t for component i
$N(t)$	Number of shocks arrived to the system by time t
$f_Y^{<m>}(y)$	Pdf of m shock damage sizes
H_i^1	Failure threshold of component i
H_i^l	Threshold of l^{th} degradation state of component i
D_i	Threshold for catastrophic/hard failure of i^{th} component
$F_{W_i}(\cdot)$	The cumulative distribution function (cdf) of W_i
U_j	Random variable represents the initial degradation value of state j
$f_{U_n}(u)$	Probability density function (pdf) of initial degradation of component n
τ	Inspection interval
$P_{ss'}^a$	The transition probability of being in state s' at time t+1, if the system was in state s at time t, and the agent chooses the action a.
r_t	The reward at time t.
$\pi_t(s,a)$	The probability that action a is selected at time t given state s.
$A(S)$	The set of possible actions, and the action at time t
$Q_\pi(s,a)$	The optimal policy, the value of taking action a at any state s
$\Re_{ss'}{}^a$	The expected cumulative reward if the process starts at state s, take the action a, and move into state s'

$V^{\pi}(s)$	The state value function, the expected return if the process is continued from state *s* following policy π.
R_t	The cumulative future cost
C_R	The replacement cost
C_ρ	The penalty cost for downtime
ω	Binary variable indicating the failure states
σ_{il}	Binary variable for actions of {do nothing: σ_{i0}, repair: σ_{i1}, replace: σ_{i2}}
θ	The learning rate which can have a value between 0 and 1
γ	The discount factor which can have a value between 0 and 1.
ε	Value for exploring vs. exploiting.
$Q(s,a)$	The expected value of taking action *a* in state *s*.
$T_{\text{maintenance}}$	The maintenance contract duration
$T_{\max}$	The max value of machine's virtual age
υ_1	The parameter of maintenance level 1 improvement
υ_2	The parameter of maintenance level 2 improvement
λ_s	The arrival rate of the spare part
λ_r	The success rate of maintenance
P	Transition probability matrix for "do nothing" action
P^1	Transition probability matrix for maintenance level 1
P^2	Transition probability matrix for maintenance level 2
C_r	The shortage cost
C_s	The ordering cost
C_{pm1}	The cost of maintenance level 1
C_{pm2}	The cost of maintenance level 2
C_{cm}	The cost of replacement
C_f	The cost for failure
$C_{p(t,i)}$	The cost for production for state *i* and virtual age *t*

1. Introduction

Over the last few decades, the maintenance of systems has become more and more complex. The maintenance function is defined as a set of activities or tasks used to restore an item to a state in which it can perform its designated functions [1]. An effective maintenance policy maintains the system by achieving high safety and low cost, both of which are critical concerns in many modern industries [2]. Reliability analysis of systems plays a critical role in determining proper maintenance. This research focuses on reliability analysis and optimal maintenance plan for various types of systems, including repairable components subject to multiple dependent competing failure processes of degradation and environmental shock process. Previous research studied single-unit systems or systems with nonrepairable components under some unreliable assumption of failure processes. To be effective in providing a maintenance plan for a specific system, the cause of failures should be determined from a technical perspective, and system reliability should be analyzed based on the defined failure processes.

Traditional approaches of reliability analysis are sometimes inappropriate or inefficient for some new devices because either they are too reliable to observe failure time data in a reasonable time period, or the time period between design and product release is too short [3]. If new technologies are to be transitioned from low volume production or relatively simple design applications, new and innovative research focusing on system reliability issues must be considered [4].

For complex systems, the reliability analysis can be very complicated and challenging due to the different failure processes, environmental factors, etc. Environmental factors such as temperature, humidity, wind speed, mechanical shocks, etc., can accelerate the deteriorating process of systems or components. Moreover, aging, wearing, corrosion, mechanical fatigue, and other physical changes can occur due to the regular operational and environmental exposure [5]. Therefore, all the possible failure mechanisms and their effects on systems or components should be identified and analyzed thoroughly. Degradation is one of the common failure mechanisms that have been widely investigated [6]. For example, deterioration of reinforced concrete has become a serious problem worldwide. Rebars and reinforced concrete are the main components of bridges. The estimated cost of bridge failures due to the rebars deterioration in the USA exceeds $8 billion. Figure 1.1 (a) and (b) show the construction of rebars and reinforced concrete in bridge construction.

(a) (b)

Figure 1.1 Rebars and reinforced concrete in bridge construction

When rebars, which are located inside the concrete, deteriorate due to the aging and environmental factors, they occupy a space equal to 6 times their new condition; therefore, it causes the failure of concrete and eventually bridges. Figure 1.2 shows the process of bridge failure due to rebar deterioration.

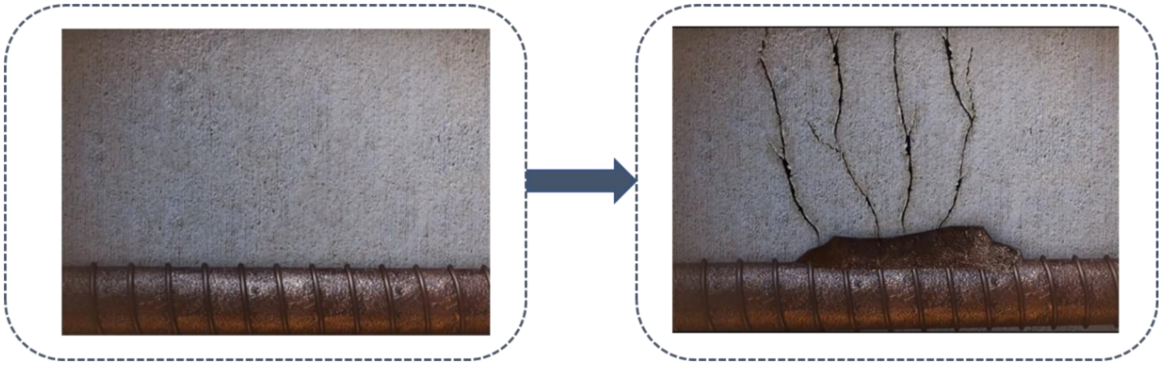


Figure 1.2 Bridge failure due to rebar deterioration

On March 14, 2019, one of the large bridges near CST station in Mumbai, India, broke down because of concrete failure. Six people died, and 30 people were seriously injured in this accident. On March 15, 2018, a complete collapse of a bridge happened in Florida, the USA, and the reason is reported as faulty design and poor equipment quality. Six people died, and nine people were injured in this collapse. There are various areas such as railroad, aircraft, pipelines, etc., with the same goal of increasing the reliability and safety of their system. In this research, the reliability models and maintenance models are developed for different types of systems, including different components subject to multiple failure processes, which can be beneficial in most of the industries to improve their system safety and diminish their total cost. This chapter starts with the problem statement that explains the system reliability problems. Then, the motivation of this research is stated for many industrial applications. Lastly, the objectives and contributions of this research are addressed.

1.1 Problem Statement

By increasing the complexity of many industrial and manufacturing systems, the importance of reliability and system safety is becoming more critical. The primary step in improving the system's reliability is analyzing the systems' failure processes. Failure mechanisms are understood from a physical perspective, and typical degradation measures include wear, drug stability, deterioration, degraded light intensity, crack propagation, resistance drift, and loss of structural strength [7].

Degradation is one of the common failure mechanisms that have been widely investigated. It can be defined as the aging processes of a system due to cumulated wear, tiredness, corrosion, etc. Degradation analysis is an approach to measure and assess system reliability. Moreover, environmental or physical shocks can be another critical factor that can cause system failure by accelerating the degradation process or causing immediate failure. These two common failure processes of degradation and random shocks are used in this research as competing failure processes, which can also be dependent. In the previous studies, mostly, they are assumed to be independent. In this study, it is assumed that each incoming shock can cause immediate failure and accelerate the degradation process of all the components, which is more realistic compared to previous studies. Moreover, for some cases, there is a mutual dependency between degradation and shock process, which is studied in this research, where degradation can increase the shock intensity, and each incoming shock can accelerate the degradation process.

Obtaining a proper maintenance policy can help most industries improve the safety and reliability of their systems. For systems whose penalty cost due to downtime is high, detecting the component status and facilitating repair/replacement decision-making before system failure, leads to low risk of failure, and subsequently, lower maintenance cost.

Time-based maintenance and condition-based maintenance are two types of preventive maintenance which have been received significant attention. There are several studies which discuss the effectiveness and provide comparisons of time-based maintenance and condition-based maintenance for different applications [8, 9]. Condition-based maintenance recommends a maintenance decision such as replacement or repair based on the observed condition of the system. For systems with each component deteriorating due to degradation and random shocks, monitoring the component status assists in providing a maintenance decision making policy. For systems of degrading components, providing a condition-based maintenance plan can be a particularly difficult problem because of the dependent degradation and dependent failure times. In previous research, preventive maintenance and periodic inspection models have been considered; however, for systems whose costs due to failure are high, it is prudent to avoid the event of failure, i.e., the components or system should be repaired or replaced before the failure happens. Providing a preventive maintenance plan is effective to avoid failure and to minimize cost, which is studied in this research for different types of systems.

Analyzing the components of a system is one of the preliminary steps in identifying system behavior. Different components configuration within the system can affect the system reliability and time of failure. Moreover, systems with repairable components or non-repairable ones should be modeled differently, which is considered in this study. For multi-component systems with individually repairable dependent components, it is not economical to replace the whole system if it fails. However, in the previous studies, the repairable systems considered as packaged and sealed together, and there is no model for reliability analysis of systems with individually repairable dependent components that are degrading and experiencing random shocks. In this study, condition-based maintenance is studied for a multi-component system with individually repairable components subject to degradation and random shocks.

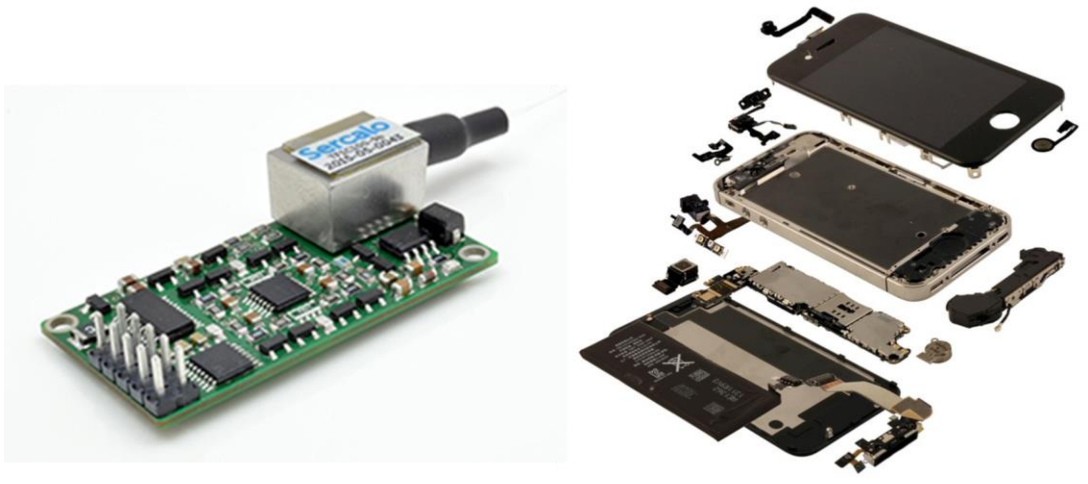
The Dependency of components degradation is another factor which affect the system reliability. Due to the similar working environment, the degradation status of components in the same system are likely to be probabilistically dependent and correlated. Without considering dependent component deterioration, reliability models are not sufficient or adequate for some systems and engineering applications. Dependent stochastic degradation paths among components represent a challenging issue because it increases the complexity of system reliability modeling and calculation. However, it is also more practical and realistic. Component degradation paths can be dependent due to different reasons:

1. Components are in close proximity, so that degradation status of some components can directly influence the degradation of other components, and in return, the degradation of other components may also affect the original instigating components and other components. Therefore, component degradation paths are dependent or correlated.
2. Components exist in a shared environment, and factors like temperature, humidity, exposure to contaminants, can affect all the component degradation paths at the same time. All components in a system that is randomly exposed to a harsher environment may degrade at a higher rate, and clustered together.
3. Components sharing load likely have dependent degradation paths. A rapid gravity filter (RGF) media degrades during the water filtration process. Typically, there are multiple RGFs configured in parallel in a water filtering system, and RGFs can be activated to simultaneously filter the incoming water. They may not be required to operate during non-peak periods, and some of them may operate in cold standby. Therefore, the degradation level and the reliability of each RGF depends on incoming water situation [10].

There are different types of maintenance policies for systems of degrading components. However, due to the inherent stochastic degradation behavior, providing a dynamic maintenance policy for a system with multiple components is potentially a more efficient and cost-effective policy than the previous maintenance plans where the inspection time or the maintenance actions are fixed and predefined for each maintenance plan. In this research, dynamic maintenance policies are also developed for systems with individually repairable components subject to multiple dependent competing failure processes.

1.2 Motivation

There have been relatively few research studies considering multiple failure processes for systems with multiple components. Most of the previous studies, consider systems as a single component or as a packaged together in a system without individual component repair. Most of the modern products consist of various components that degrade differently, so considering all of them as one-unit system which degrades over time is not practical. Moreover, considering multi-component systems with components packaged and sealed together is not practical and useful for many industrial applications. There are different product including various components where each of them can be individually maintained within the system. Analyzing the system reliability and providing a maintenance policy for systems with individually repairable components, where each component degrades over time, is a unique challenge which is studied in this research. In this study, to provide a more practical and beneficial maintenance policy for degrading system, different types of systems are studied separately. The first type is a system with multiple components which each of them degrades individually but it is not practical or beneficial to maintain them individually. Examples of this kind of systems are cellphones, or micro-electro-mechanical system (MEMS) which are shown in Figure 1.3(a) and (b) that consist of different components which degrade individually but for maintenance perspective, they should be packaged together.



(a) (b)

Figure 1.3(a) Micro-electro-mechanical, (b) Cellphone

The second type of system consists of different components that degrade differently, but they can be maintained individually within the system. A wind turbine is an example of a multi-component system that consists of different components with different mechanical functions such as blades, brake, gearbox, generator, rotor, shafts, and tower, which degrade distinctively. Since each component degrades differently within the system, they may have different failure times, which should be considered in a maintenance model and system reliability. Figure 1.4 shows the main components of a wind turbine and its location within the system.

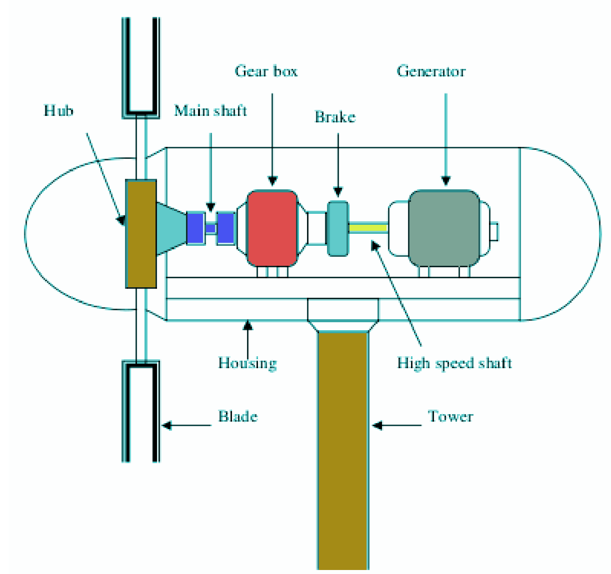
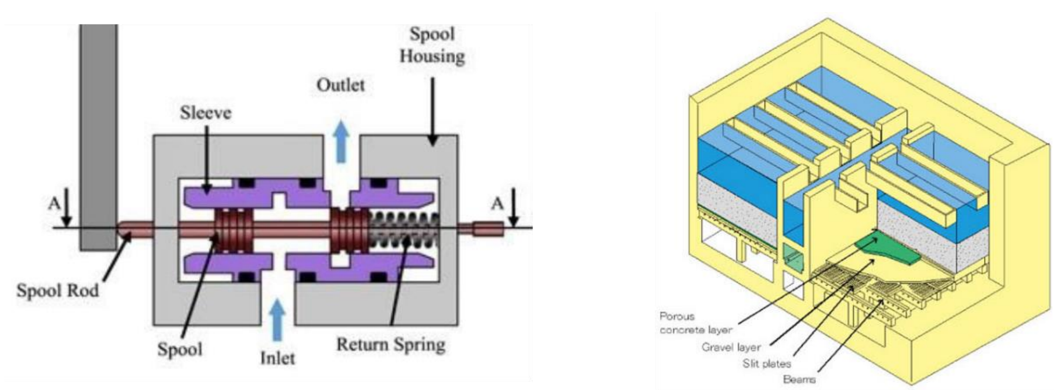


Figure 1.4 Main parts of a wind turbine

The third type of systems which are studied in this research is a system with different components, where they are degrading in a cluster due to the shared environment, factors like temperature, humidity, exposure to contaminants, can affect all the component degradation paths at the same time. A sliding spool in electrohydraulic servo-valve is used in wide range of applications from metal forming and wood processing to aircraft applications. Figure 1.5(a) shows the configuration of components in a electrohydraulic servo-valve. A rapid gravity filter (RGF) media is another example of system with dependent component, which is shown in Figure 1.5 (b)



(a) (b)

Figure 1.5 (a) Electrohydraulic servo valve, (b) Rapid gravity filter

Different types of systems have different degradation behavior, which should be considered in calculating the system reliability and subsequently obtaining a maintenance model. In this study, system reliability is analyzed for different types of systems where in each case, each component is subject to dependent competing failure processes of degradation and random shocks. Based on the system reliability which can be calculated for each specific system, an optimal condition-based maintenance can be provided to avoid the system failure and reduce the maintenance cost.

1.3 Research contributions

In this research, new models are developed to analyze the reliability of complicated multi-component system subject to multiple dependent and competing failure processes and subsequently a maintenance model is developed for each system based on the proposed reliability model. One of the main contributions is consideration and analysis of a multi-component system subject to multiple dependent competing failure processes instead of single unit system experiencing simple failure process where each component can be individually repairable. Different types of systems are studied with various components degradation behavior and different maintainability design. New reliability models are developed for different degradation behavior and different maintenance design. The other critical contribution of this study is considering different condition-based maintenance models such as static and dynamic to avoid system failure by suggesting the maintenance actions for the system or each individual component. Moreover, different types of maintenance actions and their stochastic effects on the components are also considered in this research work.

1. A reliability model is developed for a multi-component system, where each

component degrades separately, and it is subject to random shock arrivals. However, the system is packaged and sealed together, and the maintenance actions should be implemented on all the components. An optimal maintenance plan is developed to find the optimal inspection time for the whole system and the optimal threshold of preventive maintenance for all the components simultaneously. There is no model in the previous studies that can provide the inspection time and on-condition threshold of a multi-component system simultaneously. The previous studies are either considered as a single component system or focus on the time-based maintenance and condition-based maintenance distinctly.

2. System reliability is calculated for a system with multiple components, where components degrade in a cluster/groups with the same degradation behavior due to the shared environment. In this case, the degradation of components in the same clusters are dependent. Considering the dependency of components' degradation, a new reliability model is developed and analyzed.
3. In this research, system reliability is analyzed for a system of multiple components where each component can degrade separately and be maintained individually within the system. Subsequently an optimal maintenance model is found for such system to provide when the whole system should be inspected, and at each inspection interval, each component should be maintained to have the minimum cost. In most of the previous studies, the whole system is considered as packaged together and there is no reliability and maintenance model for systems with individually repairable degrading components.
4. Different types of maintenance policies are developed for a degrading system with

individually repairable components. A condition-based maintenance plan and an opportunistic maintenance policy are developed for system which can provide some maintenance thresholds for each component within the system to implement preventive maintenance and subsequently prevent the system failure.

5. Dynamic maintenance models are developed for multi-component systems with individually repairable components. By using the proposed maintenance model, the next inspection time for system can be found dynamically at the beginning of the inspection that can minimize the total cost and increase the availability of the systems. Moreover, a machine learning algorithm called reinforcement learning is used to find the optimal maintenance actions dynamically for each component within the system based on its degradation level.
6. Different types of maintenance actions are considered for degrading systems, such as perfect maintenance actions and imperfect actions. All the maintenance actions cannot be completed as expected, the maintenance actions which cannot improve the system or components to the expected level are called imperfect maintenance actions. Moreover, some actions, such as replacement need a spare part that should be ordered. Spare parts ordering and the uncertainty of the order arrivals are also considered. In this study, different types of maintenance actions and their stochastic effects are considered for the maintenance modeling of degrading systems.

Background and literature review

In this section, the literature is reviewed on reliability analysis of systems with single and multiple components subject to different failure processes and the various maintenance models for systems.

1.4 System reliability analysis

Significant and meaningful prior research has been performed on reliability analysis of systems with degradation, shocks, and independent or dependent failure processes. Using a degradation model provides a good understanding of physics-of-failure and method to predict the reliability. Therefore, analyzing the reliability by using the degradation models has been studied for several years.

1.4.1 Degradation modeling

One of the most common failure mode of industrial and manufacturing systems result from gradual cumulative deterioration of systems over time which is known as degradation [11]. There are two type of degradation: natural and forced degradation [12, 13], where natural degradation is due to the gradual deterioration of systems over time, and forced degradation is external to systems due to the load or stress which gradually increases. There are four types of degradation modeling, experienced-based approaches, model-based approaches, knowledge-based approaches, and data-driven approaches.

Experienced-based approaches are the simplest forms which are based on the distribution of event records of a population of identical items. The most popular approach of experienced-based approaches is the Weibull distribution due to its ability to conduct different types of behaviour including infant mortality and wear-out in the bathtub-tube curve [14]. In knowledge-based methods there is no degradation model due to the difficulty of mathematical models. These approaches are suitable for solving problems usually solved by human specialists. Model-based approaches use mathematical and statistical model to measure the deterioration of systems. Data-driven approaches are based upon statistical and learning techniques which come from the theory of pattern recognition.

These range from multivariate statistical methods to black-box methods based on neural networks (e.g., probability neural networks, decision trees, multi-layer perceptrons, radial basis functions and learning vector quantization, graphical models (e.g., Bayesian networks, hidden Markov model), self organising feature maps, filters, autoregressive models) [15-17].

Research shows that degradation measures often provide more information than failure time data to assess and predict the reliability of systems [18, 19]. System degradation has a stochastic property which can be modeled in different ways. Singpurwalla [20] reviewed the degradation models in a dynamic environment. Meeker and Escobar [18] reviewed different degradation modeling and compared them with failure time models. Degradation models in reliability analysis can potentially be classified as it is shown in Figure 2.1 [15].

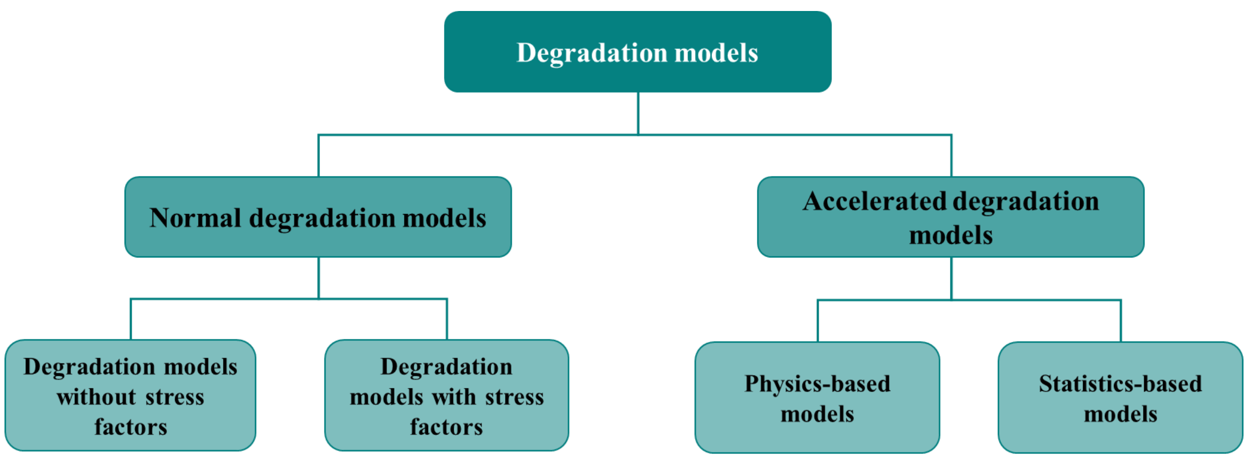


Figure 0.1 Degradation models in reliability analysis

The difference between normal degradation models and accelerated degradation models is, in normal degradation models the reliability is estimated using degradation data from normal operating condition, while in the second models the models inferences about reliability at normal condition using data obtained at accelerated time. Therefore, to analyze the system degradation, it is practical to use accelerated life models. Accelerated degradation models consist of physics-based models and the statistics-based models. Nelson [21] extensively describes both the physics-based models and statistics-based models. Gorjian et al. [22] reviewed the statistical models with accelerated failure time in more detail. General degradation path model, random process models, time series models, stress-strength interference model, and stochastic degradation models are different normal degradation models.

(1) General degradation path: This model is suitable for samples which are tested in a homogeneous environment and fit as linear or nonlinear regression models on degradation observations. Haghighi et al. [23] reviewed the nonparametric, semi-parametric, and parametric estimation of survival function using general degradation path. Lu and Park [24] studied regression-type method, general degradation path models for analyzing linear degradation data from semiconductors. Bagdonavičius et al. [25] used general nonparametric, nonlinear path models for degradation process.

(2) Random process model: In this model, there is no assumption about degradation paths, and it is a suitable model for reliability estimation with multiple observation at certain time points. The idea of random effect is borrowed from Bayesian linear regression. Lu and Meeker [26] studied a random effect model to make inferences about failure-time distribution using fatigue degradation data.

(3) Time series model: This model is appropriate for reliability estimation in a dynamic environment where systems have multiple performance measures. Lu et al [27] developed a time series model to predict system reliability considering multiple failure modes. The performance measures across time are treated as multivariate time

series, and mean vector and variance matrix of performance is predicted and used to calculate the system reliability.

(4) Stress-strength interference models: In this model, the stress from applying loads to system is considered, and it can be used to develop the reliability which corresponds to the possibilities event that strength exceeds stress. An et al. [28] investigated a model using discrete random variables as stress and strength.

(5) Stochastic degradation models: These models have specific physical interpretation about the system, and it can accommodate various kinds of uncertainties which may occur in interaction of systems with environment. Markov model is usually used for discrete degradation states for degradation modeling. However, stochastic models with continuous states are more commonly used recently, such as Inverse Gaussian processes, Wiener process (also called Brownian motion with drift), the compound Poisson process, and the gamma process [29].

Markov process model has been extended to the semi-Markov process model and the hidden Markov model to address more general reliability analysis problems [30]. Li et al. [31] used a non-homogeneous Markov process with different condition states to model the deterioration of a system. Mishalani and Madanat [32] developed a Markov process model, considering material properties, environmental conditions. Kleiner et al. [33] used a fuzzy rule-based, non-homogeneous Markov process to model the degradation of buried infrastructure assets. Various stochastic models recently attract consideration. Wiener process (Brownian motion with drift), gamma process, inverse gaussian process are the most commonly used stochastic process models recently.

Wiener process (Brownian motion with drift) [34] is a stochastic process with independent, real-valued increments and decrements having a normal distribution. Wang et al [35] predicted the residual life of a system using an adapted Brownian motion-based approach with a drifting parameter, where the drifting parameter is adapted every time a new observation ins available. A limitation of this model is that it is not appropriate for many applications whose deterioration is monotonic.

The compound Poisson process [36] is a stochastic process with independent and identically distributed jumps which occur as a Poisson process with randomly distributed jump sizes. It is suitable for modelling some examples such as damage due to sporadic shocks. A gamma process is a stochastic process with independent, non-negative increments having a gamma distribution with a constant scale parameter and a shape parameter depending on the length of the time interval [37]. In effect, it has an infinite number of jumps in finite time intervals, and it is suitable to model gradual damage monotonically accumulating over time in a sequence of tiny increments, such as wear, fatigue, corrosion, crack growth, erosion, consumption, etc. Brian and Gabraeel [38] developed a stochastic model for a multi-component system and estimated the residual lifetime of each component based on their degradation model. Chen et al. [39] Used inverse Gaussian with random effects to model the system degradation to find the optimal maintenance planning. Gao et al [40] developed a reliability model for a multi-phase degradation system in a dynamic environment using a Wiener process. Shen et al. [41] combined different stochastic processes to model degradation of a system in a dynamic environment.

Gamma process is an appropriate stochastic process to model the degradation path of components which their degradation is monotonically increasing. In this research, a gamma process is used to model a component's degradation path.

1.4.2 Random shock arrival models

Shock models have been studied for several years and there have been various numbers of research for modeling reliability system situated in random environment by considering shock models. The way in which the time between two consecutive shocks, the damage caused by a shock, the system failure and the relationships among all these elements are modelled these characterize a shock model, while the major types are distinguished depending on whether the effect of the shock on the system is independent of its arrival time or not [42]. There are four types of shock models with independence assumptions, (1) homogeneous Poisson process, where the times between two consecutive shocks are independent, identically distributed exponential random variables. (2) non-homogeneous Poisson process, which has a counting process null at the origin with independent increments. (3) non-stationary pure birth process which is a Markov process for probability of shock arrivals. (4) renewal process, that is, the times between two consecutive shocks are independent and identically distributed random variables.

The simplest case is homogeneous Poisson process which is used by Esary and Marshall [43] for the first time. Further, A-Hameed and Proschan [44] extended the results obtained by Esary and Marshall [43] of homogeneous Poisson process for shock modeling. Klefsjö [45] studied a non-stationary pure birth process for shock modeling. Nakagawa [46] used the renewal process model for shock modeling in reliability analysis of a single unit system.

Fan et al. [47] developed a shock model when there exists dependence between the effect of the shock and its arrival time. Three principal models are considered: extreme shock model, where the system breaks down as soon as the magnitude of an individual shock exceeds some given level; cumulative shock model, where the system fails when the cumulative shock magnitude exceeds some given level and run shock model, where the system works until *k* consecutive shocks with critical magnitude occur [42]. Anderson [48] investigated a model for shock damage using limit theorems considering shock magnitude and failure threshold. Further, he developed a new model where the time interval between shocks, in the domain of attraction of a stable law-of-order, is less than a certain level or relatively stable [49]. Shanthikumar and Sumita [50] analyzed a general shock model considering the correlated pair of renewal sequences, where the system fails if any shock magnitude exceeds the predefined failure threshold. Their model considers the dependency of shock arrivals on the length of the interval since the last shock. Further, they studied the distribution of system failure in general, and shock models associated with correlated renewal sequences [51].

1.4.3 Multiple failure processes models

For most industrial and mechanical systems, different failure modes are competing with each other, which means once one of the failure modes occurs, the product is failed, and other failure modes will not happen anymore or will be delayed until a repair action is performed. These failure modes are defined as competing failure processes. Failure processes are usually classified into the two groups of (i) hard failure, which are caused by environment such as shocks, and (ii) group of soft failure, or failure due to the system degradation. There is a significant literature already dedicated to reliability analysis for systems subject to multiple failure processes.

Peng et al [52] developed a competing failure process to analyze the system reliability, considering hard failure and soft failure. In their study, each component in the system degrades with time, and when a shock arrives, if damage is greater than hard failure threshold, catastrophic failure occurs which is called as hard failure. Moreover, if the component survives the shock, total degradation containing both pure degradation and additional incremental degradation caused by shock damage is greater than a defined soft failure threshold level, soft failure occurs. Zuo et al [53] further developed a mixture model considering both catastrophic failures and degradation failures to analyze the system reliability of continuous state devices, where the degradation process is modeled using three different approaches of random process, general path model, and multiple linear regression. Lemoine and Wenocur [54] studied a new approach for failure modeling and developed a distribution for failure-time considering multiple failure processes of degradation and shocks, where system degradation is modeled by random process and the occurrences of shocks are modeled by a Poisson process whose rate function is state dependent. Their model provides a means of expressing the dynamics implicit in failure processes.

Li and Pham [55] investigated the reliability of a multi-state degraded system subject to competing failure processes of random shocks and degradation, where the degradation consisting of two independently competing causes. The model can be used not only to determine the reliability of the degraded systems in the context of multi-state functions, but also to obtain the states of the systems by calculating the system state probabilities. Hao and Su [56] developed a new failure model for a system considering multiple degradation processes and random shocks, where there is a correlation among different degradation processes. Subsequently, the system reliability is analyzed based on their new failure model. Wang et al. [57] analyzed system reliability considering both degradation and shock process where the degradation analysis is directed using fuzzy degradation data and different states are formed based on the shock damage on degradation process.

Jiang et al [58] studied a system subject to multiple failure processes of soft failure and hard failure, where the arrival of each shock impacts both failure processes, and also the shock process affects the hard failure threshold level. In this study, the system becomes more sensitive to hard failures, when it received more shocks. Two cases of damage from shocks are considered in this study. In the first case, the value of hard failure threshold reduced after m number of shocks, and in the second case, the hard failure threshold decreases to a lower level when the first shock is recorded above a predefined value. Rafiee et al [59] developed a new reliability analysis considering two failure processes of degradation and random shocks. It is considered that the degradation rate can change when the system is vulnerable to fatigue and deteriorates faster due to the shocks have been arrive to the system. Different shock patterns which are considered in this study are generalized extreme shock model, generalized δ-shock model, generalized m-shock model, and generalized run shock model.

Hao and Yang [60] investigated a new reliability model for a system with multiple competing failure processes considering the combination of extended extreme shock model and extended δ-shock model for hard failure process, where the hard threshold in the extended extreme shock model will be decreased by previous harmful shocks and the threshold level in the extended δ-shock model depends on the magnitude of the previous harmful shock. Moreover, each harmful shock can accelerate the degradation process by adding as abrupt increase on degradation path and acceleration on degradation rate. Figure 2.2 shows the competing failure processes of degradation and shock on system failure. The damage caused by shocks on components performance can be any form which is studied in the literature.

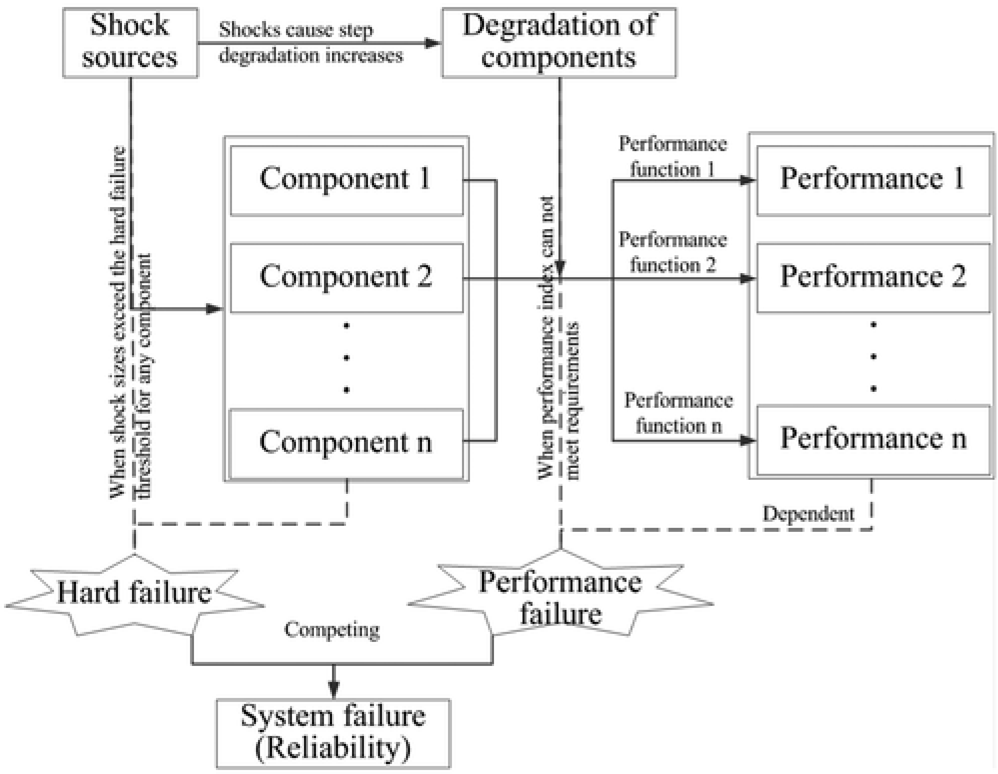


Figure 0.2 Competing failure processes on system failure [61]

1.5 Maintenance models

The maintenance function is defined as a set of activities or tasks used to restore an item to a state in which it can perform its designated function [1, 62]. An appropriate maintenance policy reduces total system costs, increases reliability and availability of systems. Maintenance optimization models focus on finding either the optimal balance between costs and benefits of maintenance or the most appropriate time to execute maintenance [63]. The development and implementation of maintenance optimization started in the early 1960s by researchers like Barlow, Proschan, Jorgenson, McCall, Radner and Hunter [64, 65]. Maintenance optimization is one of the most critical issues in production since the failure of a system during actual operation can be a costly and dangerous event [63].

Maintenance models can be divided into two basic groups of planned maintenance and unplanned maintenance. The unplanned maintenance is for the case that system is failed and immediate maintenance actions should be implemented without any before plan. On the other hand, in planned maintenance, the maintenance activities are planned well in advance to avoid system failure, and subsequently have imposed a penalty cost due to downtime of the system. There are three types of planned maintenance such as predictive maintenance, preventive maintenance, and corrective maintenance. Predictive maintenance involves the prediction of the failure before it occurs, identifying the root cause for those failure symptoms and eliminating those causes before they result in extensive damage of the equipment. Mobley [66] reviewed the predictive maintenance models for different systems. Figure 2.3 shows the main categories of maintenance plans.

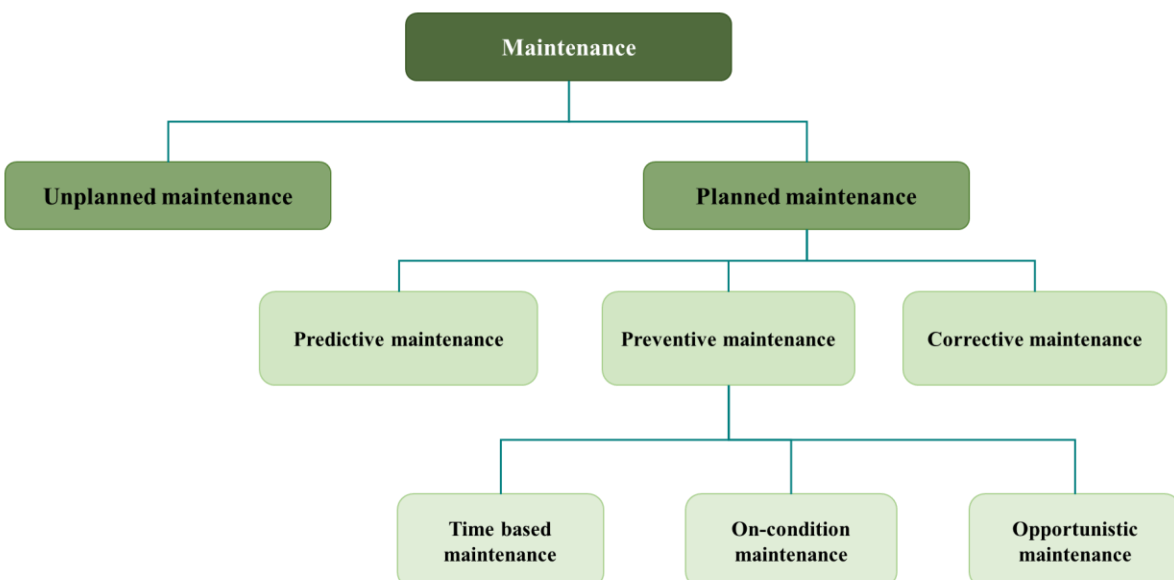


Figure 0.3 Maintenance plans

Corrective maintenance can be defined as maintenance implementation to restore the full performance of the equipment that is failed. Sherut and Krajewski [67] evaluated different corrective maintenance plans in various production systems using simulation models and economic analysis. The simulation model predicts inventory costs and delivery performance of a corrective maintenance policy in various production. Badia et al. [68] analyzed the optimal inspection policy of a single unit system which involves corrective maintenance on the system failure, and having no effect in the unit reliability otherwise.

Preventive maintenance models provide a maintenance plan for inspections, repairs, adjustments, and replacements before failure to minimize the maintenance cost by avoiding the system failure. Barlow and Hunter [69] Studied the optimization of preventive maintenance models for different cases. Mahani et al [70] investigated maintenance models for energy storage systems considering different system deterioration scenarios and market opportunities. Valdez-Flores and Feldman [71] reviewed some optimization of preventive maintenance models for repair, replacement, and inspection of systems subject to stochastic deterioration. Wu and Clements-Croome [72] investigated a preventive maintenance model by assuming the quality of a preventive maintenance action as a random variable following a probability distribution to consider the uncertainty of environments. Duarte et al [73] proposed an algorithm to find the frequency of preventive maintenance actions for a system which has linear increasing hazard rate. Based on the algorithm, the interval time between the maintenance actions for each component can be calculated which minimizes the total costs. Hsu [74] provided a new maintenance policy which combines preventive maintenance and replacement policies to minimize the replacement on failures. Dehghani et al [75] proposed a preventive maintenance model for a power distribution system to maximize the life-cycle resilience and reliability of the system.

Time-based maintenance and condition-based maintenance are two maintenance categories of preventive maintenance policy, which have been received significant attention recently. Time-based maintenance, also known as periodic-based maintenance [76]. In time-based maintenance the optimal maintenance inspection is provided using the failure time analysis. Das and Achraya [77] proposed a preventive maintenance policy for a single-unit system considering the system failure time and delay time between fault occurrence and component failure. The performance degradation during delay time is also considered to develop the preventive maintenance. Castro and Alfa [78] developed a time-based maintenance policy for a single-unit system using two approaches for replacement. In the first approach, the unit is replaced when it reaches a predetermined lifetime, and in the second approach, the repair facility is closed when the lifetime of the unit attains a predetermined quantity. Nakagawa and Yasui [79] proposed a time-based maintenance policy to provide the optimal number of units and replacement times of a multi-component system with parallel configuration. It is considered that each component can fail only due to the shocks and they can be replaced only at the scheduled inspection time. Wang [80] investigated a new preventive maintenance model by using delay-time-based models to find the optimal inspection plan of industrial systems, where the delay-time has two stages. The first stage is normal working stage, which is from new until the point of an identifiable defect, and second stage is failure delay-time stage that is from this point to failure.

Many firms still apply traditional time-based maintenance strategies, which are easy to implement as only the time that a unit is in service has to be recorded. However, substantial remaining useful life is wasted if the machine is still in reasonable condition when preventive maintenance is performed, and a breakdown might still occur if it happens to deteriorate faster than expected [81]. There are several studies that discuss the effectiveness and comparison of time-based maintenance and condition-based maintenance for different applications [82, 83].

Liu et al [84] studied a condition-based maintenance model for a continuously monitored system exposed to multiple sudden failure modes considering the system's limiting availability. A stochastic process is used to model the degradation system state. It is also considered that multiple sudden failures can occur during a system's degradation. The age of the system and the degradation state can change the system failure rate. By minimizing the long run cost per unit of time, the optimal threshold is found which maximize the system availability. Chen et al [39] obtained an optimal condition-based replacement policy where the system degradation is modeled as an inverse Gaussian process with random effects. Heterogeneities among a product population is considered in the random effect parameter of inverse Gaussian process. The problem is formulated as Markov decision process.

Do et al [85] developed a condition-based maintenance policy for a system with two components, where there are two type of dependencies between the components. It is assumed that the deterioration speed of each component depends not only on its own deterioration level but also on the degradation level of the other. The second dependency is economic dependence, which assumed that combining maintenance activities is cheaper than performing maintenance on components separately. Grall et al [86] proposed an analytical model for condition-based maintenance of a single-unit system, where the component degrades continuously. A multi-level control-limit rule is used to implement the maintenance policy. By considering the stationary law for the maintained system state, the cost model is formulated, and two decision variables of replacement threshold and the inspection schedule are found.

Dieulle et al [87] developed a new probabilistic method based on the semi-regenerative property of the evolution process to obtain the optimal condition based maintenance of a continuously deteriorating system. The degradation of the system is modeled using a gamma process, and it is considered as failed, if the degradation level exceeds a predefined failure level. Two types of replacement are also considered at each inspection time depending on whether the current system state is above a critical threshold but not failed or in the failed state. Hong et al [88] studied the optimal condition-based maintenance of a multi-component system with dependent stochastic degradation. The degradation of each component is modeled as gamma process and the dependency is represented using the copula function. The optimal maintenance policy is found for such system considering the environment condition and uncertainty of material properties.

Abdul-Malak and Kharoufeh [89] obtained an optimal maintenance model for a multi-component system using a Markov decision process model, while all components of the system experiencing the same environment. The degradation rate of each component is affected by the current state of the environment. When the cumulative degradation level of each component reaches a failure threshold, it fails. Do et al [90] introduced an adaptive condition-based maintenance policy for a deteriorating system considering both perfect and imperfect maintenance actions. Both positive and negative impacts of imperfect maintenance actions are considered. Where the positive impact of imperfect maintenance is its lower cost than perfect maintenance actions, and its negative impact is the imperfect maintenance action restores a system to a state between good-as-new and bad-as-old, and it may accelerate the speed of the system′s deterioration process. The optimal maintenance actions are found at each inspection time. Li et al [91] proposed a new maintenance grouping strategy for condition-based maintenance policy of a multi-component system considering the stochastic and economic dependences between components. Lévy copulas are used to model the stochastic dependencies due to the common environment.

Opportunistic maintenance policies suggest taking the opportunity to perform preventive maintenance on some components, along with replacement of failed components, which can lead to a lower total cost. Shafiee et al [92] investigated an optimal opportunistic condition-based maintenance policy for a multi-component system subject to degradation and random shock process. a multi-bladed offshore wind turbine system is analyzed as the case study where each blade is subject to stress corrosion cracking as the degradation and environmental shocks. Two types of shocks are considered in this study as catastrophic shocks which cause system failure immediately and minor shocks which cause instant drops in power output without any system failure. Do et al [85] proposed an opportunistic maintenance policy for a two-unit series system where deterioration speed of components is dependent on each other.

Zhang et al [93] used a Markov decision problem to formulate an opportunistic maintenance model for a multi-component system due to curse of dimensionality. The multi-component system is composed of different single-components, which are mutually influened by each other. By minimising the long-run average maintenance cost, the optimal opportunistic maintenance policy is obtained. Huynh et al [94] proposed two new opportunistic predictive maintenance strategies based on the remaining useful lifetime of the components and the remaining useful lifetimes of the system and its components. It introduced a multi-level decision-making approach that combines maintenance decisions an n-component deteriorating system with a k-out-of-n:F structure is studied for the proposed maintenance policy.

Zhu et al. [95] proposed an opportunistic maintenance for systems subject to stochastic degradation failures, and dynamically adjusted the inspection interval and maintenance thresholds. By minimizing the long-run cost rate considering the costs associated with inspection, setup, and maintenance actions, the optimal opportunistic threshold is found for the system which is proportional to the preventive maintenance threshold.

Recently, in a few research efforts, dynamic condition-based maintenance models are studied for different types of systems, mostly for a single system or multi-component systems where components are sealed and packaged together. Wu et al [96] proposed a dynamic maintenance model that can sequentially plan the system preventive maintenance schedule based on the actual maintenance history and health information. Both preventive maintenance and opportunistic maintenance models are integrated into one framework to find the next inspection dynamically for a single system. Tang et al [97] developed a dynamic maintenance model for a single degrading system using a random-coefficient autoregressive model.

Wang et al [98] proposed a dynamic maintenance scheduling model for a degrading system considering harsh external conditions. The degradation of the system is modeled as a Markov process based on physical characteristics, with the effects of harsh external conditions represented as probabilistic models. The optimal maintenance strategies are obtained by optimizing the proposed model with the cost to go, including system reliability cost and maintenance cost. Wu et al [99] proposed a dynamic condition-based maintenance model for a degrading single system. An Inverse Gaussian process with stochastic parameter is proposed to describe the change of the equipment degradation characteristics during operation.

Yousefi et al [100] developed a dynamic maintenance model using a reinforcement learning algorithm to find the maintenance actions for a system with fixed inspection times. A multi-component system is considered where each component can be repaired individually in the system. The deterioration of components is modeled as a Markov decision process and Q-learning algorithm is used to solve it an find the maintenance actions dynamically. Zhao et al [101] investigated a dynamic maintenance model to select the inspection time dynamically for a single system, using the historical degradation condition of the system, and a nonlinear Wiener process as the degradation model.

Liu et al [102] developed a dynamic inspection maintenance model for a single system subject to a continuous degradation process which is modeled as a gamma process. In addition to the degradation process, the system is subject to aging, which contributes to the increase of failure rate. An additive model is used to describe the effect of the degradation process and aging on failure of the system. Omshi et al [103] studied a dynamic auto-adoptive condition-based maintenance model to select the next inspection time for a single degrading system with unknown deterioration parameters, and Bayes theorem was used to update the prior information. They combined the remaining lifetime and a preventive maintenance threshold for making decisions about inspection scheduling.

1.6 Reliability analysis and maintenance of repairable and non-repairable systems

A repairable system is a system which after failing of one or more components it can be restored to its satisfactory performance by replacing or repairing the system or components. However, some systems are non-repairable, which means they cannot be repaired if they fail, such as light bulbs or pacemakers, while for repairable systems such as computers, air planes, wind turbines, etc., components or system can be repaired or replaced. Maintainability of components or systems has a significant impact on system availability, and reliability. Table 2.1 can show the difference of repairable and non-repairable systems.

Table 0.1 Comparison of repairable and non-repairable systems[104]

Non-repairable systems	**Repairable systems**
Discarded upon failure	Restored to operating conditions
Lifetime is random variable described by single time-to-failure	Lifetime is age of system or total time of operation
Group of systems – lifetime assumed independent & identically distributed (from same population)	Random variables of interest are times between failure and number of failures at particular age
Failure rate is hazard rate of a lifetime distribution – a property of time-to-failure	Failure rate is rate of occurrence of failures (ROCOF), a property of a sequence of failure times

Taylor and Ranganathan [105] introduced applications of Markov analysis to nonrepairable systems. Different system configuration with non-repairable components are considered in this study. Li et al [106] developed a new reliability model using a discrete time semi-Markov chain for a non-repairable system, where each component has multi-mode failures. Kim and Kim [107] analyzed the reliability of a nonrepairable system using

Markov chain, where the system is composed of heterogeneous components. A phase-type time-to-failure distribution is used for each component of the system. Mettas and Savva [108] used a simulation method to estimate reliability of non-repairable systems, and subsequently a software tool is developed that calculates the exact analytical solution for the reliability of a non-repairable system. Feng et al [109] studied the reliability of a complex non-repairable system with common cause failures using a survival signature-based simulation method. A double-loop Monte Carlo simulation is used instead of an analytical approach to enhance the propagation of the common cause failures. Bamrungsetthapong [110] analyzed the reliability of a non-repairable multi-state system where the failure rate is modeled using fuzzy Weibull distribution with uncertainty time.

Repairable systems are those systems that can be restored to fully satisfactory performance by a method other than replacement of the entire system [111]. In repairable systems, the time between the failures of a system is dependent on the repair strategy applied to the system [112]. Shu and Flower [113] investigated the stochastic behavior of the reliability of repairable systems. For a perfectly maintained system, the failed system or component is replaced by a new one. Lin et al [114] proposed a non-periodic condition-based maintenance policy for a deteriorating complex repairable system. Different condition variables are considered in the reliability model and subsequently the optimal maintenance policy is found by considering different scenarios which can assist in evaluating the maintenance cost for each scenario. Imperfect actions are considered in the maintenance policy to avoid the system failure.

Yi et al [115] studied reliability analysis of repairable systems with multiple fault modes. Markov process theory is used to model the failure process of the systems. Goal-oriented method is combined with Fussell–Vesely method for quantitative reliability analysis of a repairable system with multiple fault modes. Rafiee et al [116] proposed a condition-based maintenance policy considering imperfect repair for a repairable deteriorating system which is subject to *s*-dependent competing risks of internal degradation and external shocks. Stochastic deterioration process is used to model the internal degradation of the system and the external shocks categorized into the two types of fatal shocks and non-fatal shocks. Fatal shocks make the system fail immediately while non-fatal shocks damage the system by randomly increasing the degradation level. By minimizing the expected long-run maintenance cost rate function, the optimal inspection time for the system is found.

Fan et al [117] investigated a condition-based maintenance for a single unit repairable system subject to two statistically dependent failure modes which bidirectionally affecting each other. In this model, two failure modes are statistically dependent such that the hazard rate of one failure mode depends on the accumulated number of failures of the other failure mode. Imperfect maintenance actions are formulated for each failure mode, and the age reduction factor for each failure mode due to maintenance has some deterministic relation to the degree of resources cooperatively allocated to perform maintenance. Le and Tan [118] studied a single unit deteriorating system whose condition is inspected periodically. Each degradation level can be represented by a state, making the system a multi-state system which is modeled by continuous-time Markov process. A condition-based maintenance is proposed by finding the optimal inspection-maintenance schemes for such system.

1.7 Reinforcement learning algorithms and applications

Reinforcement learning is one type of machine learning, that trains an agent to decide how to perform an action based on the system state and associated rewards. Learning is one mechanism that could provide the ability for an agent to increase its intelligence while in operation [119]. Despite the supervised learning approaches, in reinforcement learning problems, an agent learns from examples provided by a knowledgeable external supervisor. By applying the trial-and-error to maximize the reward, the agent learns how to make decisions in an uncertain, complex environment. The applications of reinforcement learning nowadays are abundant, given the data-centric era that is approaching and the number of processes requiring accurate and optimal decision-making. A significant number of practical applications have been reported on the reinforcement learning technique. [100, 120-123].

One of the reinforcement learning algorithms is Q-learning, which is the most commonly used algorithm. Q-learning does not require any prior knowledge about the environment and state transition of the system. By iteratively experiencing trajectory paths and their corresponding sets of rewards and states, the agent learns which action should be taken at each specific state in order to maximize our expected reward. However, when the number of states or actions become very large. Q-learning algorithm would not be as efficient as before for two reasons. Firstly, the required memory required to save and update the whole table increases as the number of states increases. Secondly, the required time to explore all the states and create the Q-table would be unrealistic. Therefore, in this study, another algorithm called deep Q learning is used to approximate the Q-value function by training a neural network.

Q-learning algorithm is originated in the work of [124] and becomes one of the most commonly used algorithm to solve various problems in different fields. Zhang et al [125] used Q-learning algorithm for a energy-efficient scheduling problem to reduce the energy consumption of the processor(s), and to extend the utility time of edge computing devices. Shen et al [126] studied a ship stowage planning problem using a Q-learning algorithm. They solved several real-world production cases using their proposed model. Low et al [127] proposed an improved Q-learning algorithm to solve a path planning problem of a mobile robot. Samma et al [128] studied a new optimization method for a signal optimization model using a Q-learning algorithm. Mosadegh et al [129] proposed a new mathematical model for a stochastic mixed-model sequencing problem using Q-learning algorithm. Wang [130] investigated a scheduling strategy with adaptive features for job shops considering the dynamic and uncertain production environment using A weighted Q-learning algorithm.

Deep reinforcement learning is the combination of reinforcement learning and deep learning, which is useful for problems with a large number of states or actions. In this study, deep reinforcement learning is used to find the best maintenance policy based on the system degradation level. The degradation process of the system is modeled using the gamma process, and at each inspection time, the best action can be suggested using the proposed maintenance model to minimize the maintenance cost for the duration of the maintenance contract.

Deep Q-learning is an alternative algorithm to solve a problem with huge state and action spaces or when the state or action spaces is continuous. Deep Q-learning is a combination of Q-learning and deep leaning. In deep Q-learning, the Q-values are approximated by using a neural network. The deep Q learning method tries to recognize patterns instead of mapping every state to its best action. Tong et al [131] used a deep Q-learning algorithm in task scheduling in cloud computing. Tommy et al [132] studied the automated vehicles negotiating with other vehicles, typically human driven, in crossings with the goal to find a decision algorithm by learning typical behaviors of other vehicles using a deep Q-learning algorithm. Mohanty et al [133] used a deep learning algorithm for a path planning and obstacle avoidance problem of a wheel mobile robot. Liu et al [134] investigated on deep reinforcement learning for lung cancer detection.

There are two main advantages of reinforcement learning approaches over their traditional dynamic programming counterparts to determine a cost-effective maintenance plan. The first one is that traditional dynamic programming algorithms like value iteration or policy iteration, are model-based methods. These methods know how the environment works and so they can predict the next states that they are going to enter or the rewards that they are going to receive. Model-based approaches can become impractical in many realistic applications [135]. On the contrary, the RL algorithm used in this research work, namely Q-learning, is a model-free approach. It does not require knowledge about the environment, and therefore, can "learn" an optimal policy by iteratively experiencing trajectory paths and their corresponding sets of rewards and states.

The second, and maybe the most important, advantage has to do with computational efficiency. Traditional dynamic programming approaches require full backups in order to estimate the Q-values of different state-action pairs. On the contrary, by using Q-learning someone is able to estimate these in constant time per iteration. This constitutes a meaningful advantage, especially when considering large state and action spaces.

It is indeed true that machine learning is not new for maintenance optimization. However, in this problem the purpose is to use this specific branch of machine learning, in order to efficiently solve a sequential decision-making problem. This is by nature different than what most machine learning approaches in the maintenance field have tried in the past, like prediction of next failure. In this work, I formulate the maintenance optimization in a novel format and attempt to solve it holistically, that is, to derive directly optimal maintenance policies, without explicitly assuming failure time knowledge, such as assumed distributions.

Moreover, in most of the previous studies, model-based methods are used to establish the maintenance model of systems. In model-based methods, the system behavior and its environment are modeled using stated assumptions and parameters. In some other studies, machine learning methods such as reinforcement learning are used, and an agent learns how to make decisions through sufficient training by trial-and-error steps. There have been few research studies using reinforcement learning for maintenance models. Tang, et al. [136] applied a reinforcement learning method to select the maintenance scheduling of a fighter aircraft. Rocchetta, et al. [137] used reinforcement learning to schedule the preventive maintenance actions to maximize the power grid load. Huang et al [138] developed a new group maintenance and opportunistic maintenance for a production line using deep reinforcement learning. Wang, et al. [139] used reinforcement learning method to develop a maintenance model for a flow line system of two series machines an intermediate finite buffer in between.Correa-Jullian et al [140] developed a dynamic condition-based maintenance model for operation scheduling using reinforcement learning for a solar hot water system. In most of the dynamic models in the literature, the system is considered as one unit without considering different components, and the main focus is finding the inspection time dynamically, but not the maintenance actions. However, Yousefi et al [100] developed a dynamic condition-based maintenance model for a multi-component system using reinforcement learning to find the maintenance action dynamically. They considered different regions based on some predefined thresholds to formulate the Markov decision problem.

Establish baseline models

In this chapter, some baseline models are initially described for a multi-component system subject to degradation and random shock process. Yousefi et al. [141, 142] developed a reliability model for systems and subsequently a maintenance model can be provided based on the reliability and system configuration of each specific system. The initial baseline models are for systems of degrading components which are maintained together, and then the models can be extended to the individual component maintenance case.

In this study, it is considered that each component degrades so that irreversible damage gradually occurs, and the degradation model is monotonically increasing. In this case, it is appropriate to use the gamma process to model the degradation path. A thorough review of the gamma process model and its applications can be found in Van Noortwijk [143]. For our applications, the gamma process with positive shape parameter is linear in t, with shape parameter $\alpha_i(t)$ and scale parameter β_i is a continuous time stochastic process with the following properties:

- It starts from 0 at time 0, i.e., $X_i(0) = 0$
- $X_i(t)$ has independent increment

- for $t > 0$ and $s > 0$, $X(t) - X(s) \sim gamma(\alpha_i(t) - \alpha_i(s), \beta_i)$

In fact, the probability density function of degradation process for each component i, $X_i(t) - X_i(s)$ is given by:

$$g(x; \alpha_i(t) - \alpha_i(s), \beta_i) = \frac{\beta_i^{\alpha_i(t)-\alpha_i(s)} x^{(\alpha_i(t)-\alpha_i(s))-1} \exp(-\beta_i x)}{\Gamma(\alpha_i(t) - \alpha_i(s))} \tag{3.0.1}$$

where $\alpha_i(t)$ and β_i are the shape parameter and scale parameter for component i. In other words, the degradation process of component i between two time intervals t and s, follows gamma distribution with shape parameter of $\alpha_i(t) - \alpha_i(s)$. The shape parameter of gamma process can be in two forms of linear and nonlinear for degradation process. For expected linear degradation, the shape parameter is in form of $\alpha_i(t) - \alpha_i(s) = \alpha_i(t - s)$ and for non-linear degradation it can be expressed as $\alpha_i(t) - \alpha_i(s) = \alpha_i t^b - \alpha_i s^b = \alpha_i(t^b - s^b)$ where b is the power of time interval t. In general, b is very unlikely to be greater than 2 because the reliability of normal industrial products has no physical reason to degrade so quickly except defective goods. Gamma process can be expressed as an incremental process which can be shown on Figure 3.1. Due to the stochastic property of the gamma process, with the same shape and scale parameter, the degradation paths are different.

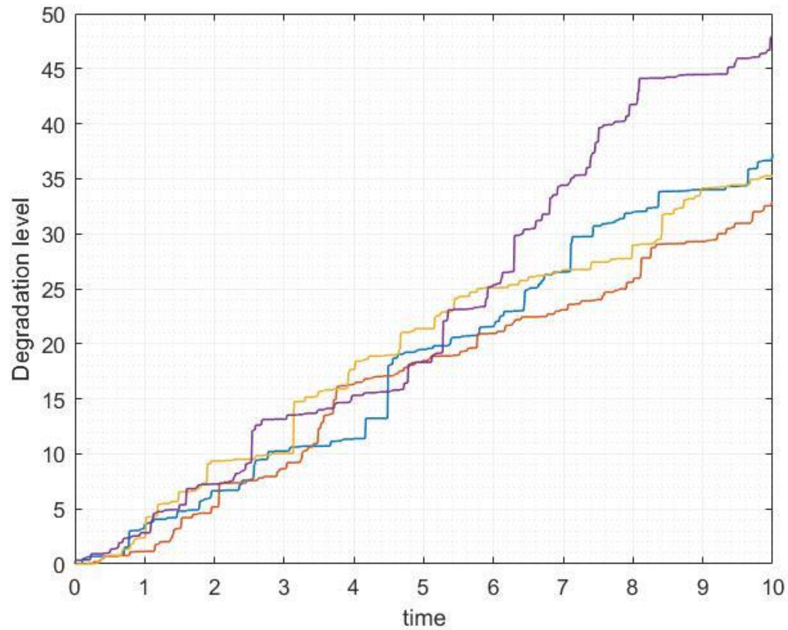


Figure 0.1 Degradation process modeled by gamma process

The shape parameter α_i controls the rate of jump arrivals (incremental increases in degradation) and when α_i equals to some specific value, gamma distribution can be a special case of another specific distribution like exponential distribution, Erlang distribution, etc. Gamma distribution is flexible and an appropriate model for simulations of many different processes. The scale parameter β_i controls the range of the gamma distribution. A gamma distribution with $\beta_i = 1$ is known as the standard gamma distribution. For a gamma process, the shape parameter is a function of the starting and ending time of the time period being considered.

One of the most common failure types is failure due to the degradation of systems or components which is called soft failure. At any time, if the cumulative degradation process of a system or component is greater than a predefined failure threshold, it is detected as failed due to soft failure, and based on the maintenance plan, a maintenance action should be implemented.

The second most common failure type is failure due to external shock arrival which is called hard failure. Due to the variety of external shocks, some of the research assumes that the shock magnitude follows some specific distributions, while others consider only normal distributions. Also, shocks arrive at random times and these time intervals could have different interarrival times which follow different distributions. Most previous research assumes these shocks arrive as a Poisson process, which is the most common arrival situation, so the time between shocks is an exponential random variable. The probability of m shocks arriving in a time interval τ is presented in the equation below:

$$P\left(N(t+\tau)-N(t)=m\right)=\frac{e^{-\lambda\tau}(\lambda\tau)^m}{m!} \tag{3.0.2}$$

The degradation process and shock process are the failure processes which are considered for this study. There are different types of dependency between these processes, which is considered in the following chapters.

Some of the assumptions which are considered in all the models of this study are as follow:

- Soft failure occurs for the ith component when the total degradation of that component exceeds its critical threshold level H_i^1. Component degradation is accumulated by both continuous degradation over time and cumulative incremental damage due to random shocks.
- When the shock size exceeds the hard failure threshold of any component i (D_i), hard failure occurs of that component.
- Random shocks arrive as a Poisson process.

1.8 System reliability and maintenance modeling for degrading multi-component systems

Due to the inevitable deterioration of many components and system, systems may fail. To restore a failed system is often time-consuming and costly. Periodic and frequent inspection and repair/replacement can reduce the probability of deterioration and failure; however, it also incurs potentially excessive maintenance cost [144]. High quality operational performance and low maintenance cost can then become two conflicting objectives. For systems whose penalty cost due to downtime is high, detecting the component status and assisting in repair/replacement decision-making before system failure, leads to low risk of failure, and subsequently, lower maintenance cost. For systems whose costs associated with failure are high, it is advantageous to repair or replace the components or system before the failure occurs.

The concept of condition monitoring and on-condition thresholds for the components is used to evaluate and measure system status, and therefore, increase the opportunity to detect the components' critical and degraded situation and to avoid costly failure events. Maintenance optimization is based on reliability modeling of system subject to dependent and competing failure processes. The maintenance optimization is challenging because of the dependent degradation and dependent failure times among all components. For some systems, the cost and consequence of failure are excessive compared to comparable preventive repair cost, replacement cost or other kinds of cost. Therefore, it is prudent to prevent failure from occurring and replace the equipment at the earliest convenience after it has sufficiently aged, rather than allowing to fail and possibly cause more severe consequences. Yousefi et al [141] developed a new condition-based maintenance model for a multi-component system by formulating the system reliability and optimizing the average maintenance cost rate.

By providing a new maintenance threshold (on-condition threshold), the components or systems which are close to failure can be detected and the corresponding maintenance plan can be implemented to avoid the system failure. By maintaining the component or system before failure happens, the maintenance cost can be reduced by avoiding the penalty cost due to the system downtime. However, finding the optimal on-condition threshold and inspection interval simultaneously is a challenge which is studied in this chapter. Low on-condition thresholds can be inefficient because they waste components life, and high on-condition thresholds are risky because the components are prone to costly failure.

In this chapter, an optimization model is developed to determine on-condition thresholds and inspection intervals for multi-component systems with each component experiencing multiple failure processes. Initially a reliability model is presented for systems in which failure processes for each component are dependent and failure times for all components are dependent. I introduce a working principle for defining the on-condition threshold and system status. A periodic inspection maintenance policy is selected so that the decision-making depends on the on-condition thresholds for all components. Finally, a maintenance cost rate model is developed.

A system is considered where each component within the system can fail due to two competing dependent failure processes that share the same shock process; soft failure process and hard failure process [2], as depicted in Figure 3.2. Each component in the system degrades with time, and when a shock arrives, if damage is greater than hard failure threshold, catastrophic failure occurs. If the component survives the shock, total degradation containing both pure degradation and additional incremental degradation caused by shock damage is greater than a defined soft failure threshold level, soft failure occurs. The two failure processes are competing and dependent.

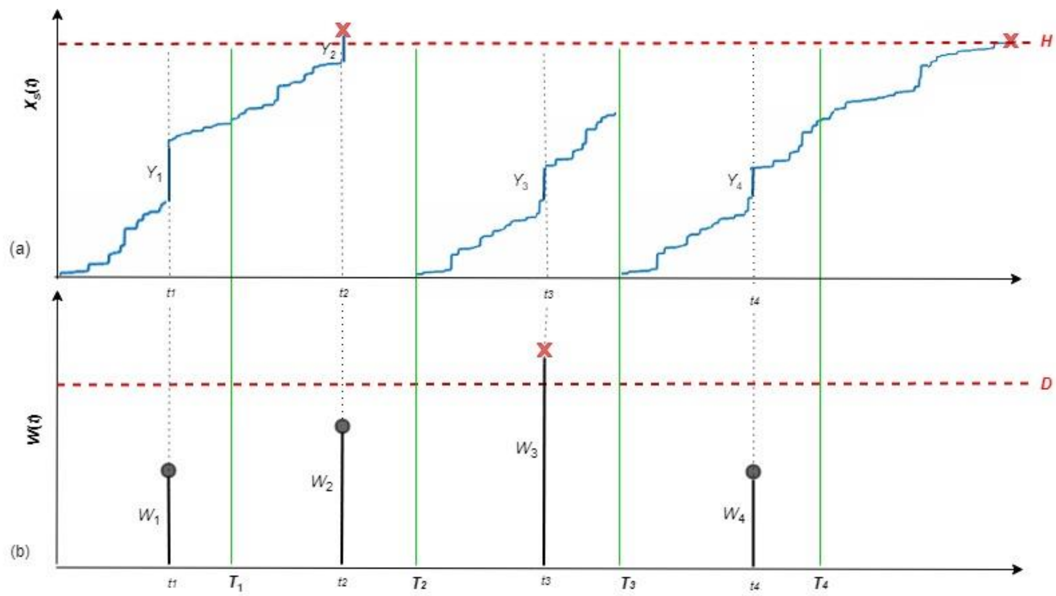


Figure 0.2 Two dependent and competing failure processes for a component (a) soft failure process and (b) hard failure process [144]

Two failure processes for each component are dependent, and failure times for all components are also dependent. Component hard failures occur when a shock load exceeds thresholds. Figure 10 shows that component *i* may fail when damage from a shock exceeds D_i. W_{ij} is the shock size and it is an *i.i.d*. random variable with some defined distribution which is assume in this study as a normal distribution, $W_{ij} \sim N(\mu_{W_i}, \sigma_{W_i}^2)$, with selected μ_{w_i} and σ_{w_i} such that the probability of having negative W_{ij} is insignificant. This is not a restriction for our model and depending on μ_{w_i} and σ_{w_i}, considering a truncated normal distribution is also an effective way to avoid having negative W_{ij}. The probability density function of truncated normal distribution is shown in Equation (3.3)

$$f_{W_i}(w)=\begin{cases}0\ , & \text{for } w<0\\ \dfrac{\dfrac{1}{\sqrt{2\pi\sigma_{W_i}^2}}e^{-\frac{(w-\mu_{W_i})^2}{2\sigma_{W_i}^2}}}{1-F_{W_i}(0)}, & \text{for } w\geq 0\end{cases} \tag{3.0.3}$$

The probability that the i^{th} component survives a shock [3] is given by:

$$P_{Li}=P(W_{ij}<D_i)=F_{Wi}(D_i)=\Phi\left(\frac{D_i-\mu_{W_i}}{\sigma_{W_i}}\right)\ \text{for } i=1,2,\ldots,n, \tag{3.0.4}$$

where $\Phi(\cdot)$ is the cdf of a standard normal random variable.

As shown in Figure 3.2, total degradation of the i^{th} component can be accumulated as $X_{S_i}(t)=X_i(t)+S_i(t)$, and when $X_{S_i}(t)>H_i^1$, soft failure occurs. Conditioning on the number of shocks and using a convolutional integral of $X_{S_i}(t)$, the probability that component i does not experience soft failure before time t can be obtained as follow:

$$\begin{aligned}P(X_{S_i}(t)<H_i^1)&=F_{X_i}(H_i^1,t)\\&=\sum_{m=0}^{\infty}P\left(X_i(t)+\sum_{j=1}^{m}Y_{ij}<H_i^1\right)\frac{\exp(-\lambda t)(\lambda t)^m}{m!}\\&=\sum_{m=0}^{\infty}\left(\int_0^{H_i^1}G_i(H_i^1-u,t)\,f_{Y_i}^{<m>}(u)du\right)\frac{\exp(-\lambda t)(\lambda t)^m}{m!}\end{aligned} \tag{3.0.5}$$

$X_i(t)$ follows a gamma process, so $G_i(\cdot)$ is the cdf for a gamma distribution. It is convenient for Y_i to be gamma or normal distributed because the sum of m iid gamma random variables is also gamma, and the sum of m iid normal random variables is normal. In Song et al. [7], the assumption was made that Y_i was normally distributed, while in this study, it is assumed Y_i is gamma distributed, but this is not a restriction.

1.8.1 Reliability Analysis for Multiple Components System with MDCFP

Our example system configuration is a series system, in which a component fails when either of the two dependent and competing failure modes occurs, and all components in the system behave similarly. Song at al. [7] developed a multi-component system reliability model when each component experiencing multiple failure processes due to each component degradation and external shock loads. The reliability of this series system can be obtained, since the system fails when the first component fails. Later, Yousefi et al [141] extended this model by developing a new condition-based maintenance model for a multi-component system subject to dependent failure processes.

Figure 3.3 shows a series system with *n* components. The reliability of this series system at time *t* is the probability that each component survives each of the $N(t)$ shock loads ($W_{ij} < D_i$ for j =1, 2, …) and the total degradation of each component is less than the soft failure threshold level ($X_{Si}(t) < H_i^1$ for all i) .

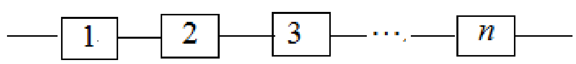


Figure 0.3 Series system example

In this model, shocks arriving at random time intervals are modeled as a Poisson process. When the system receives a shock (at rate λ), all components experience a shock. If the component survival probabilities are conditioned on the number of shocks, then the failure processes for all components become independent for a fixed number of shocks. System reliability function can be derived for the general case for a series system as shown in Equations 3.3.6 and 3.3.7. [7]:

$$R(t)=\sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[P(W_i<D_i)^m P\left(X_i(t)+\sum_{j=1}^{m}Y_{ij}<H_i^1\right)\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{3.0.6}$$

Using convolution integral, it can be obtained as follow:

$$R(t)=\sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[P(W_i<D_i)^m\int_0^{H_i^1}G_i(H_i^1-u,t)\,f_{Y_i}^{<m>}(u)du\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{3.0.7}$$

For the multi-component system considered in this research, the components are packaged and/or sealed together and it is reasonable or necessary to replace the whole system before the critical degradation thresholds are reached. The proposed model is useful to avoid failure by replacing the system before it fails. On-condition rules provide the capability to measure system status and replace the system before failure to avoid system downtime. Based on defined rules, the implementation of a lower degradation threshold can be useful to avoid failure by providing criteria to detect the degradation status of the components.

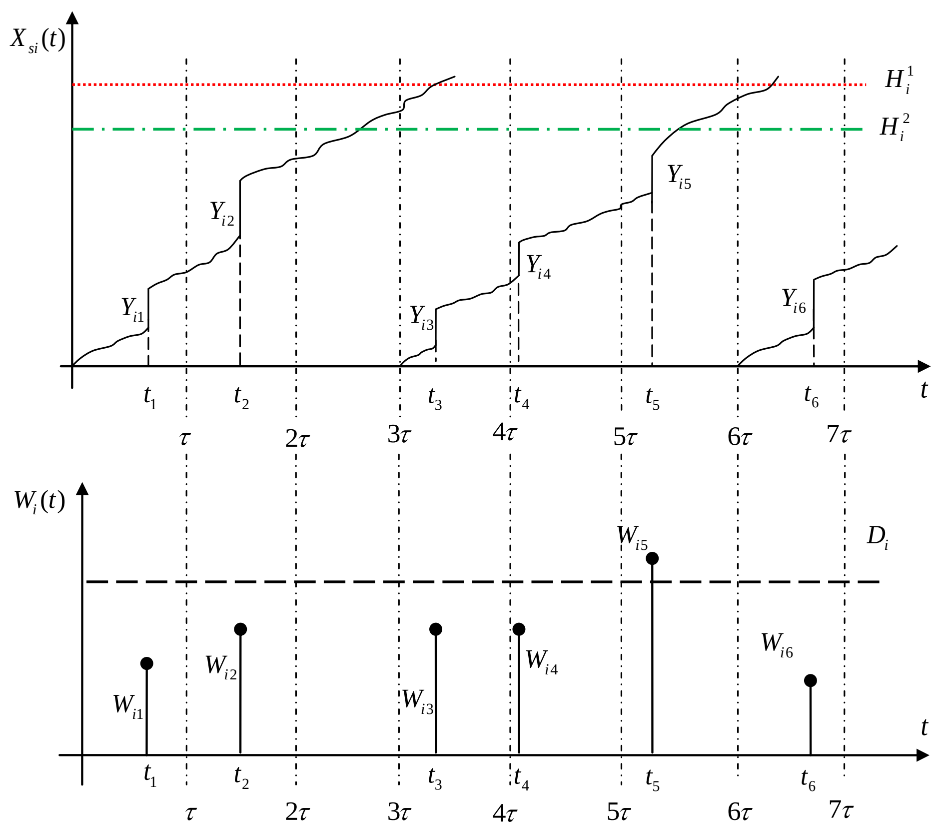


Figure 0.4 Two thresholds divide system status into three regions

H_i^1 is defined as the soft failure threshold for component *i* and H_i^2 is now defined as the on-condition threshold for component *i*, with $H_i^2 \leq H_i^1$. At each inspection time, component condition is determined for each component by inspection and compare it to a threshold. The action taken depends on a selection of condition-based operational status data and the defined maintenance condition rules. In Figure 3.4, a fixed on-condition threshold H_i^2 for component i (lower bar and dash line in soft failure process) can be observed. The rules can be adopted related to this on-condition degradation threshold to define the component degradation state.

At each inspection interval, if no hard failure occurs, and at the same time, total degradation of the i^{th} component is less than H_i^2, the on-condition threshold for i^{th} component, the component is in the safe region. The safe region is defined as the combination of soft failure process and hard failure process both below their respective thresholds and this status is defined as event A shown in Table 3.1. If no hard or soft failure occurs and total degradation is between H_i^2 and H_i^1 for any component i, this component has not failed; however, probabilistically it may fail within a short period. This status can be described by the combination of soft failure process area between H_i^2 and H_i^1, and hard failure process area below the hard failure threshold, which is defined as event B as presented in Table 3.1. If there has been a hard failure or the total degradation of any component i is greater than H_i^1 (higher dash line in soft failure process), the system has failed. The status can be defined as the union of the soft failure process area above the red dashed line, and hard failure process area above black dashed line, and this status is defined as event C.

Table 0.1 Component status defined with two soft failure thresholds and hard failure threshold

A	component is in safe region	$P(A)=\sum_{m=0}^{\infty}P(W_i<D_i)^m\left(\int_0^{H_i^2}G_i(H_i^2-u,t)\,f_{Y_i}^{<m>}(u)du\right)\frac{\exp(-\lambda t)(\lambda t)^m}{m!}$
B	component is working, but probabilistically fails soon	$P(B)=\sum_{m=0}^{\infty}P(W_i<D_i)^m\left(\int_0^{H_i^1}G_i(H_i^1-u,t)\,f_{Y_i}^{<m>}(u)du-\int_0^{H_i^2}G_i(H_i^2-u,t)\,f_{Y_i}^{<m>}(u)du\right)\times\frac{\exp(-\lambda t)(\lambda t)^m}{m!}$
C	component fails	$P(C)=\sum_{m=0}^{\infty}\left(1-P(W_i<D_i)^m\int_0^{H_i^1}G(H_i^1-u,t)\,f_{Y}^{<m>}(u)du\right)\frac{\exp(-\lambda t)(\lambda t)^m}{m!}$

Considering the safe region case for example, conditioning on *m* shocks arriving to the system by time *t* with probability $\frac{\exp(-\lambda t)(\lambda t)^m}{m!}$, the probability of no hard failure is $P(W_i<D_i)^m$, and the probability that total degradation is less than H_i^2 is $\int_0^{H_i^2}G_i(H_i^2-u,t)\,f_{Y_i}^{<m>}(u)du$. Combining both soft failure process and hard failure process, the probability for event *A*: the component *i* is in safe region is:

$$P(A_i)=\sum_{m=0}^{\infty}P(W_i<D_i)^m\left(\int_0^{H_i^2}G_i(H_i^2-u,t)\,f_{Y_i}^{<m>}(u)du\right)\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{3.0.8}$$

Similarly, for event *B*, component *i* is still working, but it may probabilistically fail within the next inspection interval, the probability of no hard failure considering on m shocks is $P(W_i<D_i)^m$ and the probability that total degradation is between H_i^1 and H_i^2 is $\int_{H_i^2}^{H_i^1}G_i(H_i^1-u,t)\,f_{Y_i}^{<m>}(u)du$. Combining both soft failure process and hard failure process; the probability for event *B* can obtained. For event *C*, either soft failure or hard failure occurs, the and probability equals to one minus the probability that neither of these two failure happens. The policy is summarized in Table 2.

Specific assumptions used for the reliability and maintenance modeling in this study are as follow:

- The model is for systems that are packaged and sealed together, making it impossible or impractical to repair or replace individual components within the system, e.g., MEMS.
- For the maintenance policy, the system is inspected at periodic intervals and no continuous monitoring is performed. Replacements are assumed to be instantaneous and perfect.
- At any inspection time, if the degradation of any component i is lower than its own on-condition threshold H_i^2, component i is in the safety level; hence, the system is within the high safety level area if all the components are in their own safety level areas. It should be noted that each component has its own unique on-condition threshold which can be distinctly different from other components.
- Upon an inspection, if the degradation of any component i is between its own failure threshold H_i^1 and its on-condition threshold H_i^2, it has not failed but it can be anticipated to fail, and for a series system, failure of any component causes system failure. Therefore, it is advantageous to replace the system to avoid downtime when any (or possibly more than one) component i exceeds H_i^2 for an increasing degradation path.
- If the system fails, that is, the total degradation of any component i is higher than its own H_i^1 before the specified inspection interval, it is not immediately detected and not replaced until the next inspection. There is penalty cost per time associated with the failure of system during downtime, e.g., cost associated with loss of production,

opportunity costs, etc.

Given this reliability model for systems with each component experiencing multiple failure processes due to simultaneous exposure to degradation and shock loads, O can define a maintenance cost optimization objective function. The system is inspected periodically, and the condition of each component is observed and compared to a threshold. Upon an inspection, the system is replaced with a new one, when it is observed that a hard failure has occurred or total degradation is greater than the on-condition threshold for any component i.

The expected number of inspections N_I, for a vector on-condition thresholds $\boldsymbol{H}^2$ = $(H_1^2, H_2^2, ..., H_n^2)$ is given by,

$$E(N_I) = \sum_{k=1}^{\infty} k(F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau)) \tag{3.0.9}$$

Where the probability that at least the degradation of one component is above its own on-condition threshold by time t can be calculated.

$$F_T^{\mathbf{H}^2}(t) = 1 - (\sum_{m=0}^{\infty} \prod_{i=1}^{n} \left[P(W_i < D_i)^m \int_0^{H_i^2} G_i(H_i^2 - u, t) f_{Y_i}^{<m>}(u) du \right] \frac{\exp(-\lambda t)(\lambda t)^m}{m!}) \tag{3.0.10}$$

System downtime is the time duration between the time a failure occurs and the next time an inspection is performed, and a failure detected, which is shown on Figure 3.5.

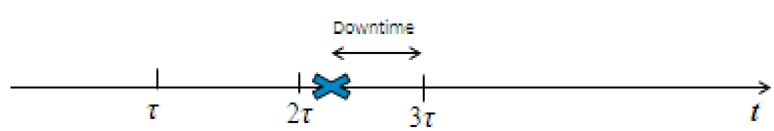


Figure 0.5 System downtime under periodic inspection maintenance policy

Conditioning on the event that there is a failure at time t between the $(k$-1)th and kth inspection $[(k$-1$)\tau$, $k\tau]$ with probability $\left(F_T^{\mathbf{H}^2}(k\tau) - F_T^{\mathbf{H}^2}((k-1)\tau)\right)$, and defining the failure time as t, the system downtime is $k\tau$ - t. The expected value of system downtime or the expected time from a system failure to the next inspection when the failure is detected, can then be determined as $\int_{(k-1)\tau}^{k\tau}(k\tau-t)dF_T^{\mathbf{H}^1}(t)$. Summing over the probability that failure can occur in any inspection interval, the expected system downtime can be estimated as shown in Equations 3.3.11 and 3.3.12.:

$$E[\rho]=\sum_{k=1}^{\infty}E[\rho\mid N_I=k]P(N_I=k)=\sum_{k=1}^{\infty}\left(\left(F_T^{\mathbf{H}^2}(k\tau)-F_T^{\mathbf{H}^2}((k-1)\tau)\right)\int_{(k-1)\tau}^{k\tau}(k\tau-t)dF_T^{\mathbf{H}^1}(t)\right) \tag{3.0.11}$$

$$F_T^{\mathbf{H}^1}(t)=1-\left(\sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[P(W_i<D_i)^m\int_0^{H_i^1}G_i(H_i^1-u,t)\,f_{Y_i}^{<m>}(u)du\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!}\right) \tag{3.0.12}$$

The expected time between two replacements or expected cycle length is

$$E[K]=\sum_{k=1}^{\infty}E[K\mid N_I=k]P(N_I=k)=\sum_{k=1}^{\infty}k\tau(F_T^{\mathbf{H}^2}(k\tau)-F_T^{\mathbf{H}^2}((k-1)\tau)) \tag{3.0.13}$$

1.8.2 Condition-Based Maintenance Modeling and Optimization

With the on-condition rules stated in last section, a condition-based maintenance policy can be defined for the system with multiple components each exposed to two competing dependent failure processes. Condition-based maintenance offers the promise of enhancing the effectiveness of maintenance programs. For some cases, the penalty cost due to downtime is relatively higher than the comparable corrective maintenance costs, so it is cost-effective to replace the whole system before the wear volumes of components reach their failure thresholds, However, there are some other cases that replacing the system upon failure is more beneficial because you obtain maximum system life and downtime costs are small. In this study, if the optimal on-condition threshold is the same as failure threshold, i.e., H_i^2 = H_i^1, it is the case that the implementing preventive maintenance before failure is not necessary or even beneficial.

On-condition degradation threshold can achieve our goal of replacing the system before failure by providing the criteria to detect the degradation of component beyond the on-condition threshold. If the on-condition threshold is too low and far away from the nominal threshold level, then it is beneficial to replace the whole system more frequently, and it results in extra cost due to the waste of system life. Alternatively, if the threshold is too high, then the system may fail before the next inspection leading to potentially expensive downtime cost. Therefore, on-condition degradation thresholds for all components and an inspection interval for the whole system are chosen to be decision variables in this maintenance optimization problem.

To evaluate the performance of the condition-based maintenance policy, an average long-run maintenance cost rate model is used, in which the periodic inspection interval τ for the whole system and on-condition thresholds H_i^2 for all components are the decision variables. At time τ, and subsequent inspection intervals of time τ, the entire assembled system is inspected. If the system is still operating satisfactorily with no component wear volume above the on-condition threshold, nothing is done. If degradation thresholds for all component are below the fixed critical degradation thresholds H_i^1 but some are above the on-condition threshold H_i^2, the whole system is replaced preventively. If there is a hard failure or at least one component's wear volume is above the critical degradation threshold H_i^1 prior to inspection, then the system is not replaced with a new one correctively until the next inspection. The average long-run maintenance cost per unit time can be evaluated by:

$$\lim_{t\to\infty}(C(t)/t)=\frac{\text{Expected maintenance cost between two replacements}}{\text{Expected time between two replacements}}=\frac{E[TC]}{E[K]} \qquad (3.0.14)$$

Where TC is the total maintenance cost of a renewal cycle, and K is the length of a cycle that takes a value of a multiple of τ [36]. The expected total maintenance cost is given as:

$$E[TC]=C_I E[N_I]+C_\rho E[\rho]+C_R \qquad (3.0.15)$$

Where C_I is the cost of each inspection. C_R is the replacement cost, C_ρ is the penalty cost incurred during down time, and τ is the time interval for periodic inspection. Based on Equation (3.9) to (3.11), the average long-run maintenance cost rate is given as

$$CR(\tau,\mathbf{H}^2)=\frac{C_I\sum_{k=1}^{\infty}k\left(F_T^{\mathbf{H}^2}(k\tau)-F_T^{\mathbf{H}^2}((k-1)\tau)\right)+C_\rho\sum_{k=1}^{\infty}\left(\left(F_T^{\mathbf{H}^2}(k\tau)-F_T^{\mathbf{H}^2}((k-1)\tau)\right)\int_{(k-1)\tau}^{k\tau}(k\tau-t)dF_T^{\mathbf{H}^1}(t)\right)+C_R}{\sum_{k=1}^{\infty}k\tau(F_T^{\mathbf{H}^2}(k\tau)-F_T^{\mathbf{H}^2}((k-1)\tau))}$$

(3.0.16)

For our maintenance optimization problem, if there are n components in a series system, there are n+1 decision variables; namely n on-condition thresholds for all components and the periodic inspection interval for the whole system. Our objective is to minimize maintenance cost rate, and constraints are that on-condition thresholds for all components should be less than or equal to their critical failure thresholds, and inspection interval should be a positive value. Therefore, our maintenance optimization problem can be formed as Equation 3.3.17:

$$\begin{aligned}
\min \quad & CR(\tau,\mathbf{H}^2) \\
\text{s.t.} \quad & 0\le H_1^2\le H_1^1, \\
& 0\le H_2^2\le H_2^1, \\
& \quad\ldots \\
& 0\le H_n^2\le H_n^1, \\
& \tau\ge 0,
\end{aligned} \qquad (3.0.17)$$

Yousefi et al [141] proposed a new condition-based maintenance model by solving the maintenance optimization problem which is shown in (3.17), and finding the optimal on-condition maintenance threshold for each component a long with optimal inspection time of the whole system.

It is a difficult non-linear optimization problem but with continuous decision variables and a convex feasible region. For constrained nonlinear optimization problems, there are many available algorithms to obtain optimal solutions. Interior point methods have proved to be very successful in solving many nonlinear problems in different research problems [145-150].The Interior point method consists of a self-concordant barrier function used to encode the convex set. It reaches an optimal solution by traversing the interior of the feasible region using one of two main types of steps at each iteration [151]. The algorithm first attempts to take a direct step within the feasible region to solve the Karush Kuhn Tucker (KKT) equations for the approximate problem by a linear approximation, which is also called a Newton step. By solving the KKT equations, the direct step and the solution for the next iteration is found. If it cannot take a direct step, it attempts a conjugate gradient step, and minimizes a quadratic approximation to the approximate problem in a trust region, subject to linearized constraints. It does not take a direct step is when the problem is not locally convex near the current iteration. At each iteration, the algorithm decreases a merit function, and a new solution point is reached after taking the step and start a new iteration. It continues until stopping criterion is met. In this study, to solve the optimization problem, an interior point method is used (as implemented as the *fmincon* algorithm in the MATLAB optimization toolbox). *Fmincon* in Matlab is easy to use, robust and has wide variety of options. The built-in parallel computing support in *fmincon* accelerates the estimation of gradients. There have been some studies show the preference of using *fmincon* in solving nonlinear optimization problems.

1.8.3 Numerical example for condition-based maintenance of multi-component system

I consider some numerical examples; the first one is a series system with four components where component 1 and 2 have the same parameters and component 3 and 4 are also the same. The second one is for a system with four different components, and the third one is a series system with four identical components with replacement cost dependent of number of aged and failed component. The parameters for reliability analysis of these examples are provided in Table 3.2. Y_{ij} follows gamma distributions and W_{ij} follows normal distributions in both examples. For the first example, it is assumed that those parameters of component 1 and 2 are the same, and parameters of component 3 and 4 are the same. This is a conceptual example to demonstrate the reliability function and maintenance models. However, although the example is conceptual, H_i^1 and D_i are estimated based on documented degradation trends [152]. In this part, maintenance optimizations are performed for both series system and all the individual components making up the system separately, and the results are discussed.

Table 0.2 Parameter values for multi-component system reliability analysis for the first example

Parameter	component 1 & 2	component 3 & 4	Sources
H_i^1	0.00125 μm^3	0.00127 μm^3	Tanner and Dugger [152]
D_i	1.5 Gpa	1.4 Gpa	Tanner and Dugger [152]
α_i	0.7	0.8	Assumption
β_i	0.3	0.3	Assumption
λ	2.5×10^{-5}	2.5×10^{-5}	Assumption
Y_{ij}	$Y_{ij} \sim gamma(\alpha_{Y_i}, \beta_{Y_i})$ $\alpha_{Y_i} = 0.4, \beta_{Y_i} = 1$	$Y_{ij} \sim gamma(\alpha_{Y_i}, \beta_{Y_i})$ $\alpha_{Y_i} = 0.5, \beta_{Y_i} = 1$	Assumption
W_{ij}	$W_{ij} \sim N(\mu_{Wi},\sigma_{Wi}^2)$ μ_{Wi} =1.2 GPa, σ_{Wi} =0.2 GPa	$W_{ij} \sim N(\mu_{Wi},\sigma_{Wi}^2)$ μ_{Wi} =1.22 GPa, σ_{Wi} =0.18 GPa	Assumption

First, the maintenance policy for the whole series system is considered with four components and a predetermined inspection interval, i.e., the whole system is inspected at one interval of τ and replace the system when the wear volume is above H_i^2 for any component. For some systems, there is a fixed or known inspection interval that is imposed by the application, the decision-maker or availability of the system for inspection. The system can only be inspected at those fixed intervals, which be far from optimal. Therefore, to compare these cases with proposed model, two fixed values are selected as possible inspection intervals and the optimal on-condition thresholds and cost rate functions are found for these cases.

The first case has a very long inspection interval of τ=120 hours, choosing C_I=\$1, C_p=\$20000 and C_R=\$100, and the minimum average long-run maintenance cost rate for system is \$3.054×10^2 and on-condition degradation threshold are H_1^{2*}= H_2^{2*}=0.0001556, H_3^{2*}= H_4^{2*}=0.0001370. Moreover, by considering a shorter fixed inspection interval of τ=24 hours, the minimum average long-run maintenance cost rate for system reduces to

\$2.2796×10^2^ and on-condition degradation threshold are $H_1^{2*}= H_2^{2*}$=0.0004637, $H_3^{2*}= H_4^{2*}$=0.0004204. When the system is inspected more frequently, there are higher on-condition degradation thresholds, i.e., closer to the failure threshold. Since the system status is detected more often, it can be replaced preventively, so on-condition degradation thresholds is closer to failure thresholds.

The contribution of this paper is to now simultaneously determine the optimal on condition thresholds and inspection interval. The minimum average long run maintenance cost rate for the system is \$1.9023×10^2^ found after 22 steps of iteration. The inspection interval is τ^*=44.7129 hours, and on-condition degradation thresholds are $H_1^{2*}= H_2^{2*}$=0.0003055 and $H_3^{2*}= H_4^{2*}$=0.0002728. Figure 5 illustrates the iteration process of decision variables: inspection interval, on-condition degradation threshold for component 1 and 2, and on-condition degradation threshold for component 3 and 4. From Iteration 10 on Figure 3.6 and 3.7 the optimal values do not change; however, the algorithm continued to confirm that there is no additional improvement and the optimal solutions are converged.

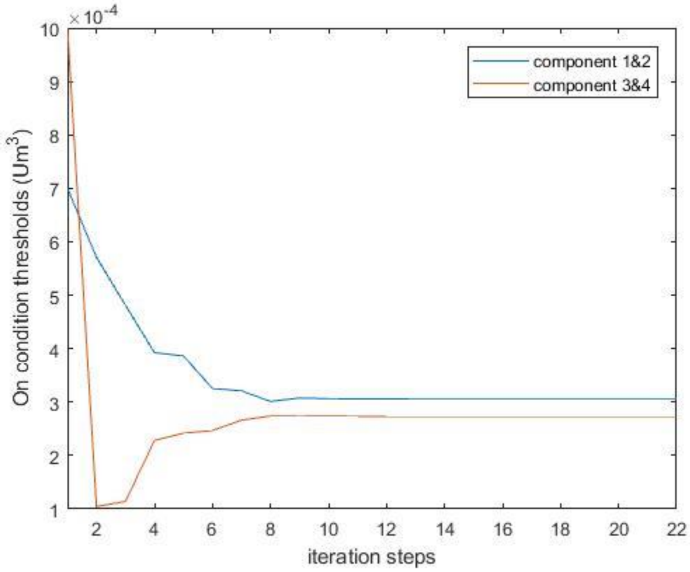


Figure 0.6 Iteration process for on-condition threshold for all components

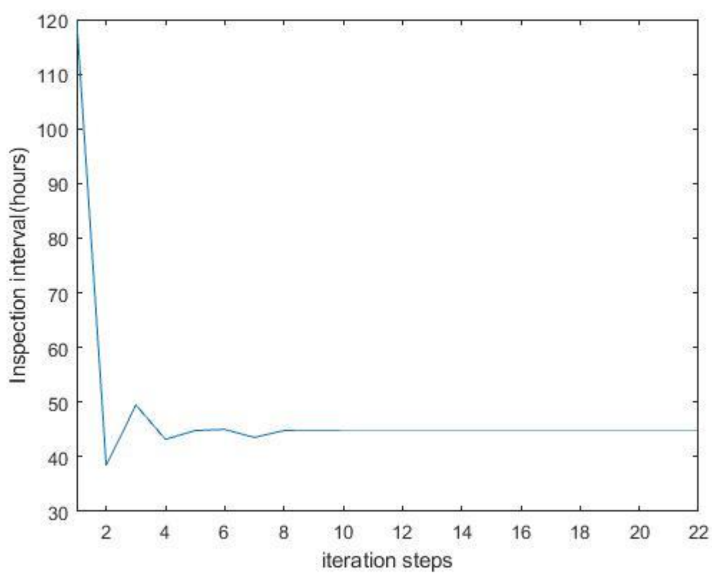


Figure 0.7 Iteration process for inspection interval τ*,

To show the preference of the proposed model, the optimal maintenance cost rate of this example is compared to optimal cost rate values for different maintenance policies such as time-based maintenance and replace-on-failure maintenance. In fact, both these policies are special cases of our proposed model. For replace-on-failure model, failure is detected by inspection, and if failures are not detected promptly, there is costly downtime.

Therefore, replace-on-failure still requires inspections, but by setting $H_i^2 = H_i^1$ for all i, the lowest cost replace-on-failure policy is found by solving an optimization problem where $CR(\tau)$ is the objective function with $H_i^2 = H_i^1$, and all the costs are the same. The optimal inspection interval is found as τ*=9.43 and minimum average long run maintenance cost rate is \$3.271×10^2. In the case, the inspection interval is small, because the only way to avoid costly downtime is to inspect frequently; while, when there are on-condition thresholds for each component to avoid failure and downtime, the minimum average long run maintenance cost rate for the system is \$1.902×10^2 which shows the proposed method can provide a beneficial maintenance policy for cases with high downtime costs by replacing the system before failure and avoiding system downtime.

Similarly, time-based preventive maintenance is investigated by setting $H_i^2 = 0$ for all i, so, the whole system will be replaced on the first inspection. The optimal inspection interval for this case is τ*=52.45 with the minimum average long run maintenance cost rate of \$2.427×10^2 which shows this policy is costly compared to our proposed model.

To further evaluate the results, an inspection and maintenance policy is also considered for the individual components. That is, four components are considered as individual systems, and inspect individual four components at their own inspection intervals. Since component 1 and 2 share the same parameters, the maintenance optimization for them are the same, and the minimum average long-run maintenance cost rate for component 1 and 2 as \$1.367×10^2 after 20 steps of iteration, with a solution of the periodic inspection interval $\tau_{1,2}$*=65.044 hours, and on-condition degradation threshold for components $H_1^{2*}= H_2^{2*}$=0.0002465. Figures 3.8 and 3.9 illustrate the iteration process of two decision variables, inspection interval and on-condition degradation threshold, for component 1 and 2.

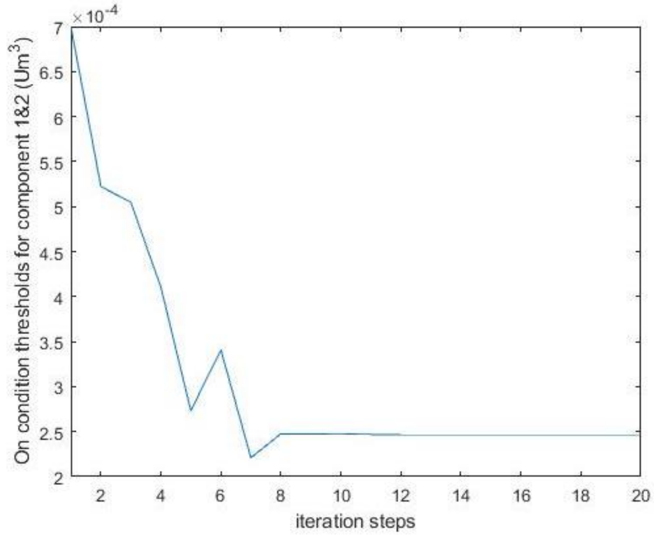


Figure 0.8 Iteration process for on-condition threshold for component 1 and 2

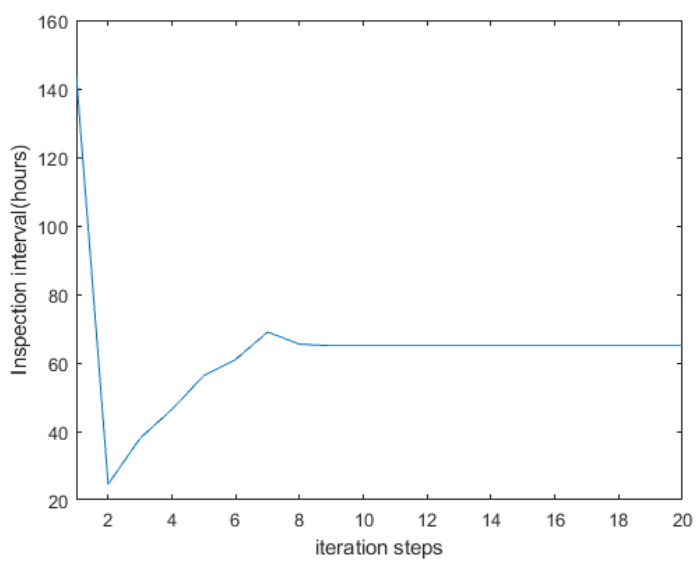


Figure 0.9 Iteration process for inspection interval τ^*

Similarly, individual component 3 or component 4 are inspected at their own inspection intervals. The minimum average long-run maintenance cost rate for component 3 and 4 is $1.762×10^2 after 13 steps of iteration, with the periodic inspection interval $\tau_{3,4}$*=71.55 hours, and on-condition degradation threshold for components

$H_3^{2*}= H_4^{2*}$=0.0002169. Figure 3.10 and 3.11 illustrate the iteration process of two decision variables: inspection interval and on-condition degradation threshold for component 3 and 4.

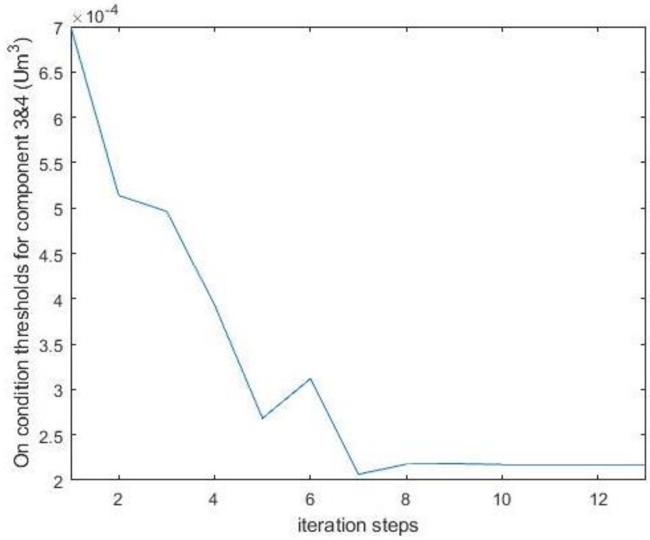


Figure 0.10 Iteration process for on-condition threshold for component 3 and 4

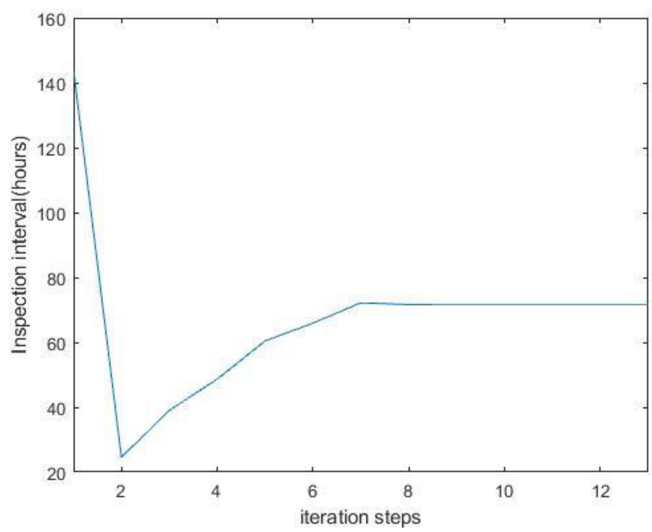


Figure 0.11 Iteration process for inspection interval τ^*

I can observe that inspection intervals for either component 1 and 2 or component 3 and 4 are greater than the inspection interval for the series system, which means there is a need to compromise to inspect the system more frequently if there are more components in the system. Since time to failure for all components are different, and series system reliability is less than individual component reliability given any fixed time, the system should be inspected more often to increase probability of avoiding failure and relative high downtime cost.

The second example is a series system with four different components. Table 3 presents the parameters of each component. Given the same cost C_I=\$1, C_ρ=\$20000 and C_R=\$100, the minimum average long-run maintenance cost rate is found for the system as \$1.8356×10^2, which is obtained at periodic inspection interval τ*=49.86 hours, and on-condition degradation threshold for components are H_1^{2*}=0.0002904, H_2^{2*}=0.0002656, H_3^{2*}=0.0007362, H_4^{2*}=0.0012359. As the results illustrate, component 4 has the highest optimal on-condition threshold that is very close to its failure threshold. This is mainly because the degradation rate and shock load damage for component 4 is lower than other components which means its reliability is higher compared to all other three components. Accordingly, its optimal on-condition threshold is higher.

1.9 Multi-component systems with dependent degrading components

Reliability describes the ability of a system to function for a specified period of time, and provides strategic information for determining effective maintenance activities, spares provisioning, warranties, etc. Reliability has a key role in achieving and maintaining the cost-effectiveness of systems. Significant prior research has been performed considering the reliability for systems with degradation, random shocks, independent or dependent failure processes, etc. The available models usually pertain to a single component or simple system, with either independent or dependent component failure times. The problem has also been studied with dependent shock damage for specific failure processes and other dependent patterns. Dependent degradation paths among multiple components in a system, especially models based on the stochastic degradation, have not been considered sufficiently in reliability modeling for systems subject to dependent competing failure processes. However, for certain classes of systems and applications, the component degradation paths can be considered as clusters with similar degradation patterns. Yousefi et al [142] studied the system reliability of multi-component systems, considering the degradation path of some components are dependent within the system.

The rapidly increasing application of many new technologies and systems demand high reliability because failures, leading to even small functional errors, can result in unaffordable consequences and effects. For example, approximately 20% of stents implanted in human bodies eventually deteriorate and fail [153]. Different stents within the same body share a similar working environment, i.e., same heart rate, blood flow velocity, rate of physical activities, etc., and the degradation status of stents in the same human body are likely to be probabilistically dependent and statistically correlated. Without considering dependent component deterioration, reliability models are not sufficient or adequate for some systems and engineering applications.

Dependent stochastic degradation paths among components represent a challenging issue because it increases the complexity of system reliability modeling and calculation. However, it is also realistic for some problems that have not been sufficiently addressed or which can be divided into two (or more) clusters, as in Figure 3.12. Mathematically, if two or more component degradation paths are dependent, the covariance of degradation is positive for some time *t*. Within a complex systems, there can be several groups or clusters of components with similar behavior.

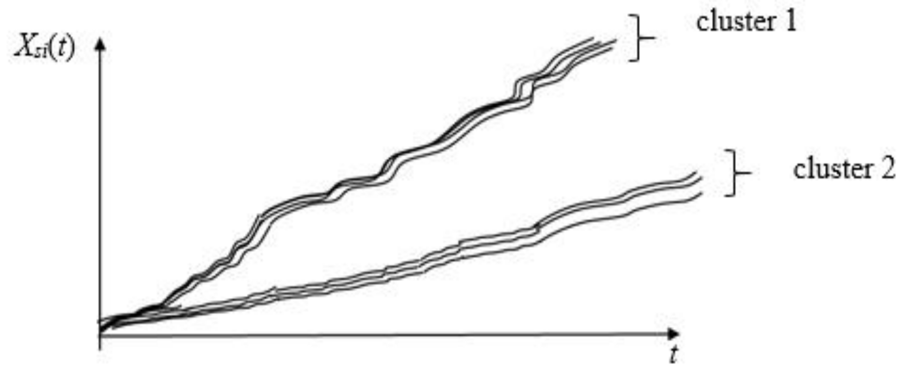


Figure 0.12 Two clusters of degradation paths for seven identical components in a system

It is assumed that each component can fail due to two competing dependent failure processes that share the same system shock process. There is a soft failure process and hard failure process. When a shock arrives, if the shock magnitude is greater than the hard failure threshold D_i, catastrophic failure occurs immediately. If the component survives the shock, and total degradation containing both pure degradation and cumulative shock damage is greater than a soft failure threshold level H_i, then soft failure occurs. The failure processes are competing and dependent, and all components in the system behave similarly. Random shocks arriving to the system occur as a Poisson process with rate λ. When the system is shocked at rate λ, all components experience a shock. The model can be readily generally to any other shock arrival model. The probability that the i^{th} component survives a shock can be obtained using Equation (3.1.4).

For each component, the gamma process is used to model stochastic degradation path.

$X_i(t_2) - X_i(t_1) \sim Ga\left(X_i; v_i(t_2) - v_i(t_1), \theta(i)\right)$ is the incremental degradation from t_1 to t_2 for component i. Yousefi et al [142] defined the scale parameter in the gamma process for component i as:

$$\theta(i) = \alpha_{0,i} + \alpha_{1,i}\theta_1 + \alpha_{2,i}\theta_2 + \ \ldots + \alpha_{k,i}\theta_k$$

θ_q is defined as a random clustering variable affecting all components in cluster q ($\alpha_{q,i} \neq 0$). The common item θ_q is not specific to any particular component, and it causes the scale parameter $\theta(i)$ for all components in cluster q to be probabilistically dependent, leading to the dependence among the degradation paths $X_i(t)$ of components in cluster q. If the scale parameter $\theta(i)$ for component i contains the item θ_q, i.e., $\alpha_{q,i} \neq 0$, then component i belongs in cluster q. The magnitude of $\alpha_{q,i}$ determines how much the component i degradation can be affected by the common factor in cluster q. For example, if θ_q is relatively high, then all $\theta(i)$ with $\alpha_{q,i} > 0$ are also likely to be high due to the dependence in degradation paths. Therefore, the probability density function of the degradation process for each component is given by:

$$g(x; v_i(t_2) - v_i(t_1), \theta(i)) = \frac{x^{v_i(t_2) - v_i(t_1) - 1} \exp(-x / \theta(i))}{\Gamma\left(v_i(t_2) - v_i(t_1)\right) \theta(i)^{v_i(t_2) - v_i(t_1)}} \tag{3.0.18}$$

$Ga\left(x_i; v_i(t), \theta(i)\right)$ is a gamma cdf for component i from time 0 to t with shape parameter $v_i(t)$, scale parameter $\theta(i)$, and $v_i(0) = 0$. Considering that $\theta(i)$ is random, $Ga\left(x_i; v_i(t), \theta(i)\right)$ is a gamma cdf, conditional on $\theta(i)$. Empirical studies [154] often show the deterioration at time t is proportional to a power law, or $v_i(t) = c_i t^{b_i}$. As examples, (1) degradation of concrete due to corrosion has been observed to be linear ($b_i = 1$); (2) sulphate attack parabolic ($b_i = 2$); (3) diffusion-controlled ageing ($b_i = 0.5$), and other cases [143].

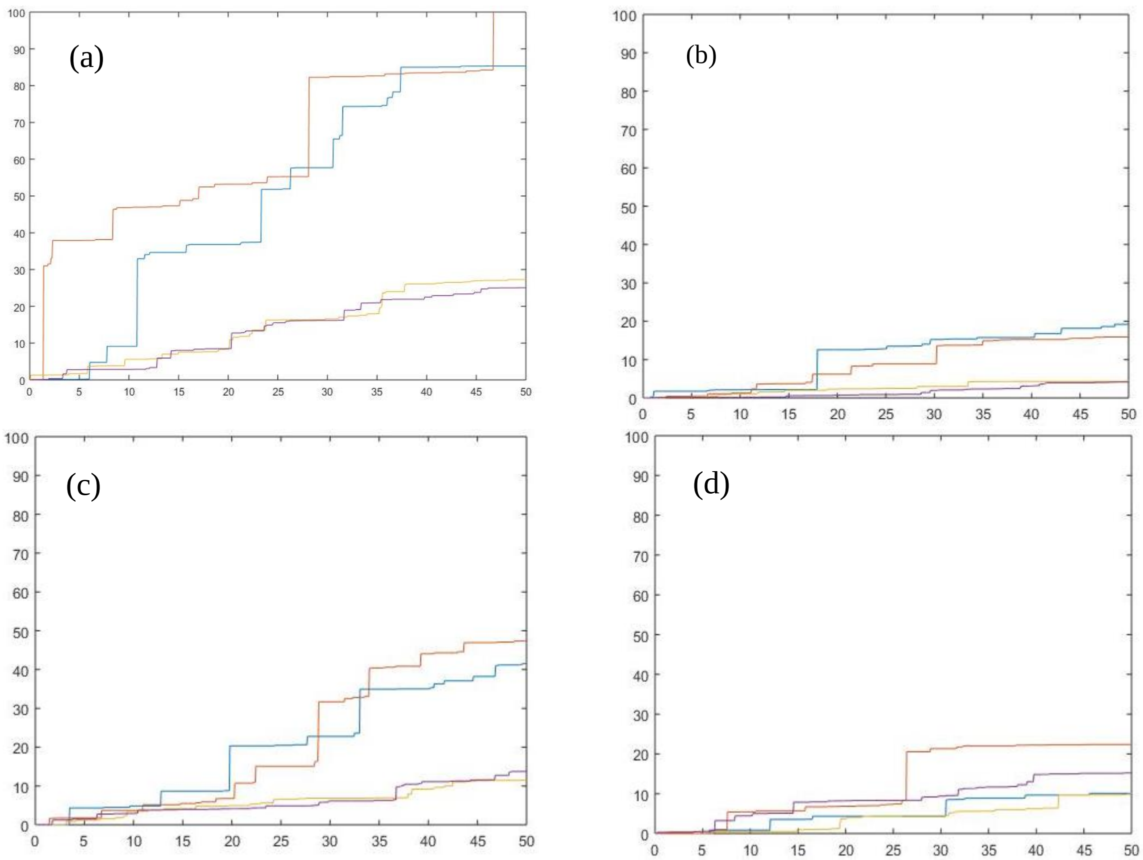


Figure 0.13 Degradation paths of four components

Figure 3.13 (a), (b), (c) and (d) show simulated degradation paths of four components in a system where degradation of each component is modeled as a gamma process. Each figure represents a different simulation of the four same components. Component 1 and 2 degrade similarly as a cluster and component 3 and 4 belong to another cluster. Notice in the figure that components 1 and 2 (and 3 and 4) tend to degrade together as partners. Occasionally the degradation of components 1 and 2 is similar to components 3 and 4 (Figure 3.13(b) and 313(d)), and in other instances, they are degrading at a higher rate (Figure 3.13(a)), but in all cases, 1 and 2 tend to be together, as well as 3 and 4. This is indicative of clusters of component degradation.

The total degradation of the i^{th} component can be accumulated as $X_{S_i}(t) = X_i(t) +$

$S_i(t)$, and when $X_{S_i}(t) > H_i$, a soft failure occurs. $X_i(t)$ is the pure degradation of component i and $S_i(t)$ is the summation of all the shock damages to the degradation path by time t. When there is no shock consideration in the system $S_i(t)$=0. Conditioning on the number of shocks and using a convolutional integral of $X_{S_i}(t)$, the probability that component i does not experience soft failure by time t , conditional on $\theta(i)$, is as follow:

$$P(X_i(t) < H_i) = \sum_{m=0}^{\infty} \left(X_i(t) + \sum_{j=i}^{m} Y_{ij} < H_i \mid N(t) = m \right) P(N(t) = m) \tag{3.0.19}$$

$$= \sum_{m=0}^{\infty} \int_0^{H_i} (Ga\left(H_i - u; v_i(t), \theta(i)\right) \mid \sum_{j=1}^{m} Y_{ij} = u)\, f_{Y_i}^{<m>}(u)du \, \frac{e^{(-\lambda t)}(\lambda t)^m}{m!}$$

A system reliability model can be developed for a system with clusters of dependent component degradation paths, with or without a system shock process. In this section, components experience no shock processes, i.e., only soft failure occurs to components, while in Section 5, components are subject to both hard and soft failure processes. That is, a component can fail either due to soft failure or due to hard failure, whichever happens first.

1.9.1 System reliability of dependent component degradation paths without shocks

The reliability for a series system at time t, without system shocks, is the probability that the total degradation of each component is less than the soft failure threshold ($X_{Si}(t) < H_i$),

$$R(t) = P\left\{ \left[X_1(t; v_1(t), \theta(1)) < H_1 \right] \cap \left[X_2(t; v_2(t), \theta(2)) < H_2 \right] \cap ... \cap \left[X_n(t; v_n(t), \theta(n)) < H_n \right] \right\} \tag{3.0.20}$$

In the new model, degradation paths among components can be dependent. By defining $\theta(i) = \alpha_{0,i} + \alpha_{1,i}\theta_1 + \alpha_{2,i}\theta_2 + \ \ldots + \alpha_{k,i}\theta_k$, with the common items θ_q as random variables, dependency among component degradation paths within a cluster can be achieved. For example, considering θ_q a as random variable for cluster q, if θ_q, for a particular system is relatively high, then all $\theta(i)$ with $\alpha_{q,i} > 0$ are also likely to be high due to the dependence in degradation paths. System reliability is then given by Equation (3.21)

$$R(t) = P\left\{\left[X_1(t;v_1(t),\alpha_{0,1}+\alpha_{1,1}\theta_1+...+\alpha_{k,1}\theta_k) < H_1\right]\cap\left[X_2(t;v_2(t),\alpha_{0,2}+\alpha_{1,2}\theta_1+...+\alpha_{k,2}\theta_k) < H_2\right] \cap...\cap\left[X_n(t;v_n(t),\alpha_{0,n}+\alpha_{1,n}\theta_1+...+\alpha_{k,n}\theta_k) < H_n\right]\right\} \tag{3.0.21}$$

As θ_q varies, the component degradation paths vary accordingly. $\alpha_{q,i}$ is a transmission parameter that determines whether component i is in cluster q, and the relative degree of dependency if $\alpha_{q,i} \neq 0$. As an example, with k = 2, and conditioning on θ_1 and θ_2:

$$R(t) = \int_{\theta_1}\int_{\theta_2} P\left\{\left[X_1(t;v_1(t),\alpha_{0,1}+\alpha_{1,1}\theta_1+\alpha_{2,1}\theta_2) < H_1\right]\cap\left[X_2(t;v_2(t),\alpha_{0,2}+\alpha_{1,2}\theta_1+\alpha_{2,2}\theta_2) < H_2\right] \cap...\cap\left[X_n(t;v_n(t),\alpha_{0,n}+\alpha_{1,n}\theta_1+\alpha_{2,n}\theta_2) < H_n\right] \mid \theta_1=\vartheta_1,\theta_2=\vartheta_2\right\} f_{\theta_1}(\vartheta_1)\, f_{\theta_2}(\vartheta_2)\, d\vartheta_1 d\vartheta_2. \tag{3.0.22}$$

Equation (3.22) can then be extended for any $k > 2$. However, most systems have two or less clusters in practice. With k = 2, the system reliability is given as:

$$R(t) = \int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n} P\left(X_i(t;v_1(t),\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2) < H_i\right) f_{\theta_1}(\vartheta_1)\, f_{\theta_2}(\vartheta_2)\, d\vartheta_1 d\vartheta_2. \tag{3.0.23}$$

The components are now conditionally independent, and the gamma distribution for each component degradation results in:

$$R(t) = \int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n} Ga\left(H_i;v_i(t),\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right) f_{\theta_1}(\vartheta_1)\, f_{\theta_2}(\vartheta_2)\, d\vartheta_1 d\vartheta_2. \tag{3.0.24}$$

$$R(t) = \int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\left(1-\frac{\Gamma\left(v_i(t),H_i/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)\right)}{\Gamma\left(v_i(t)\right)}\right) f_{\theta_1}(\vartheta_1)\, f_{\theta_2}(\vartheta_2)\, d\vartheta_1 d\vartheta_2. \tag{3.0.25}$$

where $\Gamma(a,x) = \int_x^\infty z^{a-1}e^{-z}dz$, $\Gamma(a) = F(a,0) = \int_0^\infty z^{a-1}e^{-z}dz$, and $v_i(t)$ is a nondecreasing, right-continuous function for $t > 0$. Therefore, Equation 3.3.26 can be derived.

$$R(t) = \int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\left(1 - \int_{H_i/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)}^{\infty} z^{v_i(t)-1}e^{-z}dz \Big/ \Gamma\left(v_i(t)\right)\right) f_{\theta_1}\left(\vartheta_1\right) f_{\theta_2}\left(\vartheta_2\right) d\vartheta_1 d\vartheta_2. \tag{3.0.26}$$

Equation (3.26) is the general reliability model for systems with dependent component stochastic degradation processes without a shock process and $k = 2$. As an example, replacing $v_i(t)$ with $c_i t^{b_i}$ and with $b_i = 0.5$.

$$R(t) = \int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\left(\int_{0}^{H_i/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)} z^{c_i\sqrt{t}-1}e^{-z}dz \Big/ \Gamma\left(c_i\sqrt{t}\right)\right) f_{\theta_1}\left(\vartheta_1\right) f_{\theta_2}\left(\vartheta_2\right) d\vartheta_1 d\vartheta_2. \tag{3.0.27}$$

Figure 3.14 shows a parallel system composed of n components. The reliability of the parallel system at time t is the probability that at least one component degradation is less than the threshold level ($X_{S_i}(t)<H_i$).

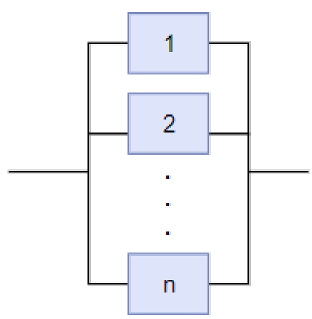


Figure 0.14 Parallel system example

Following a similar model development process, the system reliability for a parallel system is given by the following equations, still considering two clusters in the system, i.e., $k = 2$:

$$R(t) = 1 - \int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\left(1 - Ga\left(H_i; v_i(t), \alpha_{0,i} + \alpha_{1,i}\vartheta_1 + \alpha_{2,i}\vartheta_2\right)\right) f_{\theta_1}\left(\vartheta_1\right) f_{\theta_2}\left(\vartheta_2\right) d\vartheta_1 d\vartheta_2. \tag{3.0.28}$$

$$R(t)=1-\int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\left(\frac{\Gamma\left(v_i(t),\ H_i/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)\right)}{\Gamma\left(v_i(t)\right)}\right)f_{\theta_1}\left(\vartheta_1\right)f_{\theta_2}\left(\vartheta_2\right)d\vartheta_1 d\vartheta_2. \tag{3.0.29}$$

$$R(t)=1-\int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\left(\int_{H_i/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)}^{\infty} z^{c_i\sqrt{t}-1}e^{-z}dz\Big/\Gamma\left(c_i\sqrt{t}\right)\right)f_{\theta_1}\left(\vartheta_1\right)f_{\theta_2}\left(\vartheta_2\right)d\vartheta_1 d\vartheta_2. \tag{3.0.30}$$

Figure 3.15 depicts a series-parallel system with *s* subsystems arranged in parallel. Similar to the model in Song et al [155], S_l is defined the set of components in subsystem *l* with no component being used in more than one subsystem ($S_l \cap S_k = \phi$ for all *l*, *k*), and each subsystem has m_l components with $m_l = | S_l |$.

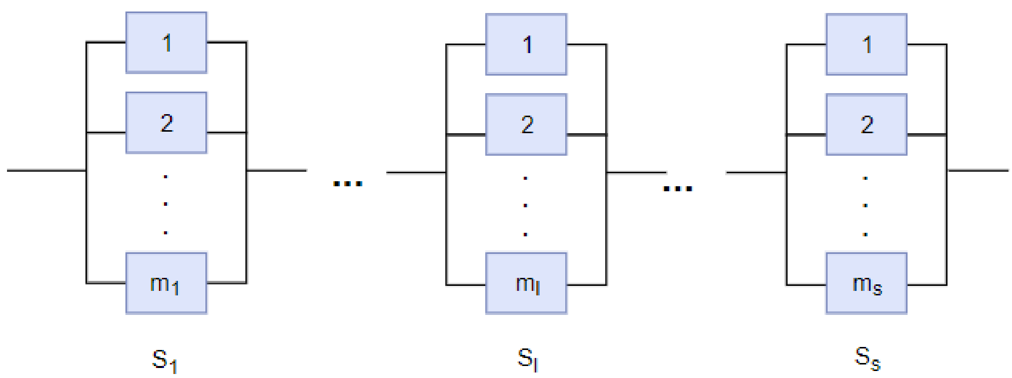


Figure 0.15 Series-parallel system example

The reliability of a series-parallel system at time *t* is the probability that at least one component within each subsystem has the total degradation less than the threshold level ($X_{S_i}(t) < H_i$). The system fails when all components for at least one parallel subsystem experience soft failure. System reliability for a series-parallel system is given by the following equation for the case of $k = 2$

$$R(t)=\int_{\theta_1}\int_{\theta_2}\prod_{l=1}^{s}\left(1-\prod_{i\in S_l}\left(1-Ga\left(H_i;v_i(t),\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)\right)\right)f_{\theta_1}\left(\vartheta_1\right)f_{\theta_2}\left(\vartheta_2\right)d\vartheta_1 d\vartheta_2. \tag{3.0.31}$$

$$R(t)=\int_{\theta_1}\int_{\theta_2}\prod_{l=1}^{s}\left(1-\prod_{i\in S_l}\left(\Gamma\left(v_i(t),H_i/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)\right)\Big/\Gamma\left(v_i(t)\right)\right)\right)f_{\theta_1}\left(\vartheta_1\right)f_{\theta_2}\left(\vartheta_2\right)d\vartheta_1 d\vartheta_2. \tag{3.0.32}$$

$$R(t)=\int_{\theta_1}\int_{\theta_2}\prod_{l=1}^{s}\left(1-\prod_{i\in S_l}\left(\int_{H_i/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)}^{\infty}z^{c_i\sqrt{t}-1}e^{-z}dz\Big/\Gamma\left(c_i\sqrt{t}\right)\right)\right)f_{\theta_1}\left(\vartheta_1\right)f_{\theta_2}\left(\vartheta_2\right)d\vartheta_1 d\vartheta_2. \tag{3.0.33}$$

1.9.2 System reliability of dependent component degradation paths with shock process

For a system with n components where each component is subjected to degradation process and random shocks, a component fails when either of the two dependent and competing failure modes occurs. The models in Section 3 pertain to systems with components experiencing pure degradation only. In this section, the system under study is exposed to shocks, and all components are impacted accordingly.

The reliability of a series system at time t is the probability that each component survives each of the $N(t)$ shock loads ($W_{ij} < D_i$ for j=1, 2, …) and the total degradation of each component is less than the soft failure threshold level ($X_{Si}(t) < H_i$) . System reliability can be expressed as:

$$\begin{aligned}R(t)=P\Big\{&\left[W_{11}<D_1,W_{12}<D_1,...,W_{1N(t)}<D_1,X_{s1}(t;v_1(t),\theta(1))<H_1\right]\cap\left[W_{21}<D_2,W_{22}<D_2,\right.\\&\left.....,W_{2N(t)}<D_2,X_{s2}(t;v_2(t),\theta(2))<H_2\right]\cap...\\&\cap\left[W_{n1}<D_n,W_{n2}<D_n,...,W_{nN(t)}<D_n,X_{sn}(t;v_n(t),\theta(n))<H_n\right]\Big\}.\end{aligned} \tag{3.0.34}$$

Conditioning on the number of shocks, the system reliability is as follow

$$\begin{aligned}R(t)=\sum_{m=0}^{\infty}P\Bigg\{&\left[W_{11}<D_1,W_{12}<D_1,...,W_{1N(t)}<D_1,X_1(t;v_1(t),\theta(1))+\sum_{j=1}^{N(t)}Y_{1j}<H_1\right]\cap\\&\left[W_{21}<D_2,W_{22}<D_2,...,W_{2N(t)}<D_2,X_2(t;v_2(t),\theta(2))+\sum_{j=1}^{N(t)}Y_{2j}<H_2\right]\cap...\\&\left[W_{n1}<D_n,W_{n2}<D_n,...,W_{nN(t)}<D_n,X_n(t;v_n(t),\theta(n))+\sum_{j=1}^{N(t)}Y_{nj}<H_n\right]\mid N(t)=m\Bigg\}P(N(t)=m).\end{aligned}$$

(3.0.35)

Separating the hard failure process and soft failure process, Equation (3.35) can be then re-written as:

$$R(t)=\sum_{m=0}^{\infty}P\left(W_{1m}<D_1,W_{2m}<D_2,...,W_{nm}<D_n\right)P\left(X_1(t;v_1(t),\theta(1))+\sum_{j=1}^{m}Y_{1j}<H_1,\right.$$
$$\left.X_2(t;v_2(t),\theta(2))+\sum_{j=1}^{m}Y_{2j}<H_2,...,X_n(t;v_n(t),\theta(n))+\sum_{j=1}^{m}Y_{nj}<H_n\mid N(t)=m\right)P(N(t)=m)$$

(3.0.36)

Considering $\theta(i)=\alpha_{0,i}+\alpha_{1,i}\theta_1+\alpha_{2,i}\theta_2+\ \ldots+\alpha_{k,i}\theta_k$, and setting $k = 2$ as an example, i.e., two clusters in the system, and conditioning on θ_1 and θ_2, results in:

$$R(t)=\sum_{m=0}^{\infty}\prod_{i=1}^{n}P(W_{im}<D_i)^m\int_{\theta_1}\int_{\theta_2}P\{\left[X_1(t;v_1(t),\alpha_{0,1}+\alpha_{1,1}\theta_1+\alpha_{2,1}\vartheta_2)+\sum_{j=1}^{m}Y_{1j}<H_1\right]\cap$$
$$\left[X_2(t;v_2(t),\alpha_{0,2}+\alpha_{1,2}\theta_1+\alpha_{2,2}\theta_2)+\sum_{j=1}^{m}Y_{2j}<H_2\right]\cap...$$
$$\left[X_n(t;v_n(t),\alpha_{0,n}+\alpha_{1,n}\theta_1+\alpha_{2,n}\theta_2)+\sum_{j=1}^{m}Y_{nj}<H_n\right]\mid\theta_1=\vartheta_1,\theta_2=\vartheta_2\}\times$$
$$f_{\theta_1}(\vartheta_1)\,f_{\theta_2}(\vartheta_2)\,d\vartheta_1 d\vartheta_2\times\frac{e^{(-\lambda t)}(\lambda t)^m}{m!}$$

(3.0.37)

$$R(t)=\sum_{m=0}^{\infty}\prod_{i=1}^{n}F_{W_i}(D_i)^m\int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}P\left(X_i(t;v_i(t),\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2)+\sum_{j=1}^{m}Y_{ij}<H_i\right)$$
$$\times f_{\theta_1}(\vartheta_1)\,f_{\theta_2}(\vartheta_2)\,d\vartheta_1 d\vartheta_2\,\frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

(3.0.38)

The shock damages affecting the soft failure process Y_{ij} are *i.i.d*. random variables. Conditioning on the sum of Y_{ij} and using the convolutional integral for the sum of Y_{ij}, produces Equation 3.3.39:

$$R(t)=\sum_{m=0}^{\infty}\prod_{i=1}^{n}F_{W_i}(D_i)^m\int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\int_{0}^{H_i}P\left(X_i(t;v_i(t),\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2)+\sum_{j=1}^{m}Y_{ij}<H_i\mid\sum_{j=1}^{m}Y_{ij}=u\right)$$
$$\times f_{Y_i}^{<m>}(u)du\, f_{\theta_1}(\vartheta_1)\, f_{\theta_2}(\vartheta_2)\,d\vartheta_1 d\vartheta_2\,\frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

(3.0.39)

$$R(t)=\sum_{m=0}^{\infty}\prod_{i=1}^{n}F_{W_i}(D_i)^m\int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\int_{0}^{H_i}P\left(X_i(t;v_i(t),\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2)<H_i-u\right)$$
$$\times f_{Y_i}^{<m>}(u)du\, f_{\theta_1}(\vartheta_1)\, f_{\theta_2}(\vartheta_2)\,d\vartheta_1 d\vartheta_2\,\frac{e^{-\lambda t}(\lambda t)^m}{m!} \qquad (3.0.40)$$

Considering the gamma distribution for each component degradation from 0 to t results in:

$$R(t)=\sum_{m=0}^{\infty}\prod_{i=1}^{n}F_{W_i}(D_i)^m\int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\int_{0}^{H_i}Ga\left(H_i-u;v_i(t),\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)$$
$$\times f_{Y_i}^{<m>}(u)du\, f_{\theta_1}(\vartheta_1)\, f_{\theta_2}(\vartheta_2)\,d\vartheta_1 d\vartheta_2\,\frac{e^{-\lambda t}(\lambda t)^m}{m!} \qquad (3.0.41)$$

$$R(t)=\sum_{m=0}^{\infty}\prod_{i=1}^{n}F_{W_i}(D_i)^m\int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\int_{0}^{H_i}\left(1-\frac{\Gamma\left(v_i(t),(H_i-u)/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)\right)}{\Gamma\left(v_i(t)\right)}\right)$$
$$\times f_{Y_i}^{<m>}(u)du\, f_{\theta_1}(\vartheta_1)\, f_{\theta_2}(\vartheta_2)\,d\vartheta_1 d\vartheta_2\,\frac{e^{-\lambda t}(\lambda t)^m}{m!} \qquad (3.0.42)$$

As an example, replace $v_i(t)$ with $c_i t^{b_i}$ where b_i =0.5:

$$R(t)=\sum_{m=0}^{\infty}\prod_{i=1}^{n}F_{W_i}(D_i)^m\int_{\theta_1}\int_{\theta_2}\prod_{i=1}^{n}\int_{0}^{H_i}\left(\int_{0}^{(H_i-u)/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)}z^{c_i\sqrt{t}-1}e^{-z}dz\Big/\Gamma\left(c_i\sqrt{t}\right)\right) \qquad (3.0.43)$$
$$\times f_{Y_i}^{<m>}(u)du\, f_{\theta_1}(\vartheta_1)\, f_{\theta_2}(\vartheta_2)\,d\vartheta_1 d\vartheta_2\,\frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

The reliability of the parallel system at time t is the probability that at least one component of this system survives each of the $N(t)$ shock loads ($W_{ij} < D_i$ for j=1, 2, …), and the total degradation of that same component is less than the threshold level ($X_{S_i}(t) < H_i$). The system fails when all components experience either soft failure or catastrophic failure. Consider $k = 2$, i.e., two clusters in the system,

$$R(t) = 1 - \sum_{m=0}^{\infty} \int_{\theta_1} \int_{\theta_2} \prod_{i=1}^{n} \left(1 - \int_0^{H_i} F_{W_i}(D_i)^m Ga\left(H_i - u; v_i(t), \alpha_{0,i} + \alpha_{1,i}\vartheta_1 + \alpha_{2,i}\vartheta_2 \right) \right) \times f_{Y_i}^{<m>}(u) du \, f_{\theta_1}(\vartheta_1) \, f_{\theta_2}(\vartheta_2) \, d\vartheta_1 d\vartheta_2 \frac{e^{-\lambda t} (\lambda t)^m}{m!} \tag{3.0.44}$$

$$R(t) = 1 - \sum_{m=0}^{\infty} \int_{\theta_1} \int_{\theta_2} \prod_{i=1}^{n} \left(1 - \int_0^{H_i} F_{W_i}(D_i)^m \left(1 - \Gamma\left(v_i(t), (H_i - u) / \left(\alpha_{0,i} + \alpha_{1,i}\vartheta_1 + \alpha_{2,i}\vartheta_2 \right) \right) \Big/ \Gamma\left(v_i(t) \right) \right) \right) \times f_{Y_i}^{<m>}(u) du \, f_{\theta_1}(\vartheta_1) \, f_{\theta_2}(\vartheta_2) \, d\vartheta_1 d\vartheta_2 \frac{e^{-\lambda t} (\lambda t)^m}{m!} \tag{3.0.45}$$

As an example, consider $v_i(t) = c_i t^{b_i}$, and replace $v_i(t)$ with $c_i t^{b_i}$, and $b_i = 0.5$:

$$R(t) = 1 - \sum_{m=0}^{\infty} \int_{\theta_1} \int_{\theta_2} \prod_{i=1}^{n} \left[1 - \int_0^{H_i} F_{W_i}(D_i)^m \left(\int_0^{(H_i - u)/\left(\alpha_{0,i} + \alpha_{1,i}\vartheta_1 + \alpha_{2,i}\vartheta_2\right)} z^{c_i \sqrt{t} - 1} e^{-z} dz \Big/ \Gamma\left(c_i \sqrt{t} \right) \right) \right] \times f_{Y_i}^{<m>}(u) du \, f_{\theta_1}(\vartheta_1) \, f_{\theta_2}(\vartheta_2) \, d\vartheta_1 d\vartheta_2 \frac{e^{-\lambda t} (\lambda t)^m}{m!} \tag{3.0.46}$$

The reliability of a series-parallel system at time t is the probability that at least one component within each subsystem survives each of the $N(t)$ shock loads ($W_{ij} < D_i$ for j= 1, 2, …), and the total degradation is less than the threshold level ($X_{S_i}(t) < H_i$) for that same component. The system fails when all components for at least one parallel subsystem experience either hard or soft failure. Consider $k = 2$:

$$R(t)=\sum_{m=0}^{\infty}\int_{\theta_1}\int_{\theta_2}\prod_{l=1}^{s}\left(1-\prod_{i\in S_l}\left[1-\int_0^{H_i}F_{W_i}(D_i)^m Ga\left(H_i-u;v_i(t),\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)\right)\right]f_{Y_i}^{\langle m\rangle}(u)du\right)$$
$$\times f_{\theta_1}(\vartheta_1)\,f_{\theta_2}(\vartheta_2)\,d\vartheta_1 d\vartheta_2\,\frac{e^{-\lambda t}(\lambda t)^m}{m!} \tag{3.0.47}$$

$$R(t)=\sum_{m=0}^{\infty}\int_{\theta_1}\int_{\theta_2}\prod_{l=1}^{s}\left(1-\prod_{i\in S_l}\left[1-\int_0^{H_i}F_{W_i}(D_i)^m\left(1-\frac{\Gamma\left(v_i(t),(H_i-u)/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)\right)}{\Gamma\left(v_i(t)\right)}\right)\right]f_{Y_i}^{\langle m\rangle}(u)du\right)$$
$$\times f_{\theta_1}(\vartheta_1)\,f_{\theta_2}(\vartheta_2)\,d\vartheta_1 d\vartheta_2\,\frac{e^{-\lambda t}(\lambda t)^m}{m!} \tag{3.0.48}$$

As an example for a series-parallel system, consider $v_i(t) = c_i t^{b_i}$, and replace $v_i(t)$ with $c_i t^{b_i}$, and $b_i = 0.5$.

$$R(t)=\sum_{m=0}^{\infty}\int_{\theta_1}\int_{\theta_2}\prod_{l=1}^{s}\left(1-\prod_{i\in S_l}\left[1-\int_0^{H_i}F_{W_i}(D_i)^m\left(\int_0^{(H_i-u)/\left(\alpha_{0,i}+\alpha_{1,i}\vartheta_1+\alpha_{2,i}\vartheta_2\right)}z^{c_i\sqrt{t}-1}e^{-z}dz\Big/\Gamma\left(c_i\sqrt{t}\right)\right)\right]f_{Y_i}^{\langle m\rangle}(u)du\right)$$
$$\times f_{\theta_1}(\vartheta_1)\,f_{\theta_2}(\vartheta_2)\,d\vartheta_1 d\vartheta_2\,\frac{e^{-\lambda t}(\lambda t)^m}{m!} \tag{3.0.49}$$

Yousefi et al [142] developed a new reliability model for different system configurations, considering the dependency of components' degradation paths. To obtain explicit reliability predictions, it is necessary to know or estimate the parameters in the reliability equations. Appendix A shows the proposed method to estimate parameters for the reliability models.

1.9.3 Numerical example for systems with clusters of dependent degrading components

Some different examples are considered in this section to demonstrate the reliability of the proposed model. For the examples, it is assumed that W_{ij} and Y_{ij} follow normal distributions, and $X_i(t)$ follows gamma processes with parameters $v_i(t) = c_i t^{b_i}$. Firstly, it is assumed that there is just one cluster and three different components which are configurated as series and parallel. The scale parameter for component i is $\theta(i)=\alpha_{0,i}+\alpha_{1,i}\theta_1$. Then, a second realistic example of a spool system is solved to demonstrate the new model.

Example 1: The first example is a conceptual system with three components and one cluster. Table 3.3 shows the parameters for reliability analysis. System reliability is complex and difficult to solve analytically. Integration with Monte Carlo simulation with 10^5 replications is used to calculate the system reliability for different scenarios.

Table 0.3 Parameters value for system reliability

Parameters	values		
	Component 1	Component 2	Component 3
H_i	10 mm	12 mm	15 mm
D_i	5 mm	10 mm	8 mm
λ	3×10^{-4}		
W_{ij}	$W_{ij} \sim Normal(1.2,0.2^2)$	$W_{ij} \sim Normal(1,0.3^2)$	$W_{ij} \sim Normal(1.5,0.2^2)$
Y_{ij}	$Y_{ij} \sim Normal(0.5,0.1^2)$	$Y_{ij} \sim Normal(0.4,0.15^2)$	$Y_{ij} \sim Normal(0.3,0.1^2)$
$\alpha_{0,i}$	0.5	0.2	0.1
$\alpha_{1,i}$	2	0.5	1
c_i	0.2	0.3	0.5
b_i	0.5	0.5	0.5

Figure 3.16 shows the reliability of series and parallel configuration for three component considering different distribution for θ_1. Three different distribution of *Uniform*(0,2), *Normal*(2,0.1) and *Weibull*(1, 1.5) are consider for θ_1 which demonstrate that the proposed method is appropriate for any distribution for θ_1. As shown in Figure 3.16, the system reliability for series configuration is lower than parallel one.

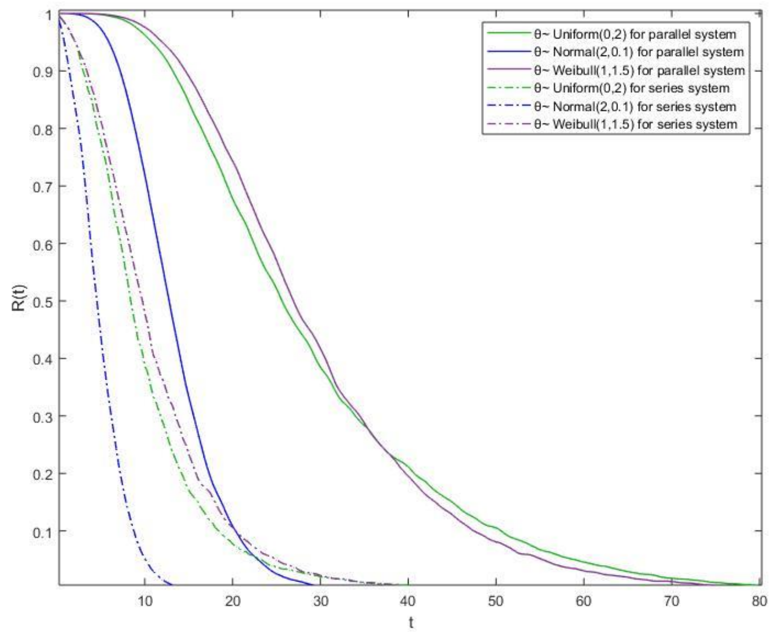


Figure 0.16 System reliability for series and parallel configuration considering different distribution for θ_1

To better understand the effect of the parameters on system reliability, sensitivity analyses are conducted on $\alpha_{1,i}$, and two parameters of gamma shape parameter $v_i(t) = c_i t^{b_i}$, i.e, c_i, b_i. For this example, it is assumed that $\theta_1 \sim$ *Uniform*(0,2), and system configuration is series. Figure 3.17 shows the effect of $\alpha_{1,i}$ on system reliability while the other parameters are fixed. The expected degradation for time t for component i is $E[X_i(t)] = c_i t^{b_i} \times \theta(i)$, so by increasing $\alpha_{1,i}$ the scale parameters increases which makes component i in the system degrade faster. Therefore, the system has a higher probability of failure for the same time duration; so, by increasing $\alpha_{1,i}$, the system reliability decreases. Figure 3.18 shows how system reliability changes according to different b_i values. By increasing b_i the degradation rate increases, and each component degrades faster, so the system reliability decreases.

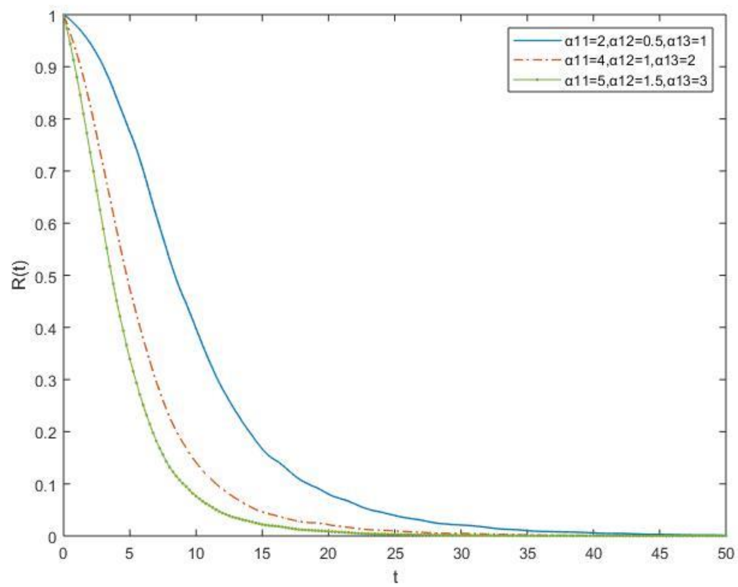


Figure 0.17. System reliability sensitivity analysis on parameter $\alpha_{1,i}$

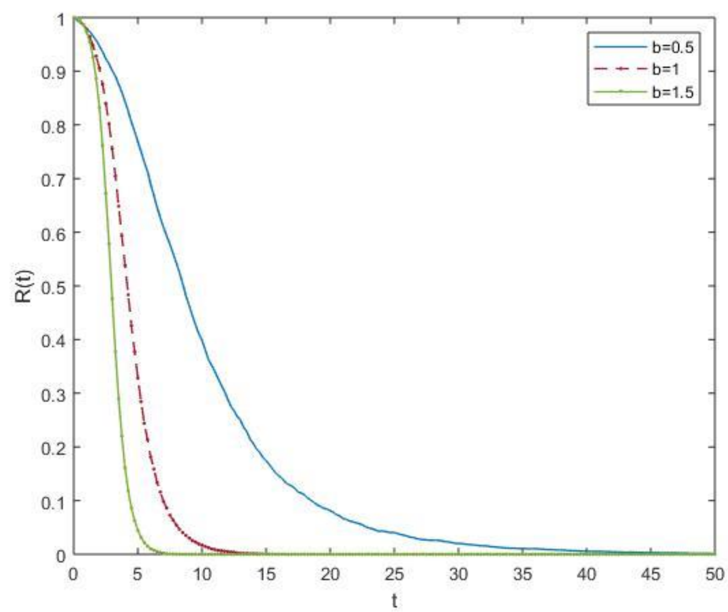


Figure 0.18 System reliability sensitivity analysis on parameter b_i

When c_i increases, the expected degradation for time *t* increases and the system has higher probability to fail for the same time; consequently, the system reliability decreases, as shown in Figure 3.19.

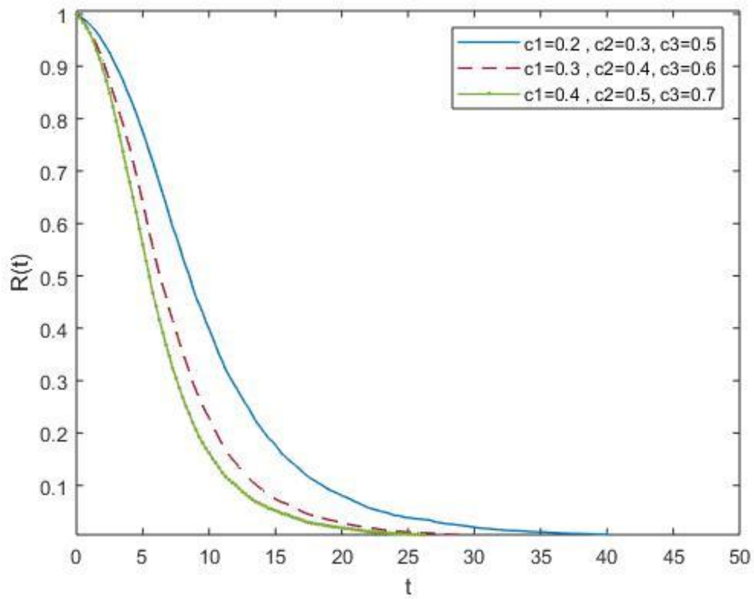


Figure 0.19 System reliability sensitivity analysis on parameter c_i

Example 2: the second example is a real case study of a sliding spool in electrohydraulic servo-valve which is used in a wide range of applications from metal forming and wood processing to aircraft applications. The valve control hydraulic oil flow and pressure by spool sliding in the sleeve. The two competing failure processes for this example are wear and contamination lock caused by oil pollution. The sudden appearance of pollutants in the hydraulic oil is considered as random shock, and deterioration of components within the servo-valve are considered as the degradation process.

The spool, sleeve, spring and spool rod as the main four components of the system which are configurated as series in the system. It is assumed that there are two different clusters (k=2) for these four components. For components who belongs to the same cluster, the degradation paths are dependent, and they degrade similarly. The parameters for this system are estimated by domain experts [156, 157] and some assumptions are made in this research study. Table 3.4 show the parameter values for reliability analysis. It is considered that spool and sleeve belong to cluster 2 and spool rod and spring belong to cluster 1. So, the scale parameter for each component is $\theta(i) = \alpha_{0,i} + \alpha_{1,i}\theta_1 + \alpha_{2,i}\theta_2$ where θ_1 and θ_2 have different uniform distribution.

Table 0.4 Parameters value for system with four components and two clusters

Parameters	Spool	Sleeve	Spring	Spool Rod
H_i	5 mm	6 mm	4 mm	4.5 mm
D_i	7.5 mm	7 mm	5 mm	5.5 mm
λ	2.5×10^{-5}			
W_{ij}	$W_{ij} \sim Normal(1.2, 0.2^2)$	$W_{ij} \sim Normal(1, 0.3^2)$	$W_{ij} \sim Normal(0.7, 0.2^2)$	$W_{ij} \sim Normal(0.8, 0.1^2)$
Y_{ij}	$Y_{ij} \sim Normal(0.5, 0.1^2)$	$Y_{ij} \sim Normal(0.4, 0.15^2)$	$Y_{ij} \sim Normal(0.25, 0.1^2)$	$Y_{ij} \sim Normal(0.2, 0.1^2)$
$\alpha_{0,i}$	0	0	0	0
$\alpha_{1,i}$	0	0	2	2.5
$\alpha_{2,i}$	1.8	1.6	0	0
c_i	0.2	0.3	0.3	0.2
b_i	0.5	0.5	0.5	0.5
θ_1	$\theta_1 \sim Uniform(0,2)$			
θ_2	$\theta_1 \sim Uniform(0,8)$			

The spool and sleeve are physically touching the hydraulic oil and share the same operational conditions, so that they can belong to the same cluster. Based on the assumptions in Table 3.4, the spool and sleeve belong to cluster 2 and consequently their scale parameters are changing based on cluster 2 (i.e., θ_2) and degrading as a group. The spring and spool rod are also degrading in a similar way. In each trial the degradation of each component is different, but the main similarity is spool and sleeve are degrading similarly and are considered as a cluster.

1.9.4 Model validation for reliability analysis of system with degradation dependency

Figures 3.20 and 3.21 show the difference of the proposed method for calculating the system reliability considering dependency of components and the system reliability using previous methods without clustering. In Figure 3.20, for calculating the system reliability considering clustering, it is assumed that θ_1~ *Weibull*(1,1.5) and θ_2~ *Weibull*(1,2). Alternatively to calculate the system reliability without clustering, the expected value of the Weibull distribution are considered for θ_1, θ_2, i.e., $\theta_1 = 1\times\Gamma(1+1/1.5) = 0.9027$ and $\theta_1 = 1\times\Gamma(1+1/2) = 0.8862$. For Figure 3.21, it is assumed that θ_1~ *Normal*(2,0.5) and θ_2~ *Normal*(4,2), and for system reliability without clustering, $\theta_1 = 2$ and $\theta_2 = 4$.

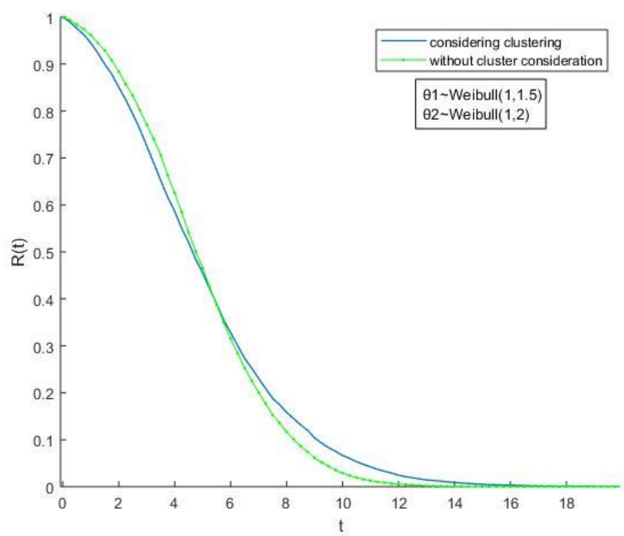


Figure 0.20 Comparison of system reliability when θ_1 and θ_2 follow Weibull distributions – Example 2

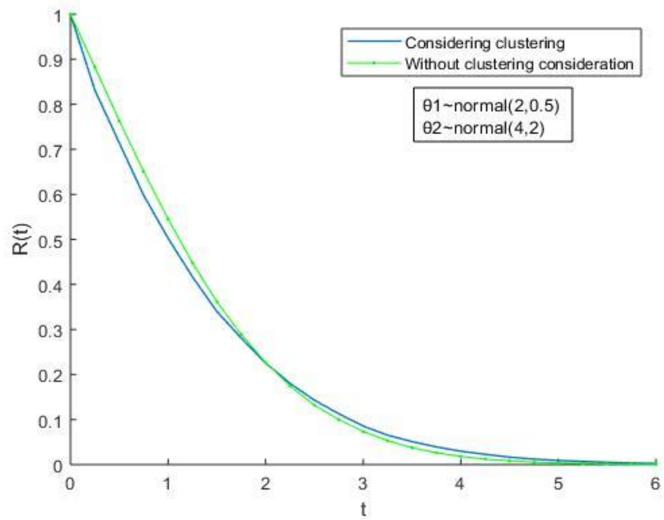


Figure 0.21 Comparison of system reliability when θ_1 and θ_2 follow normal distributions – Example 2

It can be concluded from Figures 3.20 and 3.21 that the system reliability for systems without clustering considerations have a higher decreasing rate (smaller variance of failure times), i.e., the probability density function of failure time has less variability. When components are degrading in clusters, the probability density function of system failure has more variability which cause the system reliability to decrease more gradually in time. Moreover, the system reliability considering clustering is lower than system reliability of previous systems for earlier time which is a very important information for system maintenance decisions. Therefore, for a system which has some components degrading in the same cluster due to the share environment or in close physical proximity, system reliability behaves differently, and it should be considered for any maintenance decision.

Static maintenance planning models

Many systems fail due to degradation and exposure to random shocks simultaneously. Moreover, each component is subject to two dependent competing failure processes of soft failure and hard failure. When each shock arrives, it can damage all the components in the system. As a result, all component degradation paths are dependent due to the shared shock exposure. Soft failure occurs when the cumulative degradation of each component reaches a predefined failure threshold, and hard failure occurs when the magnitude of each shock exceeds a predefined threshold. Yousefi et al [141, 142, 158, 159] developed reliability models and proposed some maintenance policies for multi-component systems where components are packaged and sealed together for maintenance perspective. However, there are some multi-component systems where each component can be maintained individually within the system.

For multi-component systems with individually repairable components, it is not economical to replace the whole system if it fails. For a multi-component system, where each component degrades by time, the failure time of components are different, and they may fail in different times. So, it is more beneficial to replace/repair each failed component individually within the system. Finding the optimal maintenance policy for systems with individually repairable components, subject to degradation and random shock arrival, is a unique challenge which is studied in this research work. Yousefi et al. [160] proposed a condition-based maintenance model for a multi-component system with individually repairable components by formulating a new reliability model and optimizing the maintenance cost rate.

A replaceable component can be restored to its initial satisfactory performance by replacing the component upon failure with a new one (or a replacement or restoration that is good-as-new). For such a component operating for a very long time (much greater than the expected failure time), the component will have been replaced several or many times, and the expected frequency of failure or replacements can reach a steady state behavior after some time. In steady-state, the expected number of failures within some time interval becomes stationary (independent of the underlying failure time distribution or degradation process). This is known as Drenick's theorem [161].

Shu and Flower [113] investigated the stochastic behavior of the reliability of repairable systems, for a perfectly maintained system when the failed system or component is replaced by a new one. It is therefore applicable to use the Drenick's theorem [161], which states that if a population of parts is put into service at time $t = 0$ and the population is perfectly maintained, as time goes on, at each socket (i.e., replaceable unit), there develops an unending sequence of failures which constitutes a random process or a renewal process[162] . A renewal process is a counting process for which the inter-arrival times are independent and identically distributed with an arbitrary distribution [163]. In a renewal process, the components or systems are renewed in a sequence, and it is assumed in the process that each renewal restarts the counting process as new [164]. Thus, it can be stated that after a very long time ($t \to \infty$) the rate of failures and replacement rate reach the steady state behavior. Jiang et al. [162] studied the steady state reliability of repairable one-unit system subjected to system modifications, while Utkin and Gurv [165] proved the limit theorems related to stationary possibility distribution functions for a repairable one-unit system.

It is assumed that all the components are new at the beginning of system life and all of them are inspected at the end of each time interval, and the failed components are only detected by inspection. For series systems when a component fails within the system, the whole system fails but the failed one can only be detected at the inspection interval.

For systems with continuous monitoring the failed components are detected immediately while in periodic inspection, the failed components can be detected just at the inspection times. The component which has already failed in an interval is replaced by a new one at the beginning of the next time interval, while the remaining operating components continue to work properly in the next time interval, and continue to degrade. In fact, the replacement of the failed component is only done after each inspection.

Drenick's theorem can be extended to consider degradation processes. By simulating a multi-repairable component (in MATLAB), recording the initial degradation of component at the beginning of previous inspection interval and calculated the average of them, it can be observed, that after a very long time, the average of initial degradation no longer changes, and it is stationary. Figure 30 and 31 show the result of a 10^5 simulation runs which indicates that after the 200th inspection interval, the average of initial degradation reaches the steady state. Indeed, this average does not change in the next inspection intervals. Figure 30 shows simulated results for a multi-repairable component system subject to only internal degradation, and Figure 4.1 is for a system when the components fail due to internal degradation and external random shocks. It can be observed that the initial degradation level of any component at the beginning of the inspection intervals oscillates in response to replacements, until steady state in reached.

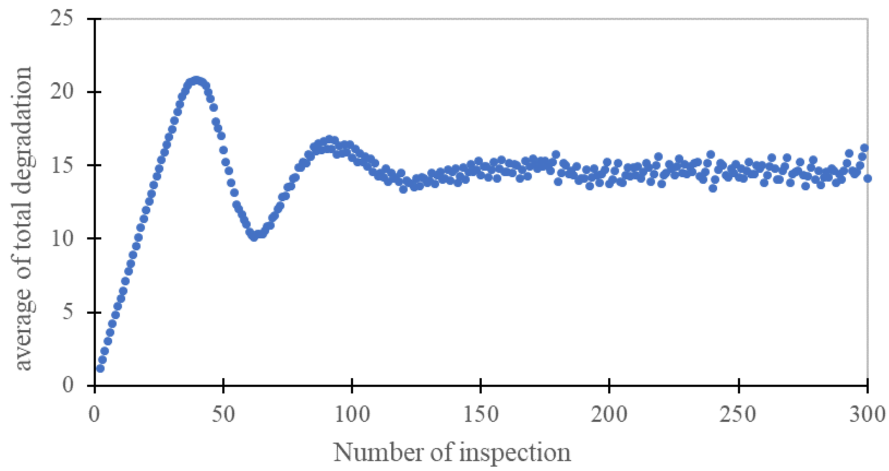


Figure 0.1 Average of initial degradation for all the previous inspections for different number of inspections for a system fails due to internal degradation

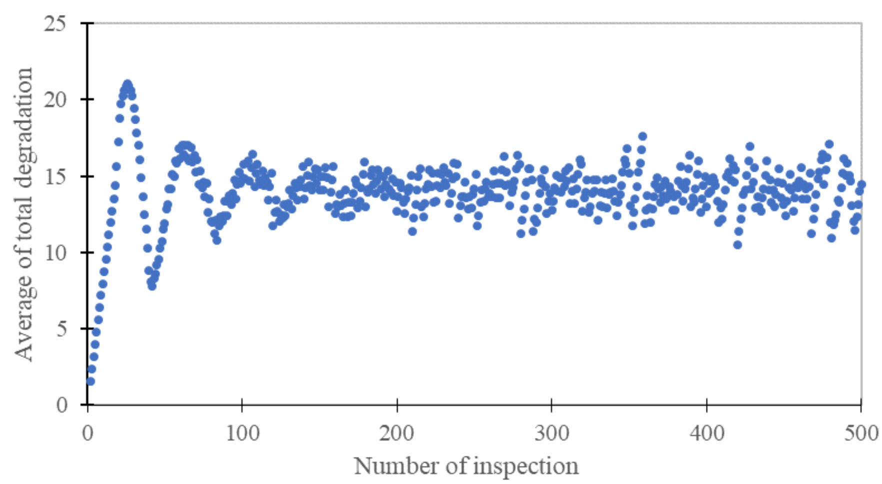


Figure 0.2 Average of initial degradation for all the previous inspections for different number of inspections for a system fails due to internal degradation and external shock arrivals

Figures 4.1 and 4.2 show how each component reaches a steady state behavior by replacing them in in the interval where it failed. In Figure 4.3, at any inspection, if a component is detected as aged or failed, it is replaced by a new one at the beginning of the next inspection interval. Therefore, after a very long time each component reaches a steady state behavior and the ages of the components is random and independent from each other. In the far right in Figure 4.3, there have now been many intervals and many component replacements. U_i is defined as a random variable representing the initial age of component *i* at the beginning of an inspection interval once the system enters into the steady state region.

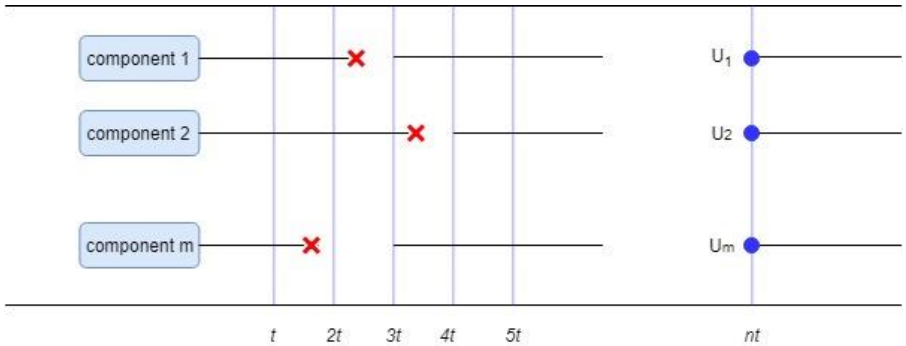


Figure 0.3 Component replacement and reaching steady state

Figure 4.4 shows an example of three components in a system subject to degradation. Each component is replaced when its degradation reaches its own failure threshold. It shows that the initial degradation is U_i for each component *i* at the beginning of inspection interval at steady state. In the first interval U_i is 0 for all components, but when the system reaches steady-state, U_i are random and independent. Yousefi and Coit [160, 166, 167] developed a reliability model for systems with individually repairable components considering the initial degradation level of each component at steady state.

If initial degradation U_i is simulated, the observed form of its distribution can be obtained as shown in Figure 4.4. In the first interval, all components are new, so U_i is 0, but in subsequent intervals, U_i is larger and has more variability as the components age, until replacements begin. At that point, the average degradation will decrease and then fluctuate as the second and third replacements take place (as in Figure 4.1), until eventually, it enters steady-state. As previously observed in Figure 4.1, the average degradation becomes constant in steady-state, as predicted by Drenick's Theorem.

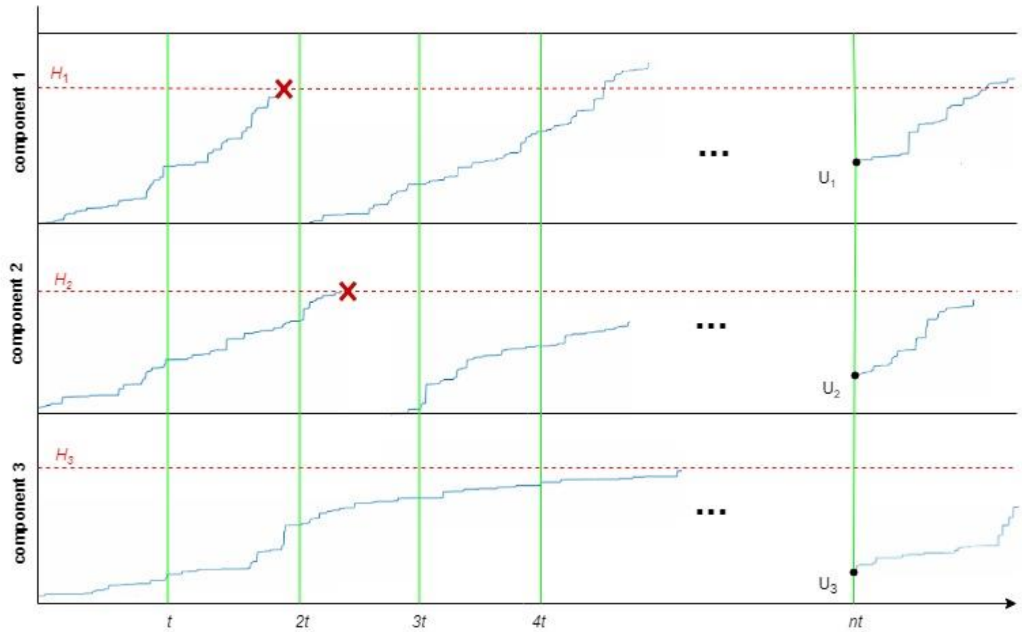


Figure 0.4 Degradation of three components in a system over time

Figure 4.5 shows a simulation of distribution of initial degradation at the beginning of inspection interval for a component which is degrading as a gamma process with shape parameter linear with time and constant scale parameter, and experiencing no shock arrival; therefore, the final distribution may change for a component experiences external shock arrival.

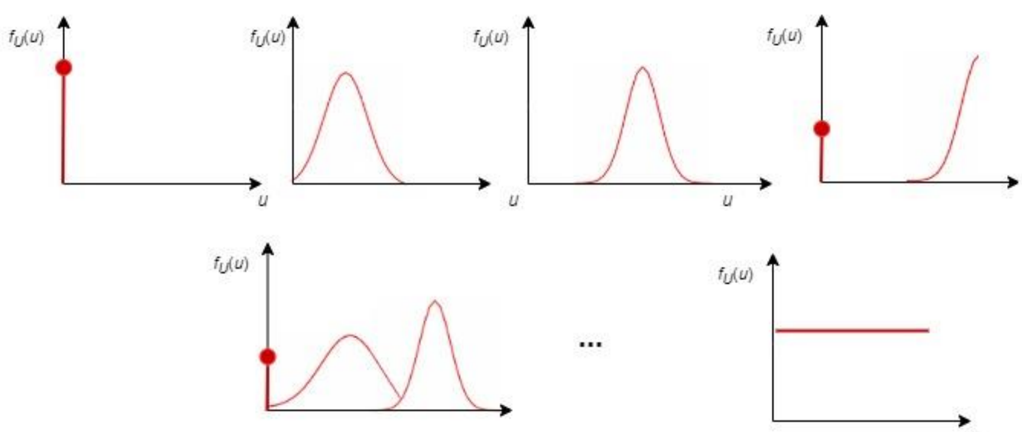


Figure 0.5 Simulated distribution of initial degradation

Until steady state behavior is reached, the initial degradation is a function of time, so it can be concluded that, as $t \to \infty$, it can be considered as a random variable of U, $\lim_{t\to\infty} U(t) = U$.

Finding the initial degradation distribution at steady state is very crucial issue for calculating the conditional reliability function of the system. In this research work, the probability density function of initial degradation at steady state is found empirically based on numerous simulations of a multi-component system with individually repairable components (in MATLAB). For each simulation replication, the initial degradation of each component is found once the it is observed in steady state. After many successive replacements, and using the result of all replications, a supposition can be made for distribution of the initial degradation based on the observed trends in the data and associated histogram plots.

To determine the intital degradation probability density funcion, extensive simulation runs and testing was performed with and without a shock process on individual component to observe these distributions empirically. Figure 4.6(a) to (d) shows some sample histogram plots of initial degradation for different systems without shock arrivals. Based on the simulation results, an accurate approximate distribution for initial degradation can be found. It is proposed that when there are no shocks arrivals to the system, the initial degradation at steady state, U_i follows a uniform distribution $U_i \sim Uniform(0, H_i^2)$. To test this hypothesis, numerous simulations (10^5) were run with different model parameters and a chi-squared test, as a goodness-of-fit test, is applied to investitgate whether the assumption that U_i follows a uniform distribution between 0 and H_i^2, can be rejected.

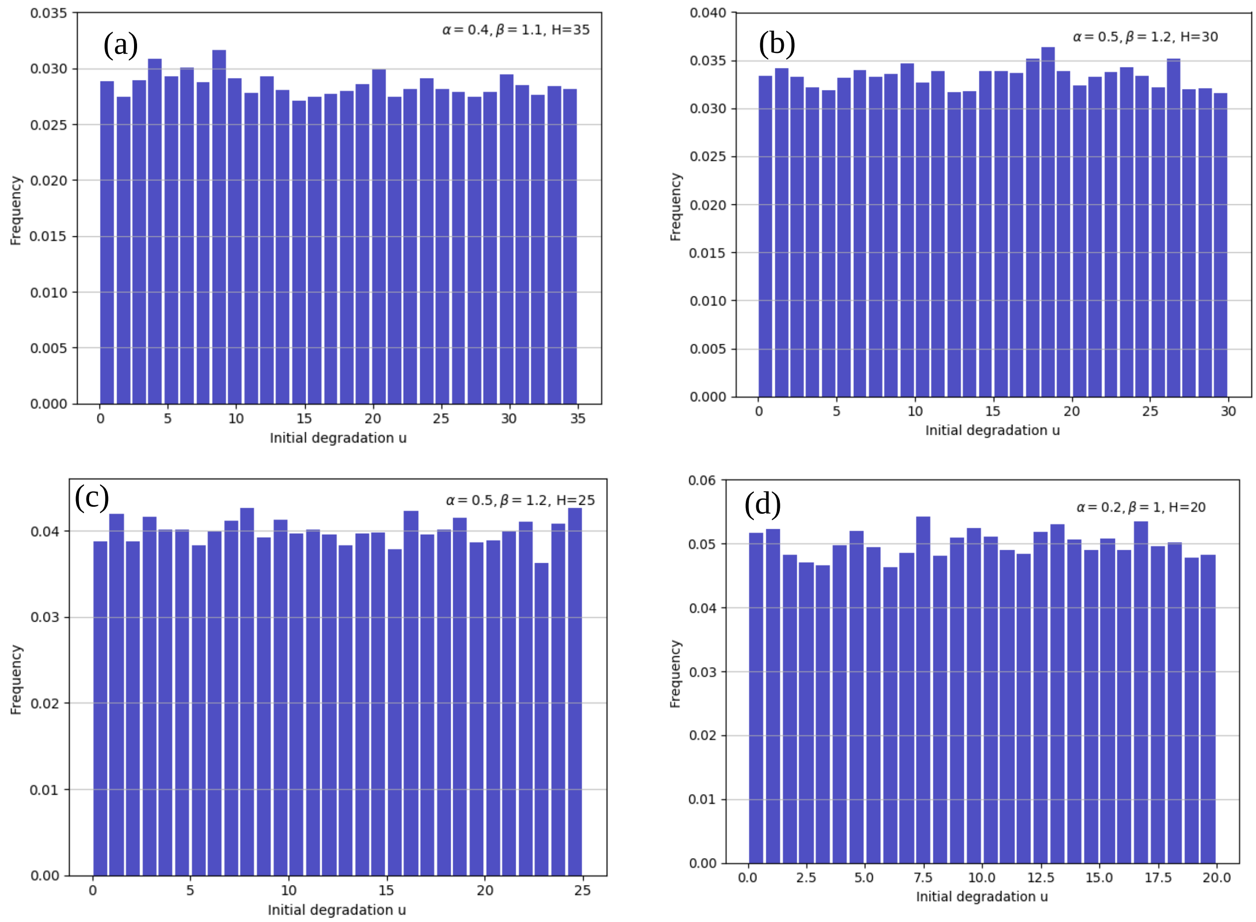


Figure 0.6 Histogram plots for initial degradation at steady state without shock arrival

Table 6 shows the result of chi-squared test for the four different systems shown in Figure 6(a) - (d). As it is shown on Table 6 using significance level of 0.05 (α=0.05) it can be concluded that there is no significant evidence to reject the null hypothesis for these four cases, i.e., the uniform distribution assumption cannot be rejected. In this research study, by changing the parameter values of H_i^2, α_i, β_i, numerous replications of simulation (10^5) are run for 1000 cases, data is collected and histogram plots and chi-squared tests are applied for these 1000 cases. Only for 3 cases out of 1000 was the uniform distribution null hypothesis rejected, Table 4.1 shows some of the examples. Therefore, it can be concluded that the distribution of initial degradation of component *i* degrading as gamma process without shock arrivals can be approximated as uniform distribution between 0 and H_i^2.

Table 0.1 Result of chi-squared tests

Scenario number	Parameter values	Squared error	Test Statistic	p-value
1	$H^2 = 35,\ \alpha = 0.4,\ \beta = 1.1$	0.000039	39.0	0.472
2	$H^2 = 30,\ \alpha = 0.5,\ \beta = 1.2$	0.000038	37.7	0.527
3	$H^2 = 25,\ \alpha = 0.5,\ \beta = 1.2$	0.000052	46.9	0.337
4	$H^2 = 20,\ \alpha = 0.2,\ \beta = 1.0$	0.000027	27.2	0.750
5	$H^2 = 10,\ \alpha = 0.1,\ \beta = 1.2$	0.000055	50.2	0.289
6	$H^2 = 10,\ \alpha = 0.6,\ \beta = 1.1$	0.000024	22.6	0.801
7	$H^2 = 15,\ \alpha = 0.5,\ \beta = 1.4$	0.000037	33.1	0.558
8	$H^2 = 15,\ \alpha = 0.1,\ \beta = 0.9$	0.000045	40.3	0.446
9	$H^2 = 20,\ \alpha = 0.3,\ \beta = 1.5$	0.000041	38.1	0.493
10	$H^2 = 20,\ \alpha = 0.6,\ \beta = 1$	0.000029	27.1	0.763

In order to not rely on only simulation, a close approximation of distribution of initial degradation at steady state is derived. It is assumed that a is the probability of failure in a random state. Therefore, based on Figure 4.7 which shows different scenarios based on the last failure , equations 4.1 – 4.3 can be obtained.

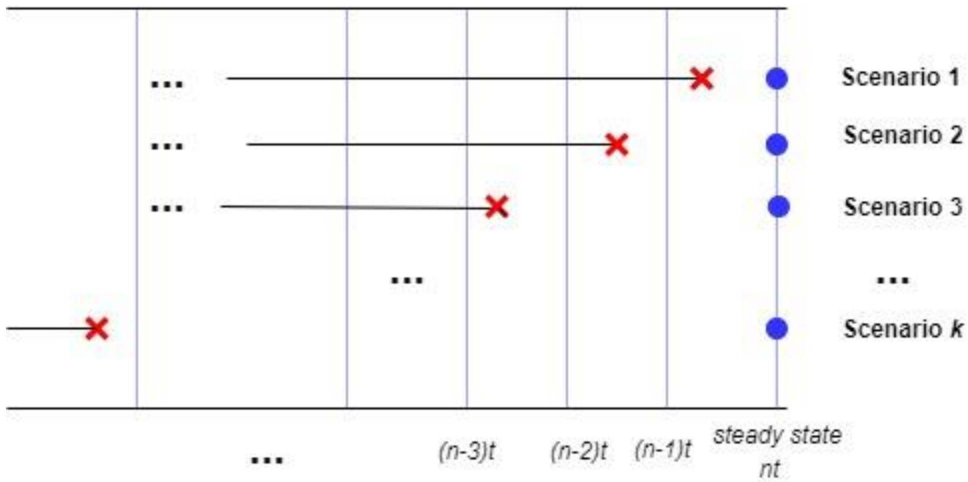


Figure 0.7 Different scenario based on the last failure

$$a + aP\{X(\tau) < H^2\} + aP\{X(\tau) < H^2 \cap X(2\tau) < H^2\} + aP\{X(\tau) < H^2 \cap X(2\tau) < H^2 \cap X(3\tau) < H^2\} + ... = 1$$
$$a + aP\{X(\tau) < H^2\} + aP\{X(2\tau) < H^2\} + aP\{X(3\tau) < H^2\} + ... = 1$$
$$a + aF_{X(\tau)}(H^2) + aF_{X(2\tau)}(H^2) + aF_{X(3\tau)}(H^2) + ... = 1$$
$$a + a\sum_{n=0}^{\infty} F_{X(n\tau)}(H^2) = 1$$
$$a = \frac{1}{1 + \sum_{n=0}^{\infty} F_{X(n\tau)}(H^2)} \tag{4.1}$$

Therefore, by using the calculated a, the probability of $f_U(u)$ is found by considering all the scenarios.

$$f_U(u) = a \cdot 0 + \frac{f_{X(\tau)}(u)}{F_{X(\tau)}(H^2)} \cdot a \cdot P\{X(\tau) < H^2\} + \frac{f_{X(2\tau)}(u)}{F_{X(2\tau)}(H^2)} \cdot a \cdot P\{X(2\tau) < H^2\} + ... \tag{4.2}$$
$$= a \cdot 0 + a \cdot f_{X(\tau)}(u) + a \cdot f_{X(2\tau)}(u) + ...$$

$$f_U(u) = \begin{cases} a & \text{For } u = 0 \\ a\sum_{n=1}^{\infty} f_{X(n\tau)}(u) & \text{For } 0 < u < H^2 \\ 0 & \text{For } u > H^2 \end{cases} \tag{4.3}$$

For a system experiencing shocks arriving as a homogeneous Poisson distribution with arrival rate of λ, each shock arrival causes some damage adding to the component degradation. The same simulation process is performed for a multi-component system with individually repairable components experiencing degradation and random shock arrival. Histogram plots are initially used resulting from the simulation to approximate the distribution of initial degradation for a multi-component system, when each component also is experiencing damage from shocks. Figure 4.6 (a) and (b) show the histogram plot of initial degradation for two different cases. The distribution of initial degradation is observed to increase linearly until the on-condition limit. The rate of increase is observed to relate to the shock arrival rate and shock damage distribution parameters.

By fitting a linear line to the histogram plots and considering that the area is 1, approximate distributions for initial distribution at steady state can be estimated. To test if the initial degradation is a linear relationship, chi-squared goodness-of-fit tests are used to test this hypothesis. By changing parameter values such as $H_i^2, \alpha_i, \beta_i, \lambda, \mu_{Y_i}$, and running numerous replications of simulation for 100 cases, based on significance level of 0.05 (α=0.05), the null hypothesis is rejected for 10% of them. So, there is an approximate linear distribution for initial degradation for multi-component system experiencing degradation and shock process.

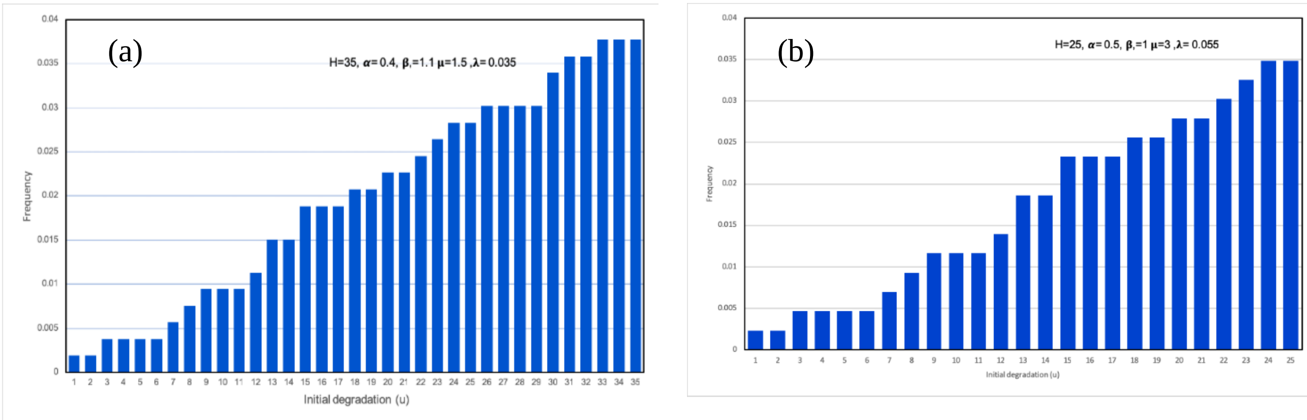


Figure 0.8 Histogram plot for initial degradation at steady state with shock arrival

The two important factors that have impacts on the slope of this fitted line are λ and μ_Y, where λ is the parameter of Poisson distribution. the shock arrival rate. To show the effects of different parameters of λ and μ_Y on the slope of initial degradation distribution, the corresponding linear slope is calculated for different parameters of λ and μ_Y and it is shown on Figures 4.9 and 4.10.

As λ increases, the shocks arriving to the system increase, and there are more shocks in the system before component failure. μ_Y represents the mean shock damage, and as μ_Y increases the shock damage on the component total degradation increases. Therefore, the effects of these two main factors were studied by using simulation on the slope of the probability density function of initial degradation at steady state. When λ has a small value, the shocks arrival rate is small, subsequently the arrival time between shocks is probabilistically a large number. For this case, the initial degradation distribution is similar as the system not experiencing external shocks. In addition, when $\mu_Y = 0$ or approaches 0, the damage caused by shocks is minimal; so, it is equivalent to the case where there are no shocks in the system, and it can be concluded that U_i can be approximated by a uniform distribution (slope is zero). As it is shown in Figures 4.9 and 4.10, increasing the shock arrival rate and expected value of shock damage, cause increases on the slope of a linearly increasing probability density function bounded by 0 and H_i^2.

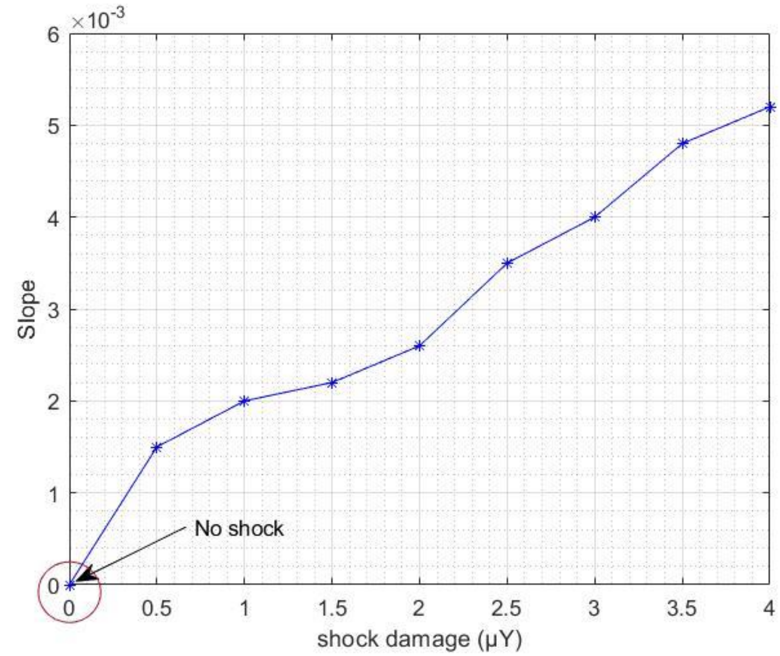


Figure 0.9 $f_U(u)$ slope vs. shock damage on total degradation

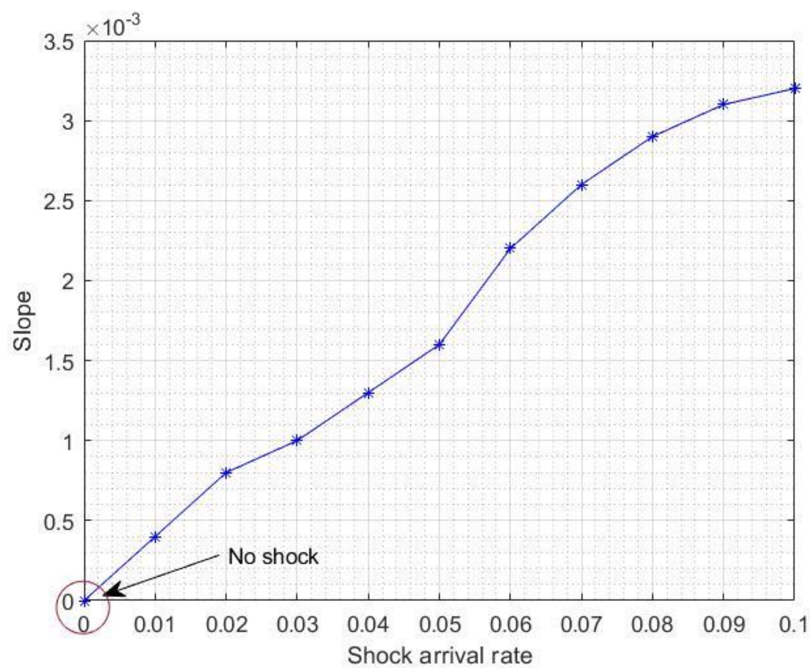


Figure 0.10 $f_U(u)$ slope vs. shock arrival rate for fixed μ_Y

4.1 On-condition maintenance for multi-component systems with individually repairable components

In this section, a condition-based maintenance model was developed for a multi-repairable component system where each component can fail due to degradation or external shock, and it can be replaced individually. Therefore, the system includes components of various ages simultaneously. The components can be replaced individually within the system and they are configured as series; So, failure of each component causes a system failure. Two failure modes are considered for each component, degradation failure and failure due to shocks. The series system fails when the first component experiences either of these two dependent and competing failure modes. Yousefi et al [160] proposed a new condition-based maintenance model for a multi-component system with individually repairable components by finding the optimal on-condition maintenance threshold for each components along with inspection time.

The conditional component reliability function for soft failure at time *t*, given successive replacements and starting from the beginning of the interval at steady state, is the probability that the total degradation of each component is less than the failure threshold. The total degradation for each component by time *t*, after the previous inspection, can be separated into pure degradation, $X_i(t)$, additional incremental degradation caused by shock damage ($S_i(t) = \sum_{j=1}^{N(t)} Y_{ij}$), and the initial degradation at steady state, U_i. The total degradation of the i^{th} component at time *t* after the previous inspection can therefore be accumulated as $X_{S_i}(t) = X_i(t) + S_i(t) + U_i$, and when $X_{S_i}(t) > H_i^1$, component soft failure occurs. H_i^1 is the failure threshold for component *i* and U_i is the initial degradation at the beginning of the interval once the system is in steady-state.

U_i must be between 0 and H_i^1 or H_i^2 depending on the maintenance policy, because if it exceeds the threshold in one interval, it is replaced with a new one prior to the beginning of the next interval. Equation (3.2.4) shows the conditional probability that the i^{th} component has not experienced soft failure by time t starting from the beginning of an inspection interval at steady state.

$$P_{NS}(t) = P(X_{S_i}(t) < H_i^1) = F_{X_i}(H_i^1, t, U_i) = \sum_{m=0}^{\infty} P(X_i(t) + S_i(t) + U_i < H_i^1) \times \frac{(\lambda t)^m e^{-\lambda t}}{m!} \qquad (4.4)$$

It is assumed that the components degrade as a gamma process, and the shock damages follow a normal distribution $Y_{ij} \sim Normal(\mu_{Y_i}, \sigma_{Y_i}^2)$ where Y_{ij} is an $i.i.d$ random variable for the j^{th} shock damage on component i. The shock magnitude is an $i.i.d$ random variable that can have any distribution; so, in this study, it is assumed that it follows a normal distribution $W_{ij} \sim N(\mu_{W_i}, \sigma_{W_i}{}^2)$, where W_{ij} is the j^{th} shock magnitude for component i. Any other distribution could be used without loss of generality. The probability that the i^{th} component survives a shock can be calculated using Equation (3.1.4). where $\Phi(\cdot)$ is the cdf of a standard normal random variable.

By using the approximating of component initial degradation distribution, the conditional probability of surviving from soft failure by time t starting from the beginning of inspection interval at steady state can be calculated easily as follow:

$$
\begin{aligned}
P_{NS_i}(t) &= P(X_{S_i}(t) < H_i^1) = \sum_{m=0}^{\infty} P\left(X_i(t) + U_i + S_i(t) < H_i^1 \mid N(t) = m\right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \\
&= \sum_{m=0}^{\infty} \int_0^{H_i^2} \int_0^{H_i^1 - u} P\left(X_i(t) < H_i^1 - u - y\right) f_{Y_i}^{<m>}(y) f_{U_i}(u) dy du \times \frac{\exp(-\lambda t)(\lambda t)^m}{m!} \\
&= \sum_{m=0}^{\infty} \int_0^{H_i^2} \int_0^{H_i^1 - u} G\left(H_i^1 - u - y; \alpha_i t, \beta_i\right) f_{Y_i}^{<m>}(y) f_{U_i}(u) dy du \times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}
\end{aligned} \tag{4.5}
$$

$G(H_i^1 - y - u; \alpha_i t, \beta_i)$ is the cumulative distribution function of a gamma distribution with parameter $\alpha_i t$ and β_i. $f_Y^{<m>}(y)$ is the probability density function of cumulative shock damage from m shocks, which in this study, follows normal distributions whose parameter are mean = $\mu_Y \times m$ and variance = $\sigma_Y^2 \times m$, μ_Y and σ_Y^2 are the parameters of normal distribution while m is the number of shock. $f_{U_i}(u)$ is the probability density function of initial degradation for component i. Shocks are arriving at random time interval as homogeneous Poisson distribution with rate of λ.

If the reliability functions are considered starting at the beginning of an inspection interval, conditioned on the number of shocks, then the failure processes for all components become conditional independent for a fixed number of shocks. Therefore, the probability that the series system with n components survives until time t from the beginning of inspection interval at steady state can be calculated as follow:

$$
\begin{aligned}
R_S(t) &= \sum_{m=0}^{\infty}\prod_{i=1}^{n}\Big[P(W_i < D_i)^m P\Big(X_i(t)+U_i+S_i(t) < H_i^1 \mid N(t)=m\Big)\Big]\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \\
&= \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[F_{W_i}(D_i)^m \int_0^{H_i^2}\int_0^{H_i^1\text{-}u} P(X_i(t) < H_i^1 - u - y)\, f_{Y_i}^{<m>}(y)\, f_{U_i}(u)\,dy\,du\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \\
&= \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[F_{W_i}(D_i)^m \int_0^{H_i^2}\int_0^{H_i^1\text{-}u} G(H_i^1 - u - y;\alpha_0 t,\beta)\, f_{Y_i}^{<m>}(y)\, f_{U_i}(u)\,dy\,du\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!}
\end{aligned}
$$

(4.6)

For parallel system, the system survives by time *t* from the beginning of inspection interval at steady state if at least one of the components survives by time *t*, so the system reliability by time *t* ($R_P(t)$) for parallel system can be calculated as follow:

$$
\begin{aligned}
R_P(t) &= 1-\sum_{m=0}^{\infty}\prod_{i=1}^{n}\Big[1-P(W_i < D_i)^m P\Big(X_i(t)+U_i+S_i(t) < H_i^1\Big) \mid N(t)=m\Big]\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \\
&= 1-\sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[1-\left(F_{W_i}(D_i)^m \int_0^{H_i^2}\int_0^{H_i^1\text{-}u} G(H_i^1 - u - y;\alpha_0 t,\beta)\, f_{Y_i}^{<m>}(y)\, f_{U_i}(u)\,dy\,du\right)\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!}
\end{aligned}
$$

(4.7)

For system with individual repairable components, it is beneficial to replace the failed component rather than replacing the whole system. Moreover, when the cost of failure is excessive compared to preventive replacement cost, it is prudent to prevent failure from occurring and replace the component before failure occurs. In this research study, I consider a system with individually repairable components where each component can be replaced individually; in fact, it is not necessary to replace the whole system when one component fails or before it reaches the critical degradation threshold. The on-condition threshold H_i^2 can be useful to avoid failure by detecting the degradation level of component *i*.

Figure 4.11 shows the degradation of component *i*, where H_i^1 is the failure threshold, and H_i^2 is the on-condition threshold. The maintenance policy used in this study is as follow:

1. The system is inspected at a periodic time interval, that is an inspection interval, and no continuous monitoring is performed.
2. At the beginning of each inspection interval the degradation of all the components are detected. If the degradation level of each component *i* is lower than H_i^2, the component is in the safe mode and nothing is done. However, if the degradation level is between its on-condition threshold H_i^2 and failure threshold H_i^1 the component is replaced with a new one immediately. The replacements are assumed to be instantaneous and perfect. Other components are not affected.
3. If the total degradation of any component is higher than H_i^1 at the beginning of the inspection interval, the component is detected as failed one and replaced. There is a penalty cost for the system for the downtime duration in the previous interval.

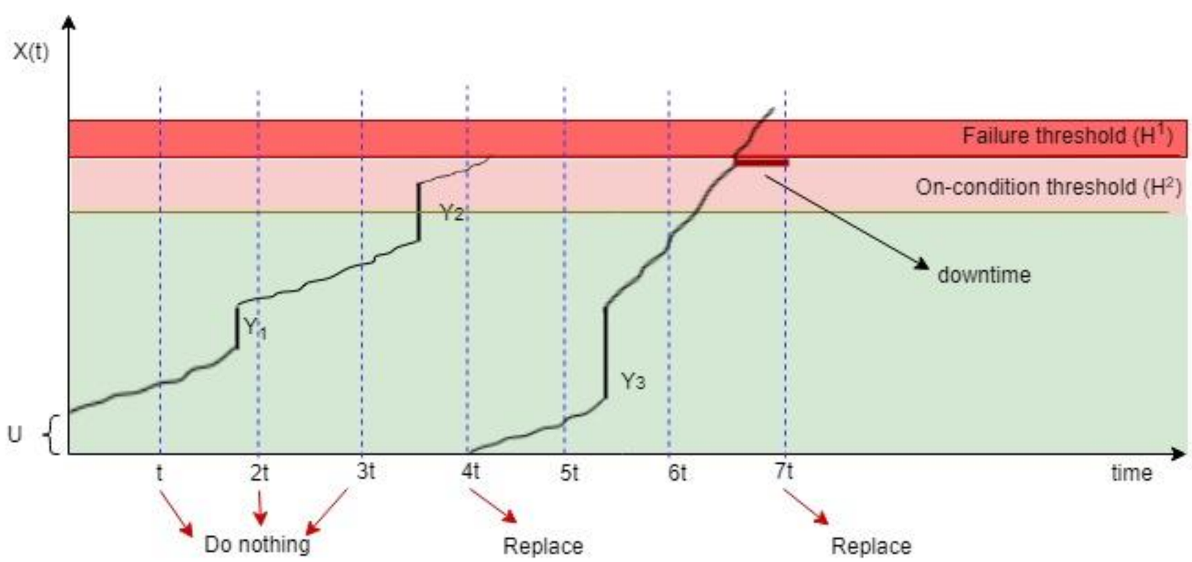


Figure 0.11 Two thresholds divide system into three regions and corresponding maintenance policy

Based on the on-condition maintenance policy of a series system, at each inspection time, if the degradation of all n components are lower than their on-condition thresholds, H_i^2, they are in the high safety level area, and nothing is done. Moreover, if the degradation of any component is between H_i^2 and H_i^1, the component is not failed, but it is prone to high failure risk; and therefore, it should be replaced with a new one preventively. If one component fails, that is, the total degradation of that component is higher than H_i^1 before the specified inspection interval or any shock magnitude in the interval is greater than hard failure threshold, it is not immediately detected and not replaced until the next inspection. There is penalty cost per time associated with downtime, e.g., cost associated with loss of production, opportunity costs, etc.

Upon the on-condition preventive maintenance model, if τ is the inspection interval, the probability that there is no replacement (P_{NRi}) from beginning of inspection interval at steady state by time τ for component i can be calculated by using equation (4.8)

$$P_{NR_i}(\tau) = \sum_{m=0}^{\infty}\left(F_{W_i}(D_i)^m \int_0^{H_i^2}\int_0^{H_i^2\text{-}u} G\left(H_i^2 - u - y; \alpha_0\tau, \beta\right) f_Y^{<m>}(y)\, f_{U_i}(u)\,dy\,du\right)\frac{\exp(-\lambda\tau)(\lambda\tau)^m}{m!} \tag{4.8}$$

It is assumed that all the components are inspected at each inspection time, and the aged component whose degradation level is above its own on-condition threshold are replaced preventively. The on-condition rules provide valuable information about the system status. Detecting all the component degradation levels at each inspection interval lead us to avoid the system failure by replacing the sufficiently aged components at the earliest convenience. The action taken at each inspection interval depends on all the component degradation levels and failure status for each component at that inspection time. At the time of inspection τ, no action is performed if the shock magnitude levels for all components are lower than their hard failure thresholds D_i, and at the same time, the total degradation level of all components are less than their on-condition thresholds H_i^2. Thus, it can be concluded that the component is in the safe region at this time. So, the probability that component *i* is in a safe mode at time τ for a system in steady state, can be derived using Equation (4.8).

Moreover, at each time inspection τ, if no hard failure occurs and total degradation for component *i* is between its on-condition threshold H_i^2 and failure threshold H_i^1, this component is more likely to fail soon. Although it has not failed yet and can still function properly, since it may probabilistically fail within a short period, replacement of this component should be performed. This region can be called aged region.

For calculating the conditional probability of being in aged region for any component *i*, it should be considered the fact that component *i* has not been failed due to hard failure for all the shocks received by the system by time *t*. Therefore it can be calculated as follow:

$$\begin{aligned}P_{aged_i}(t) &= \sum_{m=0}^{\infty}\left[P(W_i < D_i)^m P\left(H_i^2 < X_i(t) + U_i + S_i(t) < H_i^1\right) \mid N(t) = m\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \\ &= \sum_{m=0}^{\infty}\left[F_{W_i}(D_i)^m \int_0^{H_i^2}\int_0^{H_i^1\text{-}u}\left(G\left(H_i^1 - u - y;\alpha_i t, \beta_i\right) - G\left(H_i^2 - u - y;\alpha_i t, \beta_i\right)\right) f_{Y_i}^{<m>}(y)\, f_{U_i}(u)\,dy\,du\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!}\end{aligned} \tag{4.9}$$

If at inspection time τ, there has been a hard failure or the total degradation of any component *i* is greater than its failure threshold H_i^1 (soft failure), the system detected as failed one and the failed components should be replaced. In this situation there is a penalty cost for the downtime. The probability of this situation can be derived from Equation (59).

$$P_{failure_i}(\tau) = \sum_{m=0}^{\infty}\left[1-\left(F_{W_i}(D_i)^m \int_0^{H_i^2}\int_0^{H_i^1\text{-}u} G\left(H_i^1-u-y;\alpha_0\tau,\beta\right) f_Y^{<m>}(y)\, f_{U_i}(u)dydu\right)\right]$$
$$\times\frac{\exp(-\lambda\tau)(\lambda\tau)^m}{m!}$$

(4.10)

4.1.1 On-condition maintenance modeling of systems with individually repairable components

On-condition thresholds provide objective criteria to detect the component degradation status. By using the on-condition threshold and failure threshold and comparing the component degradation level with these two thresholds, the maintenance action can be determined. The penalty cost due to downtime is higher than the associated preventive maintenance cost. Therefore, it is cost efficient to replace the component before failure occurs. If the on-condition threshold is too low, the component will be replaced too frequently which leads to short component life; alternatively, if the on-condition is too high, the component may fail before the next inspection which causes more cost due to system downtime. Thus, determining the optimal on-condition threshold for each component and inspection interval is a challenge.

To evaluate the performance of the maintenance policy, the steady state cost rate is derived. Consider the inspection cost is C_I, the cost of replacement for each component is C_R, and C_ρ is the penalty cost per unit down time. The system cost rate is given by:

$$CR = \frac{C_I + \sum_{k=1}^{n}(C_s + kC_R)P_k + C_\rho E[\rho]}{\tau} \qquad (4.11)$$
$$P_k = P(k \text{ components above } H^2)$$

$E[\rho]$ is the expected downtime. The system downtime is the time duration between the time of failure t within the interval, and the next inspection time τ, which is $\tau - t$ in each inspection interval. Therefore, the expected downtime can be calculated as follow:

$$E[\rho] = \int_0^{\tau} (\tau - t) f_{T_{H^1-U}}(t)dt \tag{4.12}$$

$f_T(t;\mathbf{H}^1 - \mathbf{u})$ is the probability density function of residual failure time at time t during the time interval τ, given that the system has initial degradation U_i at the beginning of the interval. $f_{T_{H-U}}(t)$ can be calculated using Equation (4.13).

$$\begin{aligned} f_T(t;\mathbf{H^1}\text{ - }\mathbf{u}) &= \frac{d}{dt}F_T(t;\mathbf{H^1}\text{ - }\mathbf{u}) = \frac{d}{dt}(1-R(t)) \\ &= \frac{d}{dt}(1-\sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[F_{W_i}(D_i)^m \int_0^{H_i^2}\int_0^{H_i^1\text{-}u} G(H_i^1 - u - y;\alpha_0 t,\beta) f_{Y_i}^{<m>}(y) f_{U_i}(u)dydu\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!}) \end{aligned} \tag{4.13}$$

To calculate the probability of having *k* components with degradation level above their on-condition threshold, two different cases are studied. Case 1 is when all the components within the system are identical and Case 2 is a system with different components.

<u>Case 1</u>: If all the components are identical, the probability of having *k* components with condition above their own on-condition threshold (H^2) can be calculated as follow:

$$P_k = \binom{n}{k} P_{NR}(\tau)^{n-k}(1-P_{NR}(\tau))^k \tag{4.14 a}$$

Where $P_{NR}(\tau)$ is the probability that there are no replacements by time τ at steady state, which can be calculated by using Equation (4.8) , with $P_{NR_i}(\tau) = P_{NR}(\tau)\forall i$

<u>Case 2</u>: If some or all the components within the system are different. The calculation of P_k is more complex for this case. Define $S(k)$ as a set of all *n*-dimension vectors **x**=(x_1, x_2, …, x_n), whose values sum to *k* with $x_i \in \{0, 1\}$, where x_i is 1 if component *i* has degradation level greater than its own on-condition threshold. Therefore, P_k for Case 2 can be computed

considering a multinomial distribution, as:

$$P_k = \sum_{\mathbf{x}\in S(k)} \prod_{i=0}^{n} (1-P_{NR_i}(\tau))^{x_i} P_{NR_i}(\tau)^{1-x_i}, S(k) = \left\{ \mathbf{x}; \sum_{i=1}^{n} x_i = k \right\} \tag{4.14b}$$

Therefore, the maintenance cost rate is indicated in Equation 4.15:

$$CR(\tau, H_i^2) = \frac{\left(C_I + \sum_{k=1}^{n} (C_s + kC_R) P_k + C_\rho \int_0^\tau (\tau - t) f_T(t; \mathbf{H^1} \textbf{ - } \mathbf{u}) dt \right)}{\tau} \tag{4.15}$$

To find the optimal inspection interval and on-condition thresholds, a maintenance optimization model can be modeled and solved. The objective function is the maintenance cost rate. There are n+1 the decision variables, which are n on-condition thresholds for all components and one inspection interval τ. The constraints are that the on-condition threshold for all components should be greater than zero and less than their failure threshold.

The optimization problem can be formulated as Equation 4.16:

$$\min \quad CR(\tau, H_1^2, H_2^2, ...H_n^2) \tag{4.16}$$

Subject to: $0 < H_i^2 < H_i^1$ for i=1, 2, ,…, n

$\tau > 0$

Using Equation (4.15) as the objective function of the optimization problem, the problem is a non-linear optimization problem with continuous decision variables and a convex feasible region. Among all the algorithms to solve this optimization problem and find the optimal decision variables, the interior point method is used in this research. Interior point methods have proved to be very successful in solving many nonlinear problems [145-147].The interior point method consists of the iterative solution in the primal and dual variables of the Karush-Kuhn-Tucker first order optimality conditions. At each iteration, a descent direction is defined by solving a linear system. In a second stage, the linear system is perturbed to defect the descent direction and obtain a feasible direction. A line search is then performed to get a new interior point and ensure global convergence [168].

There have been some studies that show the preference of using *fmincon,* as a built-in algorithm in MATLAB to solve nonlinear optimization problems [168-170]. In this study, the interior point method used to solve the optimization problem shown in Equation (4.14) in MATLAB, using *fmincon* algorithm in the optimization toolbox.

4.1.2 Numerical study for on-condition maintenance model of systems with individually repairable components

The first example is a series system with five identical components. In this example, W_{ij} and Y_{ij} follow normal distributions. $W_{ij} \sim Normal(1.2, 0.2^2)$ and $Y_{ij} \sim Normal(0.3, 0.1^2)$, $\lambda = 2.5\times10^{-5}$, the soft failure threshold is H_i^1=10, and hard failure threshold is D_i=5. A gamma process is used to model the degradation process of all the components with $\alpha_i(t) =$ 0.5t and β_i=1.1. Assuming inspection cost $C_I = 5$, downtime cost $C_\rho = 1000$, and replacement cost for each component is $C_R = 10$, After 22 iteration steps the algorithm converged and the optimal inspection interval is obtained as $\tau^* = 4.25$ and $H_i^{2*} = 7.61\,\forall i$, with the minimum cost rate of \$ 6.63×10^2. Figure 4.12 (a), (b) and 4.13 show the iteration process for the optimization problem.

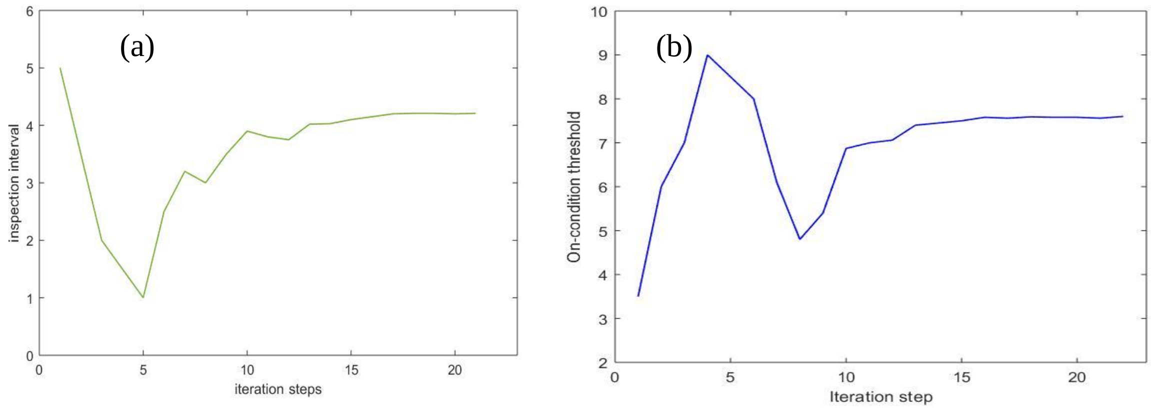


Figure 0.12 (a) Iteration process for inspection interval the first example (b) Iteration process for on-condition threshold for the first example

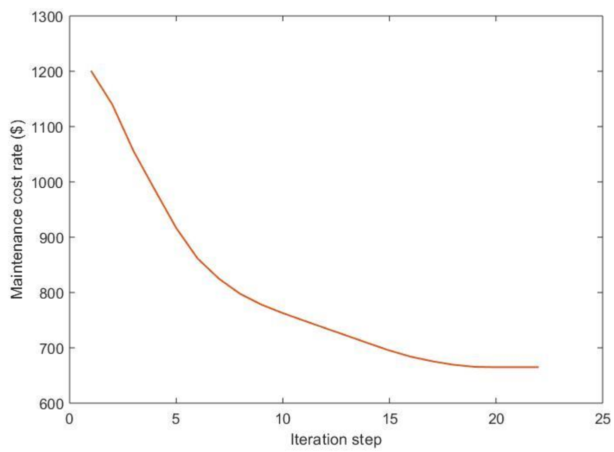


Figure 0.13 Iteration process maintenance cost rate for the first example

The second example is a sliding spool system, which is used in different applications such as electrohydraulic servo-valves. The valve controls hydraulic oil flows by the spool sliding in the sleeve [171]. As illustrated in Figure 4.14, a sliding spool is composed of a spool and a sleeve, where the spool slides in the sleeve to control hydraulic oil flows[171]. The spool is stuck in the sleeve and they work together, so they can be considered as two components configurated as series in a system. Based on the information in [172] the main cause of components failure is sliding valves because of deterioration of system fluid. Moreover, based on a survey by Sasak and Yamamote [172] one of the causes of failure is a sudden appearance of pollutant in the hydraulic oil. The oil pollution can bring some contamination lock to the system, and also some debris adds to the degradation of components. Therefore, in this study, two dependent competing failure processes are considered. The sudden contamination lock caused by oil pollution is the random shock to the system and internal deterioration of spool and sleeve is the component degradation. The degradation of spool and sleeve are considered as gamma processes.

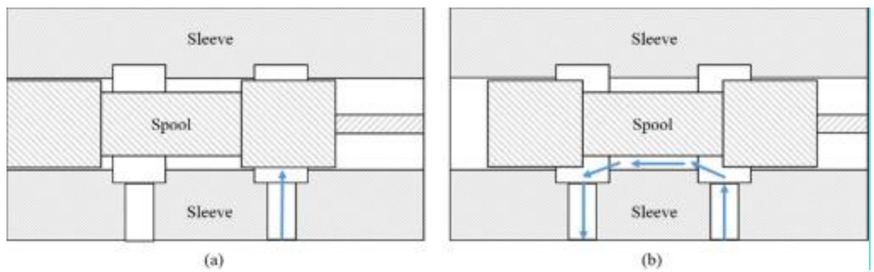


Figure 0.14 Illustration to a sliding spool: (a) Closed position; (b) Open position[173]

Table 4.2 contains the parameters value for reliability analysis of this system. Using Table 4.2 information about component degradation, the maintenance model proposed in this study is used to find the best inspection time τ and on-condition threshold H_i^2 for spool and sleeve as two components configurated as a series system. Assuming inspection cost is $C_I = 5$, downtime cost is $C_\rho = 700$, fixed setup cost for replacement is $C_s = 15$, and replacement cost for each component is replacement cost $C_R = 30$, the optimization problem to minimize the maintenance cost rate for the system can be solved.

To find the optimal inspection interval and on-condition threshold for the spool and sleeve, Equation (4.15) is used as the objective function, and Equation (4.14b) is used to calculate the probability of having *k* components above their on-condition threshold at each inspection time After 30 iterations, the optimality criteria are met and the optimal inspection interval and on-condition threshold for each component is obtained. Figure 4.15, 4.16, and 4.17 show the iteration process of the interior point method for this non-linear optimization problem. The minimum cost rate for this system is $\$2.49\times10^2$, the optimal inspection interval obtained as $\tau^*=1.37$ units of time, the on-condition threshold for spool is $H_1^{2*}=4.01$ and the on-condition threshold for sleeve is $H_2^{2*}=3.32$.

Table 0.2 Parameter values for a sliding spool system reliability analysis

Parameters	values		sources
	Spool	Sleeve	
H_i^1	5 mm	6 mm	Fan et al [157]
D_i	7.5 mm	7 mm	Fan et al [157]
λ	2.5×10^{-5}	2.5×10^{-5}	Fan et al [157]
W_{ij}	$W_{ij} \sim Normal(1, 0.2)$	$W_{ij} \sim Normal(1.5, 0.3)$	Fan et al [157]
Y_{ij}	$Y_{ij} \sim Normal(0.5, 0.1^2)$	$Y_{ij} \sim Normal(0.55, 0.15^2)$	Haiyang, et al [156]
α_i	0.5	0.2	assumption
β_i	1.2	1.6	assumption

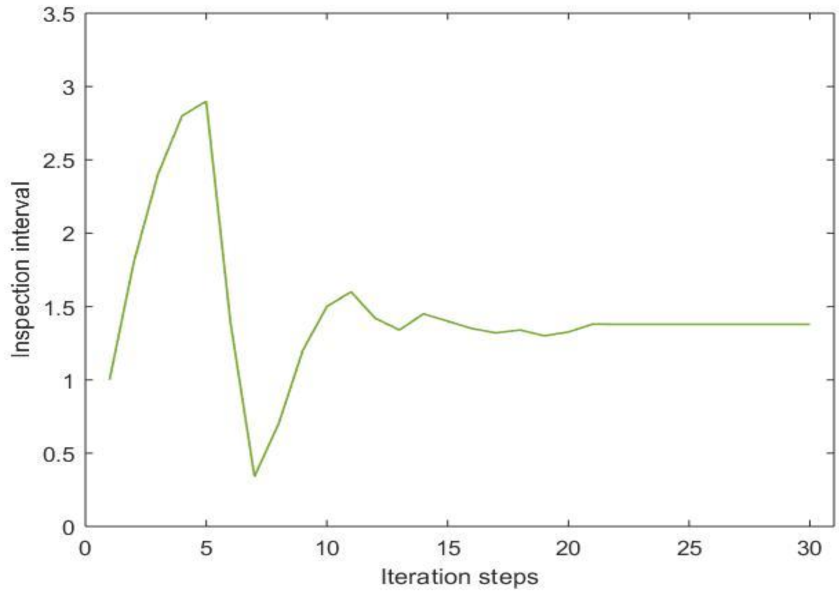


Figure 0.15 Iteration process for inspection interval

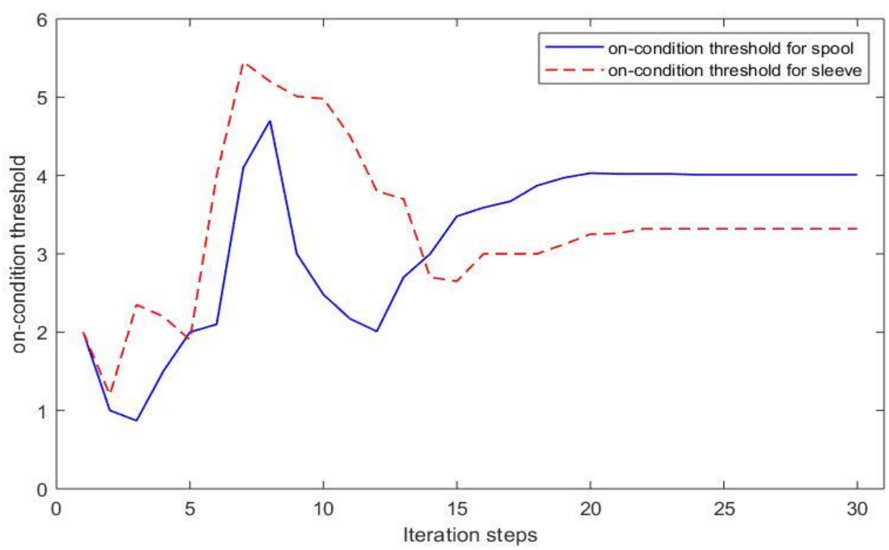


Figure 0.16 Iteration process for On-condition thresholds for spool and sleeve

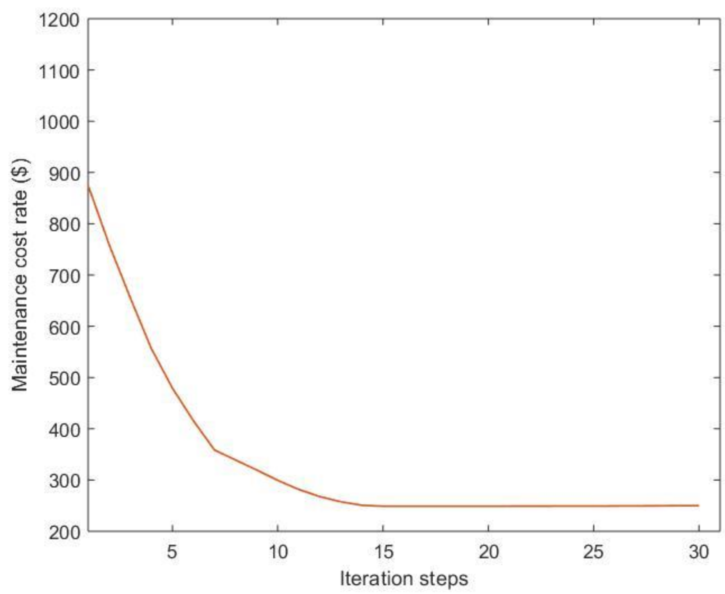


Figure 0.17 Iteration process for maintenance cost rate

For Example 3, some of the parameters of Table 7 are varied, and the maintenance optimization problem solved to find the optimal decision variables, inspection interval for system and on-condition thresholds for spool and sleeve using the new parameters. For this numerical example, there is a higher shock rate coming to the system by using shock arrival rate of $\lambda = 3\times10^{-4}$ and it is also considered that each incoming shock has more damage on one of the components. For this purpose, new parameters are used for the normal distribution for shock damage on the spool, $Y_{ij} \sim Normal(0.9, 0.15^2)$ mm, and there are the same costs as the previous example, i.e., inspection cost $C_I = 5$, downtime cost is $C_\rho = 700$, fixed setup cost for replacement is $C_s = 15$ and replacement cost for each component is replacement cost $C_R = 30$. By using the interior point method to solve this minimization problem, after 33 iteration steps, the optimal decision variables are obtained. Figure 4.18, 4.19, and 4.20 show the iteration steps for these three decision variables, inspection interval, and on-condition threshold for spool and sleeve. The minimum cost rate for new system found as $\$3.58\times10^2$ which was obtained at inspection interval $\tau^* = 0.97$, and on-condition threshold for spool $H_1^{2*} = 4.12$ and for sleeve $H_2^{2*} = 3.98$.

By increasing the shock damage and shock arrival rate in Example 3, the reliability of system decreases. In fact, at any fixed time the system reliability in Example 3 is lower than the second example system reliability. Therefore, the system should be inspected more often to increase the probability of avoiding failure by detecting the components status more frequently. As a result, the optimal inspection interval for the third example is less than the second example. Although the inspection interval for Example 3 is less than the second example, on-condition thresholds for both components in the third example are higher than the second one. Hence, it can be concluded that there is a trade-off between on-condition threshold and inspection interval. When the system is inspected more often, the opportunity for detecting the status of components are higher and there is typically higher on-condition thresholds. Conversely, if the system is inspected less often, a lower on-condition threshold prevents the system from failure.

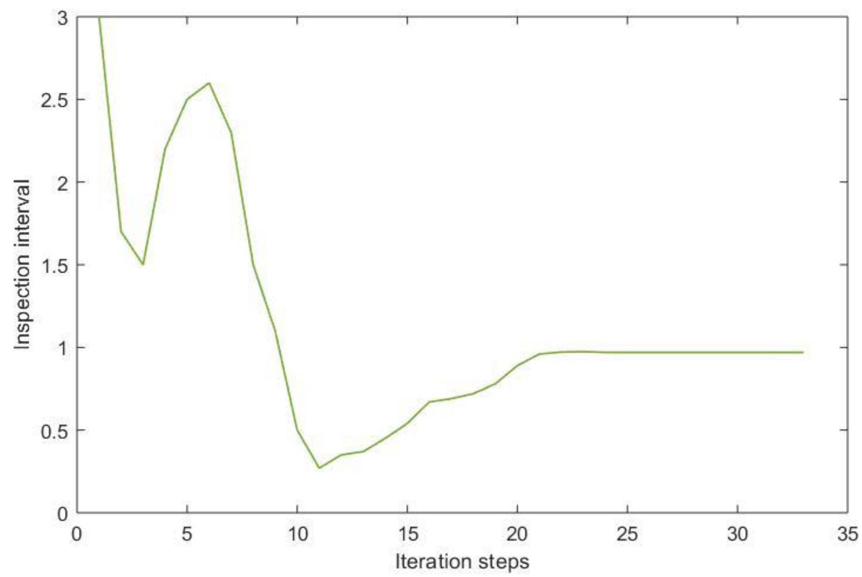


Figure 0.18 Iteration process for inspection interval

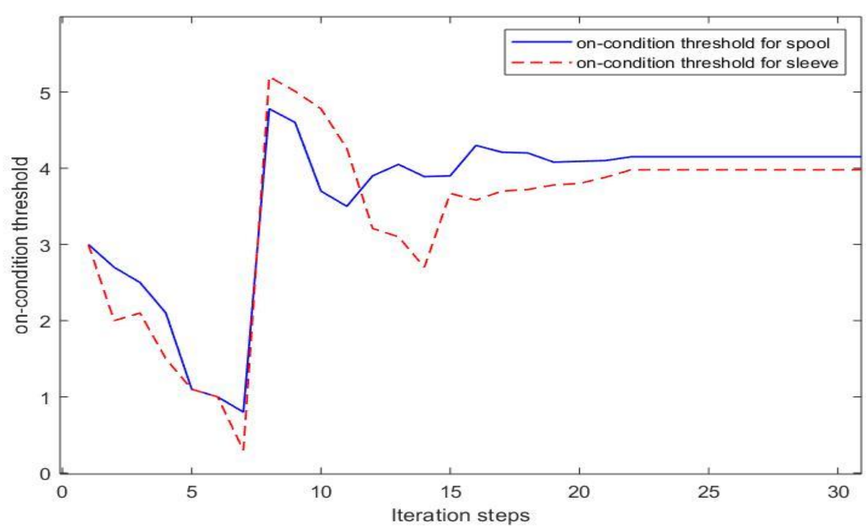


Figure 0.19 Iteration process for on-condition thresholds

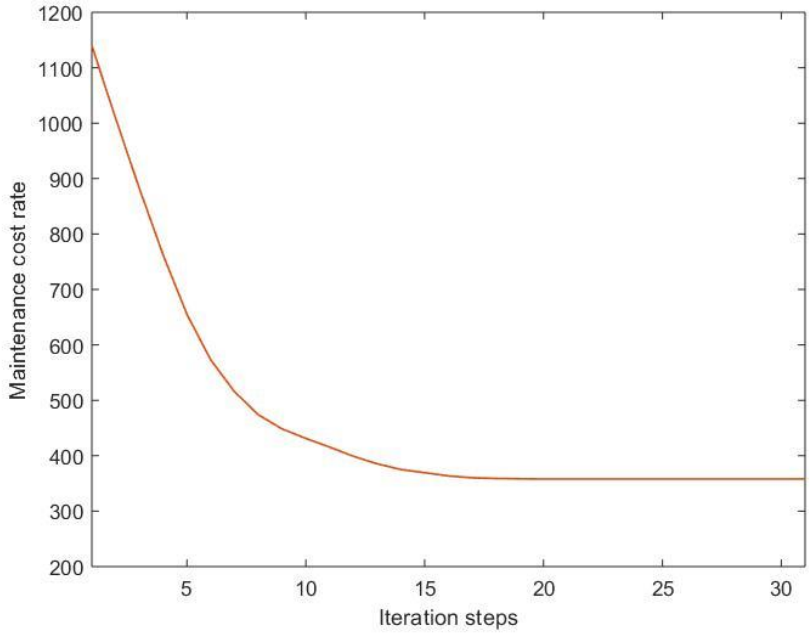


Figure 0.20 Iteration process for maintenance cost rate

Comparing the increased rate of on-condition thresholds for both components, it can be concluded that on-condition thresholds for the sleeve increased more than on-condition threshold for spool. Each arrival shock has a damage on the total component degradation, so the higher shock damage makes the component more prone to failure. Higher shock damage on the spool makes it fail sooner than sleeve. Therefore, its on-condition threshold increased less than the sleeve from the first example. In other words, during each interval, the spool reaches the failure threshold faster than before, but the sleeve has the almost the same degradation rate as before, so on-condition threshold the spool increased less.

4.1.3 Model validation for on-condition maintenance model of systems with individually repairable components

To illustrate the preference of the proposed method, the optimal cost of the numerical example of Table 4.2 is compared to cost rate of previous models such as time-based maintenance, replace-on-failure maintenance and preventive maintenance for a system with non-repairable components.

For replace-on-failure model, at any inspection time, if any component is failed it is replaced with a new one; in fact, the whole system is inspected at each inspection interval and only the failed components should be replaced, so $H_i^2 = H_i^1$ for each component *i*. By setting $H_i^2 = H_i^1$ in Equation (4.15) the optimal inspection interval is found as τ^*=0.95 and the minimum maintenance cost rate is CR^*=3.71×10^2. For time-based maintenance model, the components will be replaced on the first inspection and there is no preventive threshold. So, by setting $H_i^2 = 0$, and solving the optimization problem, the optimal inspection interval for this case is τ^*=5.23 with cost rate of CR^*=3.03×10^2. For preventive maintenance for a system with non-repairable components, the optimal inspection and maintenance cost rate is found as τ^*= 2.78, CR^*=3.12×10^2, and on-condition threshold for spool $H_1^{2*} = 3.04$ and for sleeve $H_2^{2*} = 2.14$. Therefore, comparing all the cost rates, it can be concluded that the propsoed method provide a more cost-effective maintenance polciy for systems with individually repairable components. Table 4.3 shows the result of comparision.

Table 0.3 Comparison of results

Maintenance policy	Cost rate	τ^*	H_1^{2*}	H_2^{2*}
The proposed maintenance policy	2.49 ×10^2	1.37	4.01	3.32
Time-based maintenance ($\tau, H^2 = 0$)	3.14 ×10^2	5.23	0	0
replace-on-failure maintenance ($\tau, H^1 = H^2$)	4.01 ×10^2	0.95	5	6
Preventive maintenance for nonrepairable components	3.78 ×10^2	2.78	3.07	2.15

4.2 Opportunistic maintenance model for a system with individually repairable components

In the previous section, an on-condition maintenance policy is modeled for a system with repairable components. For On-condition maintenance, it is assumed that, at any inspection time, if the degradation level of any component is above the on-condition threshold, it should be preventively replaced before failure occurs. In this section, an opportunistic maintenance policy is modeled and suggested for a multi-component system with individually repairable components. In the proposed model, it is suggested to inspect the system at periodic times and determine the condition of each component at each inspection time, three maintenance thresholds are defined for each component to help detection of the components condition by comparing their cumulative degradation with their own thresholds, failure threshold, on-condition threshold and opportunistic threshold.

For systems with high penalty cost due to downtime, it would be beneficial to replace the aged enough components with a new one to avoid the system failure and minimize costs. Using a suitable on-condition threshold for each component can help the maintenance team to detect the condition of components which are close to failure and replace them preventively. At any inspection time, if the degradation level of any component is above the on-condition threshold, it should be preventively replaced before failure occurs. Whenever a component fails, or its degradation level exceeds its on-condition threshold, the maintenance team should be sent to the field to implement the required maintenance actions. So, there is an opportunity for other components to be preventively replaced if it is necessary. In this section, opportunistic threshold is suggested for each component. Therefore, at any inspection time if the maintenance team are in the field to replace a component they should take the opportunity to preventively replace other components whose cumulative degradation is above their opportunistic threshold.

Each maintenance action due to failure, on-condition preventively replacement and opportunistic preventively replacement have different associated cost, so a maintenance cost rate is developed for a multi-component system with individually repairable components degrading by time and experiencing external shocks. By minimizing the maintenance cost rate, an optimal periodic inspection interval for system and optimal on-condition threshold and opportunistic threshold for all components are found simultaneously.

In this section, an opportunistic condition-based maintenance policy is considered for a system of multi-component, when each component is degrading by time and receives some damages from external shock arrivals. It is also considered that each component in the system can be repaired/replaced individually. Since each component replaced within the system, at each inspection time the initial ages of the components are different. Some multi-components systems are functioning for a very long time and have a very high downtime cost; so, implementing an appropriate maintenance policy for such system can reduce the total cost and provide a more reliable system. Figure 44 shows how the proposed maintenance policy works. There are three thresholds for maintenance policy for any component i, failure threshold (H_i^1), on-condition threshold (H_i^2) and opportunistic threshold (H_i^3). Using on-condition threshold (H_i^2) as a preventive maintenance would reduce the downtime cost by replacing the component before failure. Moreover, whenever a component should be replaced due to failure or on-condition maintenance rule, the maintenance team is sent to the filed to implement the required maintenance action; so, there is an opportunity to simultaneously perform preventive maintenance on other components whose degradation level is greater than their opportunistic threshold (H^3). For this paper, some specific assumptions are considered for the maintenance policy which are as follow:

1. At each inspection time, the condition of all the components in the system are determined. So, the system is inspected periodically and there is no continuous monitoring.
2. There is a failure threshold (H_i^1) for any component i in the system, whenever the component's total degradation reaches the failure threshold (H_i^1), component i fails due to the soft failure.
3. Each incoming shock j can cause failure of component i, if its magnitude (W_{ij}) is greater than a predefined threshold for component i (D_i); in fact, component i fails due to hard failure.
4. Each component i has an on-condition threshold (H_i^2) and at any inspection time, if the total degradation of each component is greater than its own on-condition threshold (H_i^2), but less than failure threshold (H_i^1), component i is still working but it may fail soon, so it is detected as aged component and should be replaced with a new one.
5. There is one more threshold for each component i as an opportunistic threshold (H_i^3). At any inspection time, if the maintenance team should go to the field to replace any component, they can take advantage of preventively replace other component i which has the total degradation level above its opportunistic threshold (H_i^3).
6. The cost of replacement is different for failure replacement, and preventively replacement due to on-condition and opportunistic maintenance rule.
7. If the system fails within the interval, it is not immediately detected and not replaced

until next inspection. But there is a penalty cost due to loss of production for the downtime duration.

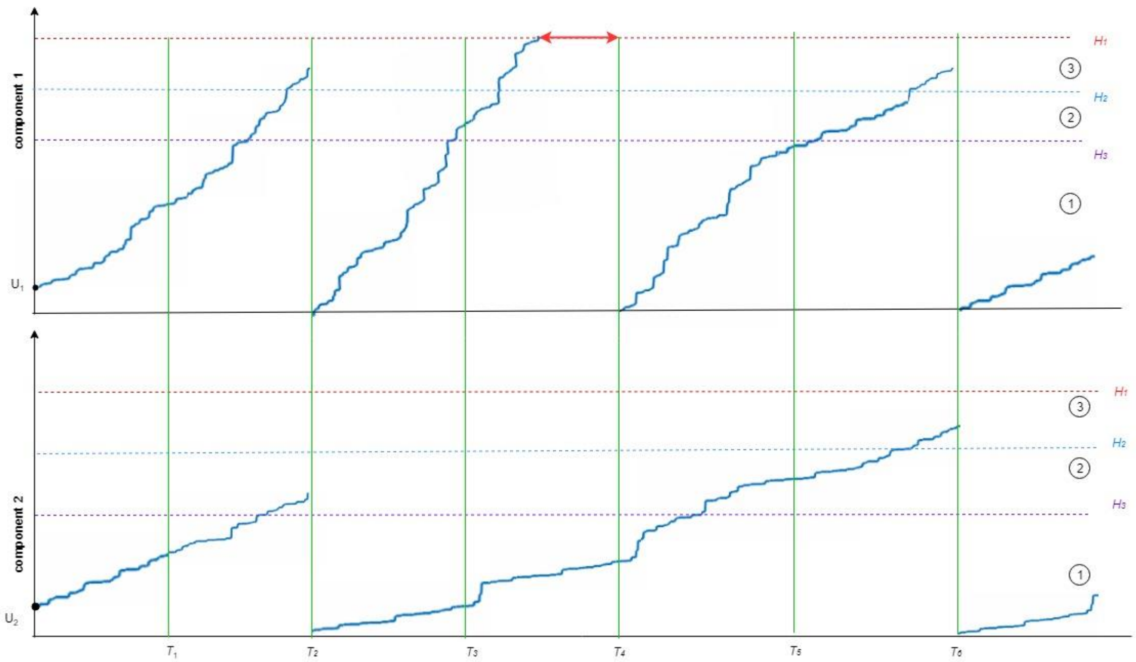


Figure 0.21 Opportunistic maintenance policy

Figure 4.21 shows a system of two components which degrading over time, and the maintenance policy for these two components. Table 4.4 shows the three regions based on comparison of total degradation of each component *i* with its own three thresholds, failure threshold (H_i^1), on-condition threshold (H_i^2) and opportunistic threshold (H_i^3).

Table 0.4 Region description for each component

region	Description for any component *i*
1	If the total degradation of component *i*, X_{S_i} is less than its own opportunistic threshold H_i^3, the probability of being in region 1 can be calculated by equation (6), $P_{1\text{-}i}$
2	If the total degradation of component *i*, X_{S_i} is greater than its own opportunistic threshold H_i^3 and less than its own on-condition threshold (H_i^2), the probability of being in region 2 can be calculated by equation (7), $P_{2\text{-}i}$

3	If the total degradation of component i, X_{S_i} is greater than its own on-condition threshold H_i^2 and less than its own failure threshold (H_i^1), the probability of being in region 3 can be calculated by equation (8), $P_{3\text{-}i}$
failure	If the total degradation of component i, X_{S_i} is greater than its failure threshold H_i^1 or any shock magnitude for component i is greater than its own hard failure threshold, $W_{ij}<D_i$, the probability of being in failure region can be calculated by equation (9), $P_{4\text{-}i}$

Table 4.5 illustrates the maintenance action at each inspection time in Figure 4.21. When both components are in their region 1 nothing should be done. When one of them should be replaced due to failure or being above the on-condition threshold, the other component should be replaced if it is in region 2. The same maintenance rules will be applied for any number of components in the system.

Table 0.5 Maintenance action for different scenarios at each inspection time

Inspection	Region for Component 1	Region for Component 2	Maintenance action
T_1	1	1	Nothing
T_2	3	2	Replace component 1 and preventively replace component 2
T_3	2	1	Nothing
T_4	failure	1	Replace component 1 and nothing should be done on component 2
T_5	1	2	Nothing
T_6	3	3	Replace component 1 and component 2

To calculate system reliability, the probability of being in any region should be calculated for any component i. Based on table 1, each component i is in region 1 if the total degradation of component i, X_{S_i} is less than its own opportunistic threshold H_i^3 , so the probability that component i is in region 1 can be calculated using Equation (4.17).

$$P_{1-i} = P(X_{S_i} < H_i^3) = \sum_{m=0}^{\infty} P(W_i < D_i)^m P\Big((X_i(t) + U_i + S_i) < H_i^3 \Big) \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$= \sum_{m=0}^{\infty} P(W_i < D_i)^m \times \int_0^{H_i^2} \int_0^{H_i^1 - u} P\Big((X_i(t) < H_i^3 - u - y \Big) f_Y^{<m>}(y) f_{U_i}(u) dy du \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$= \sum_{m=0}^{\infty} P(W_i < D_i)^m \times \left(\int_0^{H_i^2} \int_0^{H_i^1 - u} Ga(H_i^3 - u - y) f_Y^{<m>}(y) f_{U_i}(u) dy du \right) \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(4.17)

Each component i is in region 2, if the total degradation of component i, X_{S_i} is greater than its own opportunistic threshold H_i^3 and less than its own on-condition threshold (H_i^2)

$$P_{2-i} = P(H_i^3 < X_{S_i} < H_i^2) = \sum_{m=0}^{\infty} P(W_i < D_i)^m \times P\Big(\big((X_i(t) + U_i + S_i) < H_i^2 \big) - \big((X_i(t) + U_i + S_i) < H_i^3 \big) \Big)$$
$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$= \sum_{m=0}^{\infty} P(W_i < D_i)^m \left(\left(\int_0^{H_i^2} \int_0^{H_i^1 - u} Ga(H_i^2 - u - y) f_Y^{<m>}(y) f_{U_i}(u) dy du \right) - \left(\int_0^{H_i^2} \int_0^{H_i^1 - u} Ga(H_i^3 - u - y) f_Y^{<m>}(y) f_{U_i}(u) dy du \right) \right)$$
$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(4.18)

Probability of being in region 3 for each component i can be calculated as follow:

$$P_{3-i} = P(H_i^2 < X_{S_i} < H_i^1) = \sum_{m=0}^{\infty} P(W_i < D_i)^m \times P\Big(\big((X_i(t) + U_i + S_i) < H_i^1 \big) - \big((X_i(t) + U_i + S_i) < H_i^2 \big) \Big)$$
$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$= \sum_{m=0}^{\infty} P(W_i < D_i)^m \left(\left(\int_0^{H_i^2} \int_0^{H_i^1 - u} Ga(H_i^1 - u - y) f_Y^{<m>}(y) f_{U_i}(u) dy du \right) - \left(\int_0^{H_i^2} \int_0^{H_i^1 - u} Ga(H_i^2 - u - y) f_Y^{<m>}(y) f_{U_i}(u) dy du \right) \right)$$
$$\times \frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$

(4.19)

Probability of failure due to soft failure or hard failure by time t, or being in failure region can be calculated using Equation (4.18).

$$P_{4-i} = 1 - P(X_{S_i} < H_i^1) = \sum_{m=0}^{\infty}\left(1 - P(W_i < D_i)^m P\left((X_i(t) + U_i + S_i) < H_i^1\right)\right)\frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$= \sum_{m=0}^{\infty}\left(1 - P(W_i < D_i)^m \times \left(\int_0^{H_i^2}\int_0^{H_i^1-u} Ga(H_i^1 - u - y)\, f_Y^{<m>}(y)\, f_{U_i}(u)\,dydu\right)\right)\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{4.20}$$

Equation (4.17) to (4.20) show the probability that component i is in different regions. However, there is a penalty cost for any duration in an interval when the system is down. So, the probability of failure should be calculated for a system of multi-component. In this study, it is considered that the components are configurated as series in the system. Figure 4.22 shows a series system of n components.

Figure 0.22 Series system of n components

In a series system, the system fails if any component in the system fails. So, the probability of system failure can be calculated using Equation (4.19)

$$F(t) = 1 - R(t) = 1 - \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[P(W_i < D_i)^m P\left((X_i(t) + U_i + S_i) < H_i^1\right)\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!}$$
$$= 1 - \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[\sum_{m=0}^{\infty} P(W_i < D_i)^m \left(\int_0^{H_i^2}\int_0^{H_i^1-u} Ga(H_i^1 - u - y)\, f_Y^{<m>}(y)\, f_{U_i}(u)\,dydu\right)\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \tag{4.21}$$

Subsequently the expected of system downtime can be determined as $\int_0^{\tau}(\tau - t)\,dF_T(t;\mathbf{H^1}\text{ - }\mathbf{u})$. Where ($\tau$-$t$) is the duration where system is down in an interval of [0, τ] and $dF_T(t;\mathbf{H^1}\text{ - }\mathbf{u})$ is the density function of time when system fails.

To evaluate the performance of the proposed maintenance policy, a cost rate function is indicated in Equation 4.22.

$$CR = \frac{C_I + C_\rho E[\rho] + C_o \sum_{k=1}^{n-1} (C_F + kC_o) P(\text{any fails}) P_k + \sum_{k=1}^{n-1} (C_R + kC_o) P(\text{any in region 3}) P_k}{\tau} \tag{4.22}$$

Where C_I is the cost of inspection, C_p is the cost associated to downtime, C_F is the replacement cost for a failed component, C_o is the replacement cost for any component which is in the opportunity region and should be preventively replaced, and C_R is the replacement cost due to on-condition preventively maintenance. It is considered that system has n identical components, then the probabilities in Equation (4.22) can be calculated as follow, where $P_{1\text{-}i}$, $P_{2\text{-}i}$, , $P_{3\text{-}i}$ are the probability that component i is in region 1, 2 and 3 using equation (4.17), (4.18) and (4.19).

- $E[\rho] = \int_0^\tau (\tau - t) dF_T(t; \mathbf{H^1} - \mathbf{u})$ where $F_T(t; \mathbf{H^1} - \mathbf{u})$ can be calculated using equation (4.21)
- Probability of any failure is calculated using equation (4.20)
- Probability of any component is in region 3 can be calculated using equation (4.19) ($P_{3\text{-}i}$)
- Probability of having k components above their opportunistic threshold H^3 can be calculated as follow:

$$P(k \text{ above } H^3) = \binom{n-1}{k} (1 - P_{1-i})^k (P_{1-i})^{n-k-1}$$

To find the optimal inspection interval for system and optimal on-condition threshold and opportunistic threshold a maintenance optimization can be modeled and solved, where the maintenance cost rate of Equation (4.22) is the objective function. The on-condition threshold should be less than failure threshold and opportunistic threshold is less than on-condition threshold.

$$\begin{array}{ll} \min & CR(\tau, H^2, H^3) \\ \text{Subject to:} & 0 \le H^2 \le H^1 \\ & 0 \le H^3 \le H^2 \\ & \tau > 0 \end{array} \tag{4.23}$$

To solve the optimization problem *fmincon* algorithm in MATLAB toolbox is used. *Fmincon* in MATLAB is easy to use, robust and has wide variety of options. The built-in parallel computing support in *fmincon* accelerates the estimation of gradients. The optimal results are compared to the result of simulation optimization, to confirm the accuracy of interior point in *fmincon*.

4.2.1 Numerical study for the proposed opportunistic maintenance model

A conceptual numerical example is used to show the preference of proposed maintenance policy. It is considered that there are 4 identical components which are configurated as series system. The parameter values for reliability analysis are provided in Table 4.6. It is assumed that Y_{ij} follows a normal distribution of $Y_{ij} : \; Normal(\mu_{Y_i}, \sigma_{Y_i})$ and it is the same for all components, and W_{ij} is also follows a normal distribution of $W_{ij} : \; Normal(\mu_{w_i}, \sigma_{w_i})$, and it is assumed that the initial degradation U_i follows a truncated exponential distribution as Equation (4.14). Although the example is conceptual, H^1 and D are estimated based on documented degradation trends [152]

$$f_U(u) = \begin{cases} \dfrac{\theta e^{-\theta u}}{F_U(H^2) - F_U(0)}, & 0 < u < H^2 \\ 0 \,, & \text{otherwise} \end{cases} \tag{4.24}$$

It is assumed that the inspection cost is \$2, the downtime cost is \$1000, the opportunistic replacement is \$50, replacement due to failure is \$400, and replacement due to on-condition maintenance is \$200.

Table 0.6 Parameter values for reliability analysis

Parameter	value	Sources
H^1	0.00125 μm^3	Tanner and Dugger [152]
D_i	1.5 Gpa	Tanner and Dugger [152]
α_i	0.7	Assumption
β_i	0.3	Assumption
λ	2.5×10^{-5}	Assumption
θ	0.9	Assumption
Y_{ij}	$Y_{ij} : \ Normal(0.4, 0.12^2)$	Assumption
W_{ij}	$W_{ij} : \ Normal(1.1, 0.1^2)$	Assumption

The contribution of this paper is to find the inspection interval, on-condition threshold and opportunistic threshold for all the component simultaneously. By using simulation optimization for cost function in Equation (4.23), after 24 iteration steps, the optimal inspection interval is found as $\tau^* = 60.5$, and on-condition threshold is $H^{2*} =$ 0.00095 and $H^{3*} = 0.00055$, and the minimum cost rate is 1.245×10^2. Figure 4.23, 4.24, and 4.25 show the optimization process for maintenance thresholds, inspection interval and cost rate.

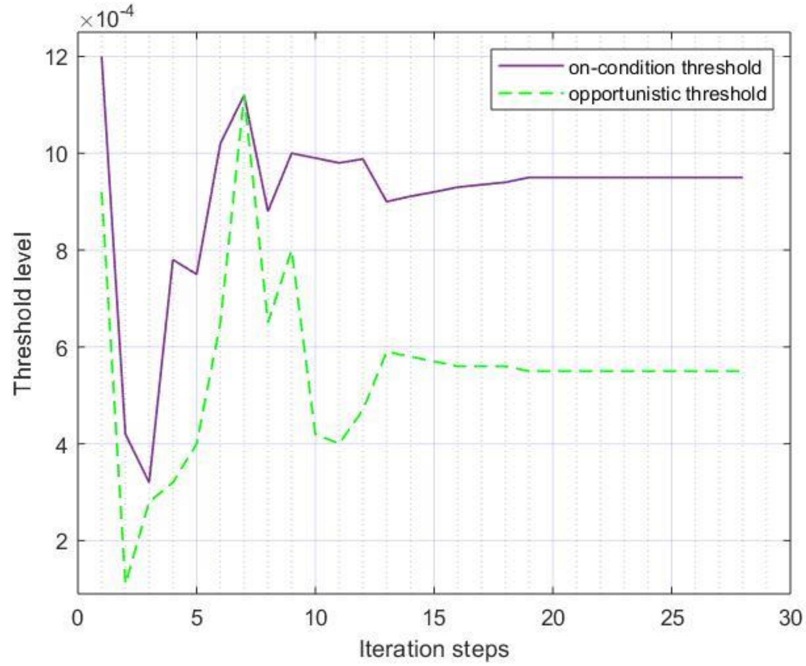


Figure 0.23 Iteration process for maintenance thresholds

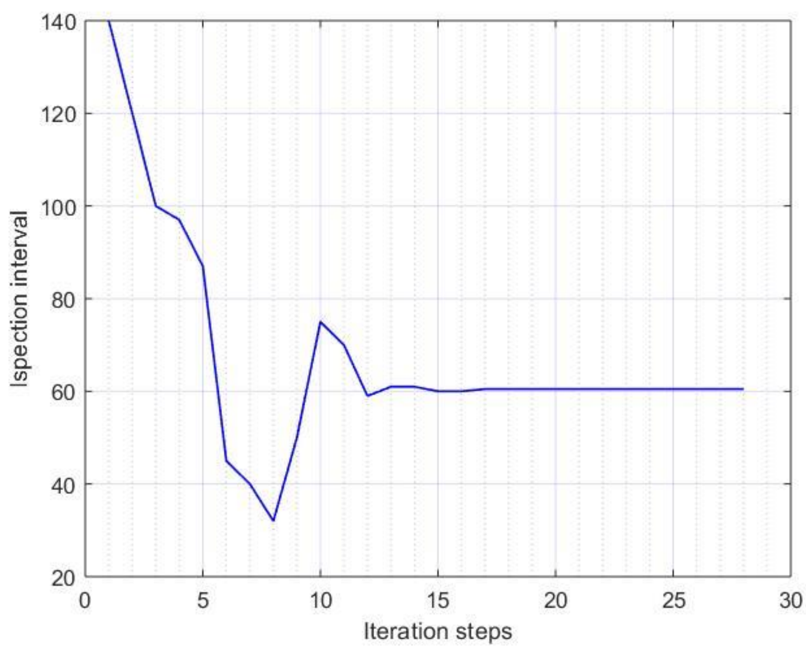


Figure 0.24 Iteration process for Inspection interval

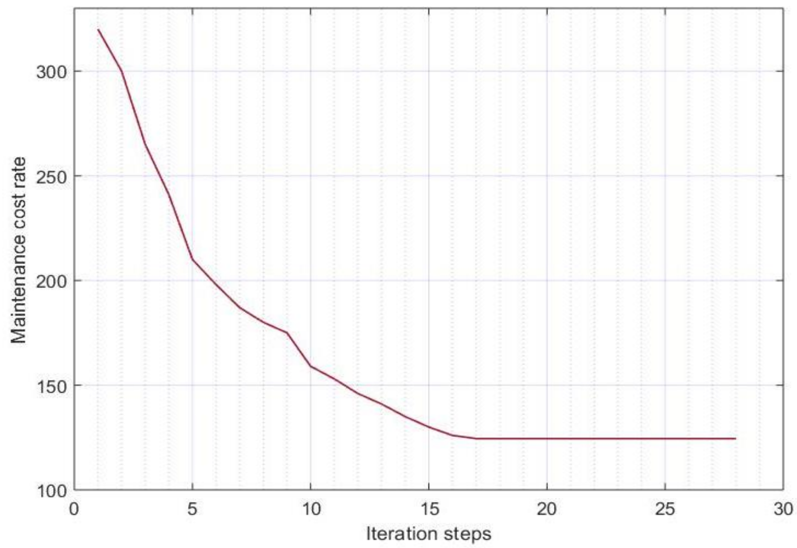


Figure 0.25 Iteration step for maintenance cost rate

4.2.2 Model validation for the proposed opportunistic maintenance model

To show the preference of the proposed maintenance model for saving cost and have a more cost-effective maintenance plan, I compare the cost rate of the proposed model and different maintenance policies such as replace-on-failure and preventively replacement on on-condition threshold but no opportunistic threshold. Table 4.7 shows this comparison. If there is no opportunistic maintenance policy for the system, and each component is replaced when its degradation is above on-condition threshold; in fact, the opportunistic threshold in this problem is exactly the same as the on-condition threshold for all the components ($H^3 = H^2$) the optimal cost rate is found as 1.975×10^2 with optimal inspection interval of $\tau^* = 45.5$. For replace-on-failure maintenance policy, the component is replaced if it is detected as failed in any inspection time; in fact, on-condition threshold and opportunistic threshold are the same as failure threshold ($H^1 = H^2 = H^3$). By solving the optimization problem for replace-on-failure policy the cost rate is found as 4.263×10^2

and the optimal inspection interval is $\tau^* = 10.3$

Table 0.7 Result comparison

Maintenance policy	Cost rate	τ^*	H^{2*}	H^{3*}
The proposed maintenance policy (τ, H^2, H^3)	1.245×10^2	60.5	0.00095	0.00055
No opportunistic maintenance ($\tau, H^2 = H^3$)	1.975×10^2	45.5	0.00086	0.00086
replace-on-failure maintenance ($\tau, H^1 = H^2 = H^3$)	4.263×10^2	10.3	0.00125	0.00125

As it is shown in Table 4.7, the maintenance cost rate is very high for replace-on-failure and the proposed model provides a more cost-effective maintenance policy for systems that their components can be individually replaced.

Dynamic maintenance planning models using optimization and neural networks

Most of the previous maintenance models are static and they do not consider dynamic information about the degradation status of components. In static maintenance models, the optimal inspection interval or maintenance thresholds are found for a specific system and will not change based on the degradation level of components, while in dynamic maintenance models the maintenance operations are redefined at each decision time. Developing a dynamic maintenance model for systems can be more cost-effective and efficient compared to previous static maintenance models. Under the dynamic maintenance policies, the maintenance manager can easily and quickly change the maintenance schedule at any moment according to the condition of the system. In this research paper, our models can be extended for dynamic maintenance plans where the inspection interval or the maintenance actions can eb dynamically determined based on the current degradation information of the components within the system to minimize the maintenance cost. Bouvard et al [174] proposed a condition-based dynamic maintenance model for a system of multiple components, based on the components states and detected failures. The new maintenance plan is found based on the updated reliability characteristic of each components. Horenbeek and Pintelon [175] developed a dynamic predictive maintenance model for a multi-component system, based on the predicted remaining useful life of the system. The maintenance schedule is updated when new information on the degradation and remaining useful life of component become available. Developing a maintenance policy for a multi-component system with individually repairable component is a unique challenged which is studied in my research work. Yousefi and Coit [166, 167] proposed dynamic maintenance models for a multi-component system with individually repairable components.

1.10 Dynamic inspection planning for systems with individually repairable components

In a multi-component system with individually repairable components, whenever each component experiences soft or hard failure, it is considered as failed and it should be replaced with a new one, but other components continue functioning until they fail. It is assumed that each failed component is detected just by inspection. Since the failed component is replaced instead of the whole system, the age of components at each inspection time is different from others. In the proposed maintenance policy, the inspection interval should be found dynamically based on the initial age of all the components. For a system with multi-components that degrading differently, a preventive maintenance model should be found considering the age of all the components at the beginning of each interval.

Figure 5.1 shows the maintenance model for a multi-component system where each component can be replaced individually at the beginning of each inspection interval, the ages of components are different from each other and subsequently the length of next inspection interval would be different based on the initial age of all the components. To calculate the system reliability for a future inspection interval, random values u_i are assumed as the initial age of each component *i* at the beginning of interval.

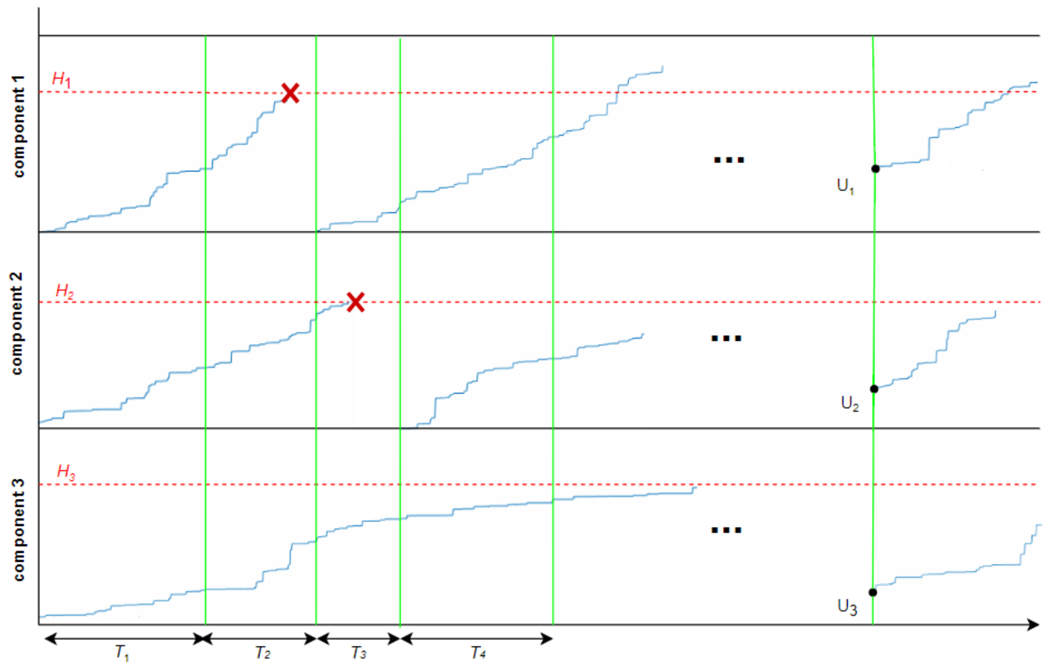


Figure 0.1 Dynamic inspection planning considering random initial age for each component

The following equation shows the probability of having no failure for component *i* in a series system from the beginning of an interval up to time *t*.

$$R_i(t;u_i) = \sum_{m=0}^{\infty}\left[P(W_{ij} < D_i)^m P\left(X_i(t)+S_i(t)+u_i < H_i\right)\right]\frac{(\lambda_0 t)^m e^{(-\lambda_0 t)}}{m!} \tag{0.1}$$
$$= \sum_{m=0}^{\infty}\left[P(W_{ij} < D_i)^m \times \int_0^{H_i} P\left(X_i(t)+y+u_i < H_i\right) f_{Y_i}^{<m>}(y)dy\right]\frac{(\lambda_0 t)^m e^{(-\lambda_0 t)}}{m!}$$

For a series configuration, the system fails if any component fails; hence, the system reliability can be calculated using Equation (5.2), and for parallel configuration, the system fails if all the components are failed, so the system reliability should be calculated using Equation (5.3)

$$R_S(t;\boldsymbol{u}) = \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[P(W_{ij} < D_i)^m \times \int_0^{H_i} P\left(X_i(t) + y + u_i < H_i\right) f_{Y_i}^{<m>}(y)dy\right]\frac{(\lambda_0 t)^m e^{(-\lambda_0 t)}}{m!} \tag{0.2}$$

$$R_P(t;\boldsymbol{u}) = 1 - \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[1 - \left(P(W_{ij} < D_i)^m \times \int_0^{H_i} P\left(X_i(t) + y + u_i < H_i\right) f_{Y_i}^{<m>}(y)dy\right)\right] \times \frac{(\lambda_0 t)^m e^{(-\lambda_0 t)}}{m!} \tag{0.3}$$

As it is demonstrated in Yousefi et al [166, 167], the inspection interval can be find dynamically at the begging of each inspection time using the reliability models of series and parallel systems.

In this section, I propose a dynamically changing inspection planning as the preventive maintenance for series and parallel systems. It is assumed that the whole system and all the components are inspected at any inspection time, and each failed component can be detected only by inspection and is replaced at the beginning of the next inspection individually, while all other components continue functioning. So, there is a penalty cost for as the production loss for the duration in any interval that the system is down. Following that, the next inspection interval should be calculated based on the initial age of all the components. To find the optimal inspection interval for the next inspection Cost rate function should be calculated and optimized dynamically.

$$CR(\tau;\boldsymbol{u}) = \frac{C_I + C_R(1 - R(\tau;\boldsymbol{u})) + C_\rho E[\rho]}{\tau} \tag{0.4}$$

Where C_I is the inspection cost, C_R is the cost of replacement, C_ρ is the downtime cost and τ is the inspection time. $E[\rho]$ is the expected of downtime which is

$E[\rho]=\int_0^{\tau}(\tau-t)f_{T_{H-U}}(t)dt$, where $f_{T_{H-u}}(t)$ is the probability density function of failure time that system fails at time t during the time interval τ starting from random value as initial degradation u. it is calculated by taking derivative of CDF of system failure time $f_{T_{H-u}}(t)=\dfrac{dF_{T_{H-u}}(t)}{dt}=\dfrac{d(1-R(t;\boldsymbol{u}))}{dt}$.

According to the system configuration, Equation (5.2) for series and Equation (5.3) for parallel, should be substituted in Equation (5.4) to calculate the system cost rate function. To find the next optimal inspection interval, an optimization problem should be solved dynamically based on the initial age of all the components, where cost rate function in Equation (5.4) is the objective function of the optimization problem. Therefore, the next optimal inspection interval is obtaining dynamically based on initial random values as age of all the components.

Two conceptual examples are considered to demonstrate the proposed reliability and maintenance model. The first example is a series system with three different components degrading with different rates. The second example is a parallel system with two components. Each component experiences two competing failure processes, soft and hard failure. Table 5.1 shows the parameter assumptions for these examples. It is assumed that the inspection cost is $5, replacement cost is $10, and downtime cost is $80. The cost rate function is calculated, and the optimization problem solved several times based on different combination of initial ages for series and parallel systems.

Table 0.1 Parameter Values for example

Parameter	component 1	component 2	component 3
H	20 mm	30 mm	35 mm
D_i	7	5	6
α_i	3	2	1

β_i	1	0.6	0.3
λ	2.5×10^{-3}		
Y_{ij}	Y_{ij}~$N(\mu_{Yi},\sigma_{Yi}^2)$ $\mu_{Yi}=2,\sigma_{Yi}=0.5$	Y_{ij} ~$N(\mu_{Yi},\sigma_{Yi}^2)$ $\mu_{Yi}=2.5,\sigma_{Yi}=0.2$	Y_{ij} ~$N(\mu_{Yi},\sigma_{Yi}^2)$ $\mu_{Yi}=3,\sigma_{Yi}=0.1$
W_{ij}	W_{ij} ~$N(\mu_{Wi},\sigma_{Wi}^2)$ $\mu_{Wi}=1.5$, $\sigma_{Wi}=0.4$	W_{ij} ~$N(\mu_{Wi},\sigma_{Wi}^2)$ $\mu_{Wi}=2$, $\sigma_{Wi}=0.3$	W_{ij} ~$N(\mu_{Wi},\sigma_{Wi}^2)$ $\mu_{Wi}=1.2$, $\sigma_{Wi}=0.15$

Figure 5.2 shows the cost rate function of three different combinations of components' initial age for the series example.

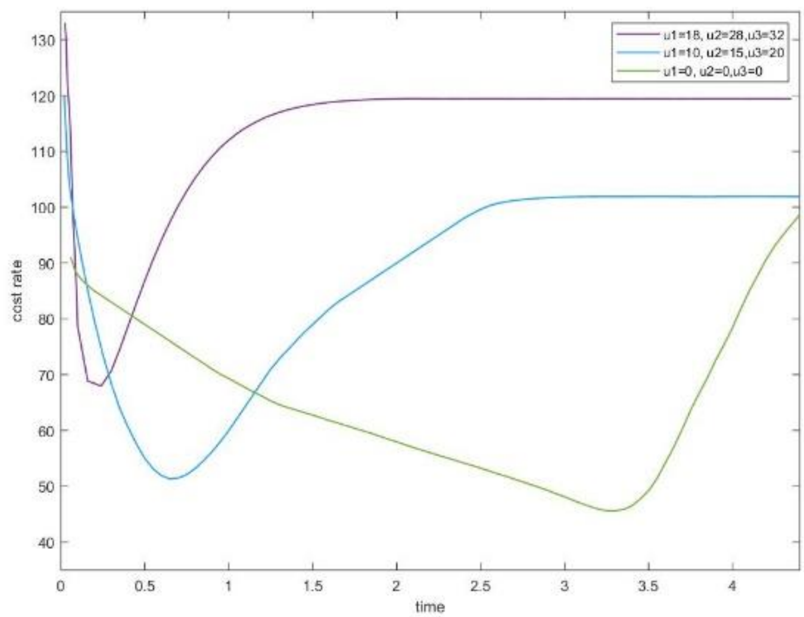


Figure 0.2 Cost rate for different combination of components' initial age

Table 5.2 and 5.3 show the optimal inspection interval for different combinations of initial ages for the series and parallel system.

As it is shown in Table 5.2. When all the three components are close to their failure threshold, the optimal inspection is very low, while when the initial ages are zero for all the components the optimal inspection is found as $\tau^*=3.3$. Moreover, it can be concluded that since component one degrades faster than other components, it has the higher impact on optimal inspection interval.

Comparing scenario 12 and 13, it can be concluded, that since the variance of component 3 is higher than component 2, it is dominant on determining optimal inspection interval. Scenarios 18-21 show than when one component or all the three components are close to their failure threshold the optimal inspection should be very low to replace them before failure.

Table 0.2 Optimal inspection interval for series system

Scenarios number	Component 1 (u_1)	Component 2 (u_2)	Component 3 (u_3)	Optimal inspection interval (τ^*)
1	0	0	0	3.30
2	5	0	0	2.61
3	0	0	5	3.09
4	0	5	0	3.15
5	5	5	0	2.41
6	0	5	5	2.97
7	5	0	5	2.30
8	5	5	5	2.06
9	10	5	5	1.56
10	5	5	10	1.87
11	5	10	5	1.94
12	0	15	25	1.45
13	0	20	20	1.58
14	10	0	20	1.34
15	10	15	0	1.41
16	10	10	10	1.04
17	10	15	20	0.72
18	0	0	32	0.14
19	0	28	0	0.15
20	18	0	0	0.14
21	18	28	32	0.14

Table 5.3 shows the different optimal inspection interval for parallel example. From Table 5.2 and 5.3 it can be concluded that when the initial degradation of all the components are high, the system will be failed very soon, and it should be inspected again in a very short inspection interval; in fact, τ^* is very small.

Table 0.3 Optimal inspection interval for parallel system

Scenario number	Component 1 (u_1)	Component 2 (u_2)	Optimal inspection interval (τ^*)
1	0	0	5.23
2	5	0	5.12
3	0	5	4.05
4	5	5	3.78
5	0	10	3.52
6	10	0	4.97
7	10	10	2.84
8	15	20	1.52
9	18	0	4.63
10	0	28	3.21
11	10	28	1.25
12	18	28	0.15

Figure 5.3 also shows the result of Table 5.3 in a 3D plot. It is obvious that when the initial age of one of the components is zero, the optimal inspection is long. Moreover, when both components are close to their failure threshold, the optimal inspection is very short.

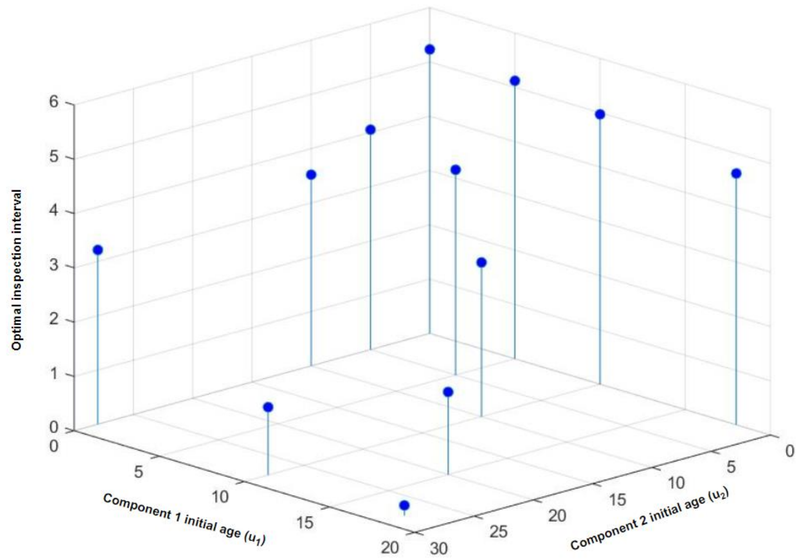


Figure 0.3 Combination of random age and inspection interval for parallel system

1.11 Dynamic thresholds and inspection times for repairable multi-component systems

For multi-component systems with individually repairable components, each component can be replaced individually within the system, and there is no requirement to replace the whole system upon failure of any components. At each inspection time, by replacing each failed or aged component, the age or the degradation level of components within the system are different; in other words, each component has its own age at the beginning of each inspection interval, and the system has a mix of different component ages. To provide an appropriate maintenance policy for such system, the initial age of all the components at the beginning of each inspection time should be considered as one of the critical factors in the proposed maintenance model of this study. For example, if all components are like new, with minimal observed degradation, then the time to the next inspection can be relatively long.

For a multi-component system with repairable components, since each component degrades separately within the system, the initial ages of the components are different and change at each inspection interval. Therefore, it is beneficial to have a dynamic maintenance plan based on the current degradation level of all the components in the system. A system of components with higher age, and thus, probabilistically higher degradation has a higher potential for failure, and it should be inspected soon, while if the ages and degradation levels of all components are low, they will likely fail in a longer time, and the inspection can be delayed. Moreover, the maintenance threshold suggests if the component should be preventively replaced or not, which can be dependent on the age of components at the beginning of the inspection, as well. Furthermore, there is an inherent trade-off and relationship between inspection time and maintenance threshold. If you inspect frequently, the maintenance thresholds can be set much closer to the failure threshold, compared to longer inspection times.

Therefore, the inspection times and maintenance thresholds should be determined at the beginning of each inspection time for all the components considering their degradation levels at the previous inspection. In the proposed maintenance model, the inspection time and maintenance thresholds are found dynamically to minimize cost rate. Yousefi and Coit [176] developed a dynamic condition-based maintenance model for a multi-component system with individually repairable components that experience degradation and shock processes. Maintenance thresholds for each component are found to suggest maintenance actions for each component at each inspection time, and the next optimal inspection time is found for the whole system dynamically based on the age of all the components at the current inspection time.

Since the failed component is replaced instead of the whole system, the age of components at each inspection time are different from the others. In this study, random variable U_i is defined as the initial age of component *i* at the beginning of each inspection interval. If the degradation is observed, and therefore known, it is expressed as u_i.

The reliability assumptions for this research work can be summarized as follow:

(1) There is no continuous monitoring for the system, and the failed or aged components can be detected only at any inspection time.

(2) At any inspection time, if the total degradation of component *i* exceeds its failure thresholds H_i^1, it is detected as failed.

(3) When the shock magnitude exceeds the hard failure threshold of any component *i* (D_i), hard failure occurs of that component, but it is not detected until the next inspection.

(4) If the system fails before the specified inspection interval, it is not immediately detected and not replaced until the next inspection. There is a penalty cost associated with the time that the system is down, e.g., cost associated with loss of production, opportunity costs, etc.

The conditional reliability of component *i* at the beginning of an inspection interval can be calculated as in Equation 5.5, based on the observed degradation at the beginning of the interval.

$$R_i(t;u_i) = \sum_{m=0}^{\infty} \left(F_{W_i}(D_i) \right)^m P\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} + u_i < H_i^1 \right) \frac{(\lambda t)^m e^{-\lambda t}}{m!} \tag{0.5}$$

If it is assumed that the components are configurated as a series system, the conditional system reliability, for a multi-component series system with individually repairable components, at the beginning of an inspection interval, can be calculated as follow.

$$R(t;\mathbf{u}) = \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left(P(W_{ij} < D_i)^m P\left(X_i(t) + \sum_{j=1}^{m} Y_{ij} + u_i < H_i^1\right)\right)\frac{(\lambda t)^m e^{-\lambda t}}{m!} \tag{0.6}$$

$$= \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left(P(W_{ij} < D_i)^m \int_0^{H_i^1 - u_i} P\big(X_i(t) < H_i^1 - y - u_i\big) f_{Y_i}^{<m>}(y)dy\right)\frac{(\lambda t)^m e^{-\lambda t}}{m!}$$

$$= \sum_{m=0}^{\infty}\prod_{i=1}^{n}\left(F_{W_i}(D_i)^m \int_0^{H_i^1 - u_i} G(H^1{}_i - y - u_i; \alpha_i t, \beta) f_{Y_i}^{<m>}(y)dy\right)\frac{(\lambda t)^m e^{-\lambda t}}{m!}$$

Where $G(H_i^1 - y - u; a_i t, b_i)$ is the cumulative distribution function of a gamma distribution with parameter $a_i t$ and β_i and u_i is the observed degradation level of component i at the beginning of inspection interval. $f_{Y_i}^{<m>}(y)$ is the probability density function of m shock damages, which in this study, follows normal distributions whose parameter are a mean of $m\,\mu_Y$ and variance of $m\,\sigma_Y^2$, μ_Y and σ_Y^2 are the parameters of normal distribution while m is the number of shocks. Shocks are arriving at random time interval as homogeneous Poisson distribution with rate of λ.

For systems with high failure costs, it is advantageous to repair or replace the components before the failure occurs. To prevent system failure, it is often preferable to replace the components which are aged enough rather than allowing them to fail and paying the high penalty cost. In some CBM models, for each component i, there is a preventive maintenance threshold (H_i^2) which is lower than the soft failure threshold (H_i^1) and determines if component i is aged enough to be replaced. The implementation of a lower degradation threshold can be useful to avoid failure by providing criteria to detect the degradation status of the components. It is assumed the failure threshold is a known fixed value, but alternatively, the preventive maintenance threshold is a decision variable and is defined as part of maintenance planning. In this research, the preventive maintenance threshold can be adjusted dynamically.

At each inspection time, I determine the conditions of each component by inspection and compare it to the maintenance thresholds, and subsequently a maintenance action can be implemented based on the comparison of thresholds and degradation level. If the degradation of any component i is lower than its preventive threshold H_i^2, component i is in the safety level and there is no maintenance action needed, so it can continue functioning. If the degradation of component i is between preventive threshold H_i^2 and failure threshold H_i^1, component i is detected as an aged component and should be preventively replaced with a new one. At any inspection time, if the degradation level of component i is greater than its failure threshold H_i^1, or any arrived shock has a magnitude greater than hard failure threshold D_i, the component is failed and should be replaced with a new one and a penalty cost due to system shutdown is incurred. In the previous studies, a fixed preventive maintenance threshold is determined for the each component and at each inspection time, the degradation level of that component is compared to the fixed threshold [141]. However, while having a fixed threshold for the entire maintenance contract is logical and practical, there are economic advantages to revising the threshold dynamically based on the observed component degradation at the previous inspection.

To minimize costs, the thresholds can be dynamically found based on the age or degradation of components. Moreover, with the same reason, the inspection time should not be fixed for the whole time, and it should also be determined dynamically based on the condition of all the components. Finding the next optimal inspection time and the preventive maintenance threshold for all the components simultaneously is a unique problem which is addressed in this study. Consider the example shown in Figure 5.4 which demonstrates the proposed dynamic maintenance model. The next inspection time and maintenance thresholds are determined dynamically based on the age of all the components in the system. The black circles show the initial age of components at the beginning of the inspection interval. As it is shown on Figure 5.4, due to the degradation and repairing components individually, the components' initial ages are different at each inspection time.

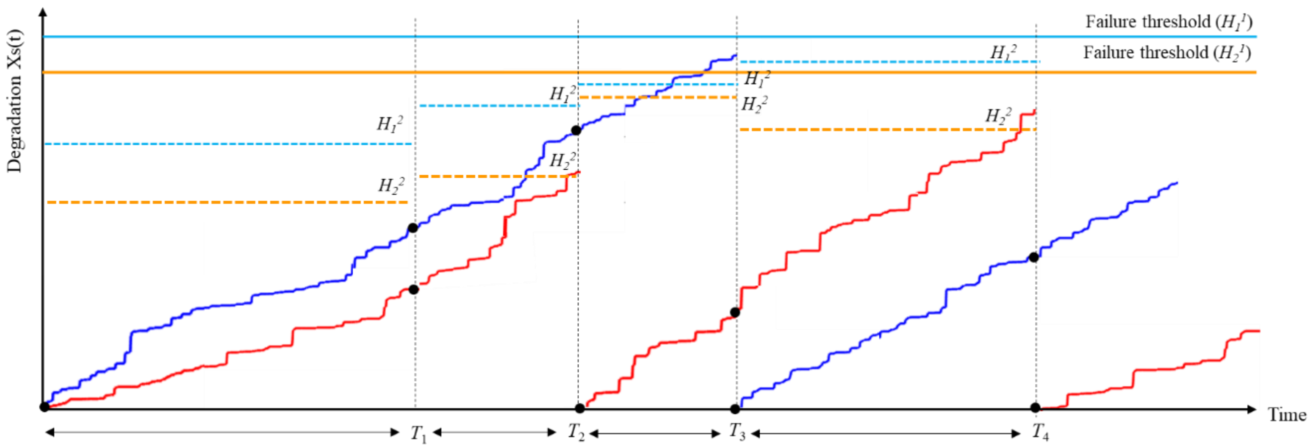


Figure 0.4 Dynamic thresholds and inspection time

Based on the CBM maintenance rules and Equation (5.6), if τ is the inspection interval, the probability that there is no replacement (P_{NRi}), either preventive or corrective, by time τ for component *i* given *m* shocks can be calculated by using Equation (5.7)

$$P_{NR_i}(\tau; u_i, m) = F_{W_i}(D_i)^m \int_0^{H_i^2 - u_i} G\left(H_i^2 - u_i - y; \alpha_i \tau, \beta_i\right) f_Y^{<m>}(y) dy \tag{0.7}$$

At each time inspection τ, if no hard failure occurs and total degradation for component *i* is between its preventive maintenance threshold H_i^2 and failure threshold H_i^1, this component is more likely to fail soon. Although it has not failed yet and can still function properly, since it may probabilistically fail within a short period, replacement of this component should be performed. This region can be called aged region, and the probability that the component is in this region can be calculated using Equation (5.8).

$$P_{aged_i}(\tau;u_i,m) = P(W_i < D_i)^m P\left(H_i^2 < X_i(\tau) + u_i + \sum_{j=0}^{m} Y_{ij} < H_i^1 \mid N(\tau) = m \right)$$
$$= F_{W_i}(D_i)^m \int_0^{H_i^1\text{-}u} \left(G\left(H_i^1 - u_i - y; \alpha_i\tau, \beta_i\right) - G\left(H_i^2 - u_i - y; \alpha_i\tau, \beta_i\right) \right) f_{Y_i}{}^{<m>}(y)dy \quad (0.8)$$

If at inspection time τ , there has been a hard failure or the total degradation of any component *i* is greater than its failure threshold H_i^1 (soft failure), the system detected as failed one and the failed components are replaced. In this situation there is a penalty cost for the downtime. The probability of this situation can be derived from Equation (5.9).

$$P_{failure_i}(\tau;u_i,m) = 1 - \left(F_{W_i}(D_i)^m \int_0^{H_i^1\text{-}u} G\left(H_i^1 - u_i - y; \alpha_i\tau, \beta_i\right) f_Y{}^{<m>}(y)dy \right) \quad (0.9)$$

To evaluate the performance of the maintenance policy, the cost rate for each inspection interval is derived. Consider the inspection cost is C_I , the cost of replacement for each component is C_R , and C_ρ is the penalty cost per unit down time. C_S is the setup cost for maintenance implementation. The system cost rate is given by:

$$CR = \frac{C_I + \sum_{k=1}^{n}(C_s + kC_R)P_k + C_\rho E[\rho]}{\tau} \quad (0.10)$$
$$P_k = P\left(k \text{ components above } H_i^2\right)$$

$E[\rho]$ is the expected downtime. The system downtime is the time duration between the random time of failure *T*, if there is a failure within the interval, and the next inspection time τ . Consider $0 < T < \tau$, if a failure occurs, the downtime is $\tau - T$ in each inspection interval. Therefore, the expected downtime can be calculated as follow:

$$E[\rho] = \int_0^{\tau} (\tau - t)\, f_T(t;\mathbf{H}^1 - \mathbf{u})dt \quad (0.11)$$
$$= \int_0^{\tau} (\tau - t)dF_T(t;\mathbf{H}^1 - \mathbf{u})$$

$f_T(t;\mathbf{H}^1 - \mathbf{u})$ is the probability density function and $F_T(t;\mathbf{H}^1 - \mathbf{u})$ is the cumulative distribution function of residual failure time, given observed degradation

$\mathbf{u}=(u_1,u_2,...,u_n)$ at the beginning of the interval. **u** is the vector of component degradation of all the components at the beginning of the interval, and $\mathbf{H}^1 = (H_1^1, H_2^1, \ldots)$. Therefore, $\mathbf{H}^1$ - **u** represents a vector of component degradations until a failure occurs. $f_T(t;\mathbf{H}^1-\mathbf{u})$ can be calculated using Equation (5.12).

$$\begin{aligned}
f_T(t;\mathbf{H}^1-\mathbf{u}) &= \frac{d}{dt}F_T(t;\mathbf{H}^1-\mathbf{u}) \\
&= \frac{d}{dt}(1-R(t;\mathbf{u})) \\
&= \frac{d}{dt}\left(1-\sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[F_{W_i}(D_i)^m\int_0^{H_i^1-u_i}G(H_i^1-u_i-y;\alpha_i t,\beta_i)\,f_{Y_i}{}^{\langle m\rangle}(y)dy\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!}\right) \\
&= -\frac{d}{dt}\sum_{m=0}^{\infty}\prod_{i=1}^{n}\left[F_{W_i}(D_i)^m\int_0^{H_i^1-u_i}G(H_i^1-u_i-y;\alpha_i t,\beta_i)\,f_{Y_i}{}^{\langle m\rangle}(y)dy\right]\frac{\exp(-\lambda t)(\lambda t)^m}{m!}
\end{aligned} \tag{0.12}$$

To calculate the probability of having k components with degradation level above their preventive maintenance threshold, define $S(k)$ as a set of all n-dimension vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, whose values sum to k with $x_i \in \{0, 1\}$, where x_i is 1 if component i has degradation level greater than its own preventive maintenance threshold H_i^2. For example, in n = 3, then $S(1)$ = {(1,0,0), (0,1,0), (0,0,1)}, $S(2)$ = {(1,1,0), (0,1,1), (1,0,1)} and $S(3)$ = {(1,1,1)}. P_k can be computed by conditioning on $N(t)$ and considering a multinomial distribution, as:

$$\begin{aligned}
P_k &= \sum_{m=0}^{\infty}\sum_{\mathbf{x}\in S(k)}\prod_{i=1}^{n}(1-P_{NR_i}(\tau;u_i,m))^{x_i}\,P_{NR_i}(\tau;u_i,m)^{1-x_i}\,\frac{\exp(-\lambda t)(\lambda t)^m}{m!} \\
S(k) &= \left\{\mathbf{x};\sum_{i=1}^{n}x_i = k\right\}, x_i \in \{0,1\}
\end{aligned} \tag{0.13}$$

Therefore, the maintenance cost, given vector **u**, can be calculated as follow:

$$CR(\tau,\mathbf{H}^2;\mathbf{u}) = \frac{C_I + \sum_{k=1}^{n}(C_s + kC_R)P_k + C_\rho\int_0^\tau(\tau-t)\,f_T(t;\mathbf{H}^1-\mathbf{u})dt}{\tau} \tag{0.14}$$

The inspection time τ and the preventive maintenance threshold H_i^2 for all the components are the decision variables and their optimal values should be found dynamically following each inspection by solving an optimization problem considering the initial age of all the components $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ observed during the previous inspection. By solving the following optimization problem, the optimal inspection time and thresholds are found sequentially for each successive inspection.

$$\begin{aligned} &\min && CR(\tau, H_1^2, H_2^2, \ldots, H_n^2; \mathbf{u}) \\ &\text{Subject to:} && 0 < H_i^2 \le H_i^1 \qquad \text{for } i = 1, 2, \ldots, n \\ & && \tau > 0 \end{aligned} \tag{0.15}$$

Using Equation (5.14) as the objective function of maintenance problem in Equation (5.15), results in a non-linear optimization problem with continuous decision variables that should be solved dynamically following each inspection to determine the optimal plan for the next interval. There are different algorithms that can be applied for solving non-linear optimization problems; however, solving a difficult nonlinear optimization problem to find the decision variable is not practical for the maintenance team in the field to derive the optimal maintenance policy. Therefore, it is desirable to have a time efficient method to find the next inspection time and maintenance thresholds for all the components to minimize maintenance cost rate.

For each multi-component system experiencing internal degradation and external shocks, the maintenance optimization problem can be solved multiple times considering different initial ages of the components to find the optimal inspection interval and preventive maintenance threshold. The initial age of components is the main factor for finding the next inspection time and maintenance thresholds.

Different scenarios can be simulated for the initial degradation of components in the system and the optimization problem can be solved for all the scenarios to find the optimal maintenance decisions. However, there are an infinite number of scenarios for the initial age of components. Therefore, by training a neural network model on a given data set, the maintenance decisions for any combination of components' initial age at the beginning of the next inspection time can be predicted. By solving the maintenance problem for different scenarios of initial age of components and finding the inspection interval and maintenance thresholds, a maintenance dataset is generated, and by training a neural network, a model is trained to predict the next inspection time and maintenance thresholds of all the components based on the initial age of components as the inputs. In this way, the dynamic maintenance plan can be updated for each successive interval without actually solving the nonlinear optimization problem in a repetitive fashion.

1.11.1 Neural Network for Maintenance Problem

A neural network is a network of connected neurons in different layers. At each neuron, the inputs of the model are combined with weights, and then the sum of all the input-weight products is passed through an activation function to transition to the next layer. At the end of this process, the last hidden layer is linked to an output layer to provide the output or prediction. One of the main strengths of machine learning algorithms is their ability to learn and improve every time in predicting an output.

$$output = f(z_1\theta_1 + z_2\theta_2 + ... + z_p\theta_p + b) \tag{0.16}$$

Equation (5.17) shows the calculation of output of each layer when there are p inputs into the model. b is the bias and $f(\cdot)$ is the activation function, which is used to convert the inputs into a predictable form of output. To evaluate the output of a neural network a cost function is calculated to measure how accurate the prediction is. One of the commonly used cost functions is a mean squared error which is calculated as follow:

$$MSE = \frac{1}{N}\sum_{e=1}^{N}(o_e - \hat{o}_e)^2 \qquad (0.17)$$

Where N is the number of samples, and o_e is the actual value of which is the inspection interval and $\hat{o}_e$ is the predicted value or the predicted next inspection interval in our problem. By calculating the cost function, the weights and biases of the network can be optimized. Stochastic gradient descent (SGD) is an effective optimization algorithm to minimize the cost function by changing the weights and biases.

Multiple-output neural networks are models which are able to predict multiple outputs simultaneously. The outputs can be dependent on multiple inputs with different weight combinations, and the convergence strategy is to calculate the weights and biases to minimize the average deviation between predict and observed outputs [177]. Multiple-output neural network utilizes multiple inputs to predict multiple outputs whereas a neural network considers a single output. Figure 5.5 shows the architecture of a multiple-output neural network. More information and mathematical details of training multiple-output neural network models are given in [178-181].

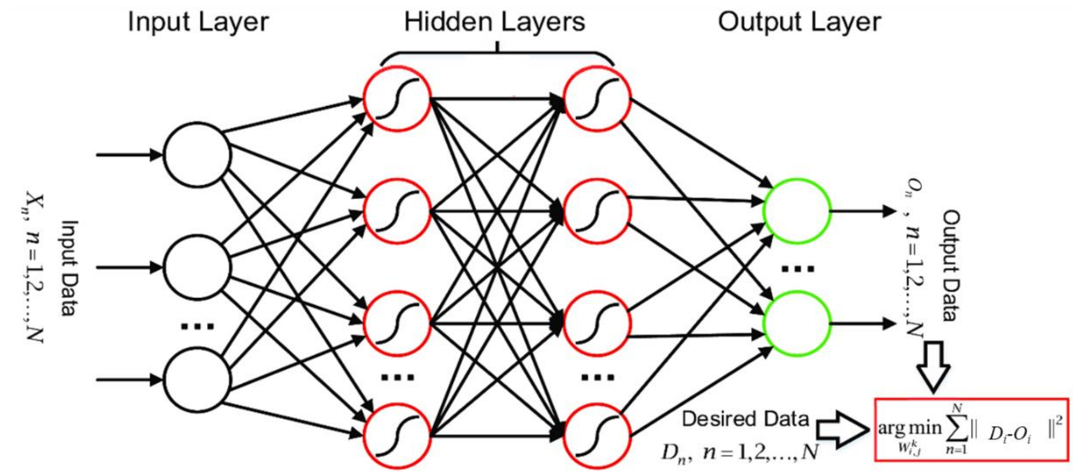


Figure 0.5 Multiple-output neural network neural network [182]

A multiple-output neural network model is considered in this study, which consists of three layers: input, output, and one hidden layer. The input layer consists of input nodes, representing the observed degradation measures. The outputs of the input nodes are normalized and transferred to the hidden layer in which they are processed through a transfer function. The minimum Mean Squared Error (MSE) method is used as the optimization criterion to train the weights of the network, and the proposed neural network is trained until MSE between the predicted and the actual values becomes minimal and remains unchanged.

In this study, there are n input variables for the initial degradation levels of n components in the system and $n + 1$ outputs for the inspection interval and the preventive maintenance thresholds for each component. Therefore, for any series system experiencing degradation and external shocks processes, the system reliability can be calculated using Equation (5.6) and the maintenance cost rate can be derived using Equation (5.14) which considers the maintenance cost for inspection, replacement, setups and penalty cost of system failure. By using the optimization problem that is formulated in Equation (5.15), the optimal inspection interval and preventive maintenance thresholds can be calculated using any nonlinear optimization algorithm. The interior point method is used in this research work. Interior point methods have proved to be very successful in solving many nonlinear problems [145-147].

Different scenarios are simulated for the initial degradation of components in the system and the optimization problem is solved for all the scenarios to find the optimal inspection interval and preventive maintenance thresholds for all the components. However, there are an infinite number of scenarios for the initial age of components. Therefore, by training a machine learning model as a black box on the given dataset, the maintenance decision can be predicted for any combination of components' initial age at the beginning of the inspection time. By using the simulated scenarios as input variables and their corresponding optimal values as the output variables, a multiple-output neural network is trained. The trained network can be provided to the maintenance team to select the optimal maintenance decisions dynamically by giving the initial degradation level of components as inputs, instead of solving an optimization problem. To evaluate the performance of the network, the dataset is split into 70% for training and 30% for testing, and the overall performances of both training and testing sets were evaluated by MSE, and the R^2coefficient is also used to determine how well the trained model can predict the target values.

1.11.2 Numerical results for dynamic maintenance thresholds and inspection times

The numerical example which is considered in this study is a sliding spool system, which is used in different applications such as electrohydraulic servo-valves. The sliding spool system has two components of a spool and sleeve with the reliability parameters shown in Table 5.4. It is also assumed that inspection cost is $C_I = 5$, the downtime cost is $C_\rho = 700$, the fixed setup cost for replacement is $C_s = 15$, and replacement cost for each component is replacement cost $C_R = 30$. To find the optimal inspection interval and preventive maintenance thresholds for spool and sleeve, the optimization problem formulated as Equation (5.14) and (5.15) was solved dynamically given the age of these two components. Since the components within the system are not identical, Equation (5.14) is used for calculating the probability of *k* replacements. By simulating the initial age of components, the optimization problem can be solved for different scenarios.

Table 0.4 Parameter values for a sliding spool system reliability analysis

Parameters	values		sources
	Spool	Sleeve	
H_i^1	5 mm	6 mm	Fan et al [157]
D_i	7.5 mm	7 mm	Fan et al [157]
λ	2.5×10^{-5}	2.5×10^{-5}	Fan et al [157]
W_{ij}	$W_{ij} \sim Normal(1, 0.2)$	$W_{ij} \sim Normal(1.5, 0.3)$	Fan et al [157]
Y_{ij}	$Y_{ij} \sim Normal(0.5, 0.1^2)$	$Y_{ij} \sim Normal(0.55, 0.15^2)$	Haiyang, et al [156]
α_i	0.5	0.2	assumption
β_i	1.2	1.6	assumption

Table 5.5 shows some examples of finding the next optimal inspection interval and preventive maintenance thresholds for components by solving the optimization problem based on the initial age of components.

As it is shown in Table 5.5, when both components are new such as Scenario numbers 1 and 2, the optimal next inspection interval is longer than scenario number 20, which both components are close to their failure thresholds. Moreover, since the system configuration is series, failure of one component in the system causes the whole system failure, so when one component is close to its failure threshold, the next inspection interval is very short, and it means that the system should be inspected frequently to avoid the penalty cost of failure. In Scenarios 9, 10 and 11, one component is almost new, and another component is close to its failure threshold, so the system should be inspected in a short time.

The optimal preventive maintenance thresholds assist the maintenance team to detect if the component is sufficiently aged to be economically replaced, and subsequently, the replacement can be done to avoid a costly failure. Therefore, for such scenarios with one new component and one aged component at the beginning of inspection time, the next inspection is relatively short. Based on the optimal preventive maintenance threshold, the new component will not be detected as aged one and there is no replacement required for new components. In fact, using the optimal inspection time and preventive maintenance threshold, there will be very little waste in the useful life of components that are new.

Solving a non-linear optimization problem dynamically is a not time-efficient or practical method for maintenance team to find the next maintenance decision variables. To provide a fast method to find the next inspection time and preventive maintenance threshold without actually solving the problem, a neural network can be trained and provided which uses the initial degradation levels of components as the input variables and predicts the next maintenance decision variable which are the next inspection time and preventive maintenance thresholds for all the components.

Table 0.5 Optimal maintenance decisions for different scenarios

Scenario number	Degradation of spool (u_1)	Degradation of sleeve (u_2)	Next inspection interval (τ)	Next threshold for	Next threshold for

				spool (H_1^2)	sleeve (H_2^2)
1	0	0	4.32	3.22	3.60
2	0.12	0.09	4.13	3.26	3.65
3	0.81	1.15	4.11	3.29	3.74
4	1.21	2.03	4.07	3.31	3.76
5	1.75	2.28	4.09	3.34	3.79
6	1.98	0.14	4.06	3.35	3.69
7	1.62	1.47	4.17	3.33	3.72
8	0.15	3.30	3.87	4.02	3.97
9	0.06	4.02	2.18	4.54	5.11
10	0.17	5.13	1.01	4.15	5.31
11	4.46	0.18	0.97	4.78	3.77
12	3.21	0.02	3.44	4.01	5.57
13	0.18	4.11	2.27	4.03	5.12
14	2.18	2.64	3.96	3.41	3.86
15	2.22	3.41	2.47	3.48	5.01
16	2.87	1.14	3.98	3.37	4.99
17	3.60	4.26	1.46	4.08	5.09
18	4.01	1.12	1.54	4.66	4.87
19	2.11	5.07	1.03	3.39	5.44
20	4.51	5.53	0.69	4.87	5.97

In this study, by simulating different scenarios for the initial age of two components of the spool and sleeve and solving the optimization problem which is formulated in Equation (5.14) and (5.15), the next maintenance decision variables are defined for 1000 different cases. By using 70% of this dataset, a multiple-output neural network is trained and tested its accuracy on 30% to evaluate the performance of the model. The coefficient of determination (R^2) indicates the percentage of the response variable variation that is explained by the trained model. R^2 is also known as “the goodness of fit” which can have a value between 0 and 1. A value of R^2 =1 indicates a perfect fit, and is thus a highly reliable model for future prediction, while a value of 0 represents a poor model for prediction. As it is shown in Figure 5.6, 5.7 and 5.8, the determination coefficient (R^2) for inspection and thresholds for spool and sleeve for both training set and testing set is high numbers (close to 1) which indicate that how well the trained model can predict the target values.

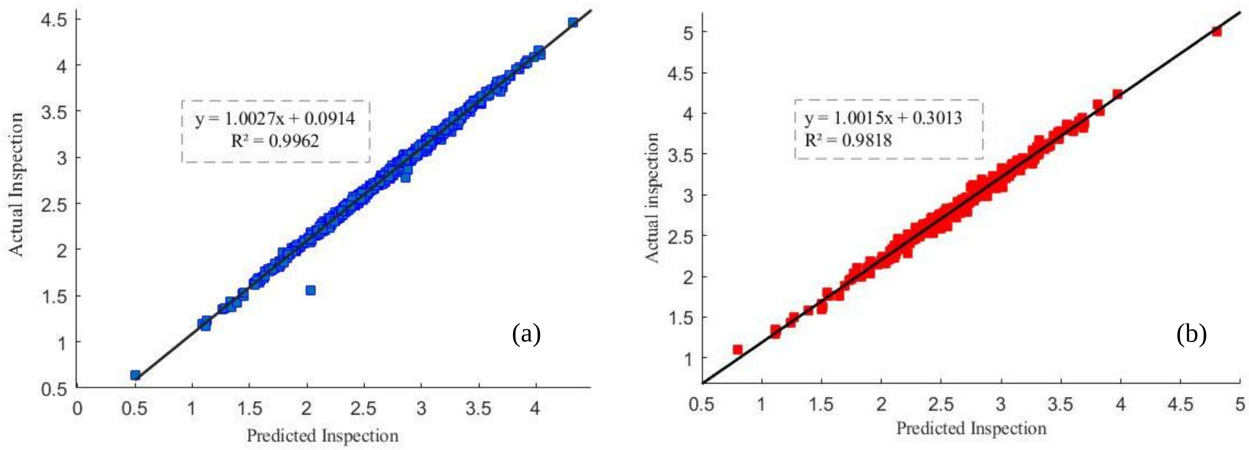


Figure 0.6 Predicted inspection vs actual inspection for (a): training (b) testing set

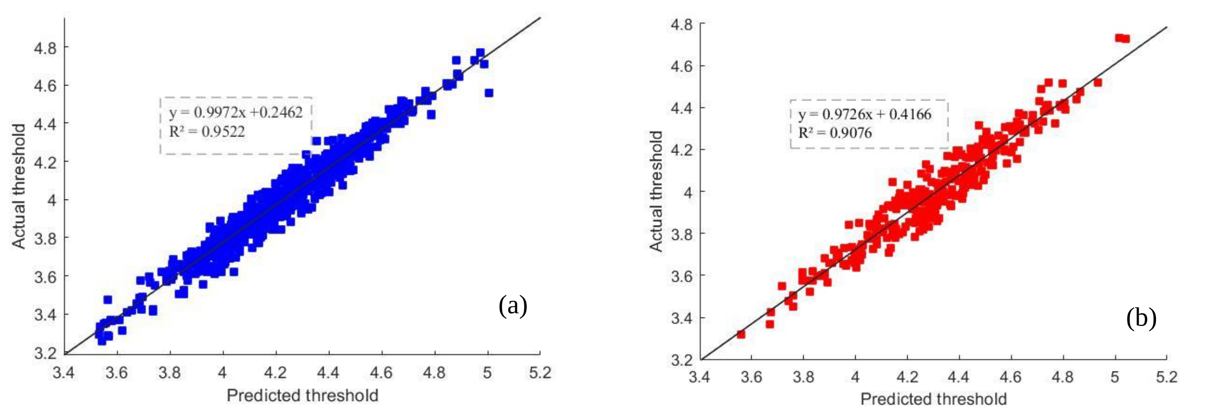


Figure 0.7 Predicted threshold for spool vs actual threshold for spool for (a): training (b) testing set

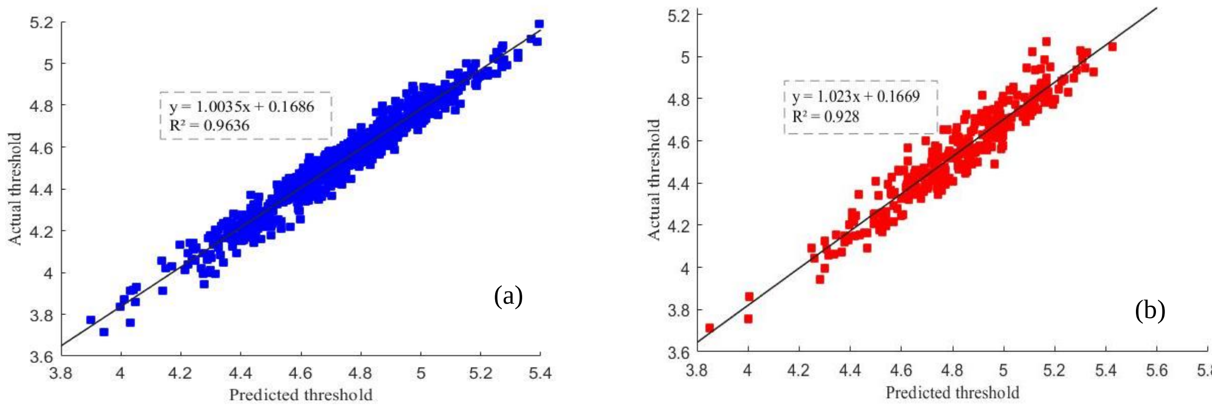


Figure 0.8 Predicted threshold for sleeve vs actual threshold for sleeve for (a): training (b) testing set

To find the best number of neurons in the hidden layers, a sensitivity analysis is done between number of neurons in the hidden layer and two performance measure of MSE_o and aRMSE for training and testing sets. Equation (5.18) and (5.19) are used to calculate the overall MSE_o and aRMSE. Where N is the number of samples, d is the number of output variables, o is the actual value and $\hat{o}$ is the predicted value.

$$MSE_o = \sum_{e=1}^{d} \frac{1}{N} \sum_{l=1}^{N} (o_e^{(l)} - \hat{o}_e^{(l)})^2 \tag{0.18}$$

$$aRMSE = \frac{1}{d} \sum_{e=1}^{d} RMSE = \frac{1}{d} \sum_{e=1}^{d} \sqrt{\frac{\sum_{l=1}^{N} (o_e^{(l)} - \hat{o}_e^{(l)})^2}{N}} \tag{0.19}$$

Figure 5.9 and 5.10 show the effect of the number of neurons on neural network performance for training and testing sets. By increasing the number of neurons in the hidden layer, the neural network becomes more complex and the errors (both MSE_o and aRMSE) for training set decrease. However, for the testing process, the errors decrease with increasing number of neurons until optimal numbers are obtained, and after that they increase again due to overfitting on the training set. As shown in Figure 5.9, the best number of neurons for the hidden layer is 6 neurons which has the MSE_o of 0.225 for testing and 0.201 for training set along with aRMSE = 0.301 for the testing set and 0.2645 for training set.

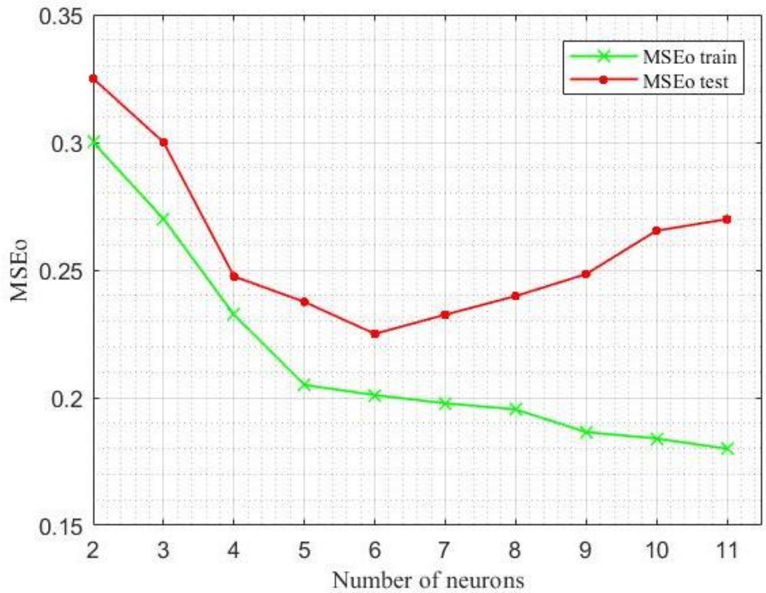


Figure 0.9 The effect of number of neurons on neural network performance for MSE

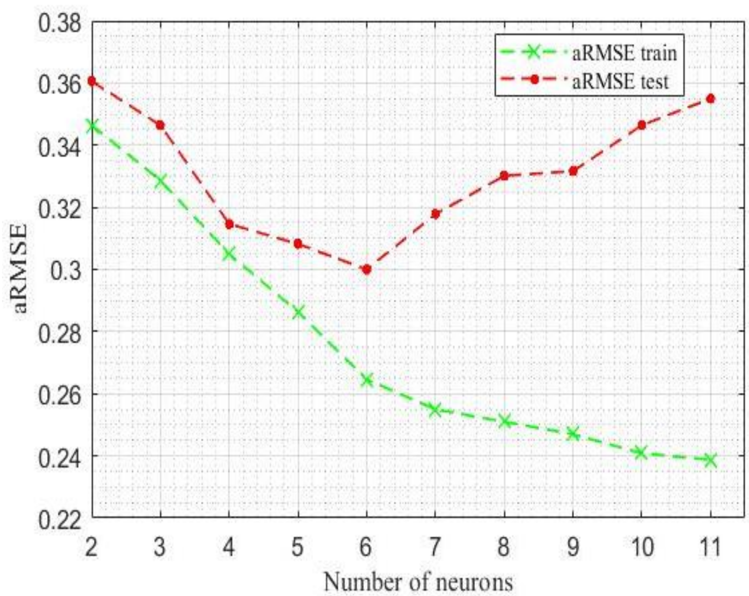


Figure 0.10 The effect of number of neurons ion neural network performance for aRMSE

The proposed maintenance model provides a neural network model as a decision-making tool for the maintenance team to find the next inspection time and the maintenance thresholds for replacing the components. The degradation paths of the components in the sliding spool system is simulated using the parameters in Table 5.4, and the trained neural network is used to show how the proposed model can suggest the maintenance decisions dynamically. The degradation simulation starts with a new system where the degradation level of both components is zero. The first step using the proposed model is feeding the initial degradation of spool and sleeve (0,0) to the trained neural network as the input and finding the next inspection time and maintenance thresholds. At the next inspection time, the degradation level of both components is computed using the parameters in Table 5.4 and are fed into the neural network to find the next maintenance decisions, and the process continues in the same way.

Figures 5.11 and 5.12 show two different simulations of the system with two components degrading based on the parameter of Table 5.4. In the first step, both components are new, and their initial degradation level is zero, and based on this information and using the neural network, the next inspection time and the maintenance thresholds are computed. In the next step, both components are degrading, and the new initial degradation levels are fed into the neural network, and the next maintenance decisions are computed. In Figure 5.11, in the third step, component 1 has a degradation level greater than the maintenance threshold, which is found using the neural network in the previous step. Therefore, it should be replaced by a new one, and its initial degradation level is zero. Since the degradation process is stochastic, as each simulation, the components degrade differently. Figures 5.11 and 5.12 show how the degradation path of components are different in different simulation runs. The stochastic nature of the gamma process is shown in different degradation levels of the same system through various simulation runs.

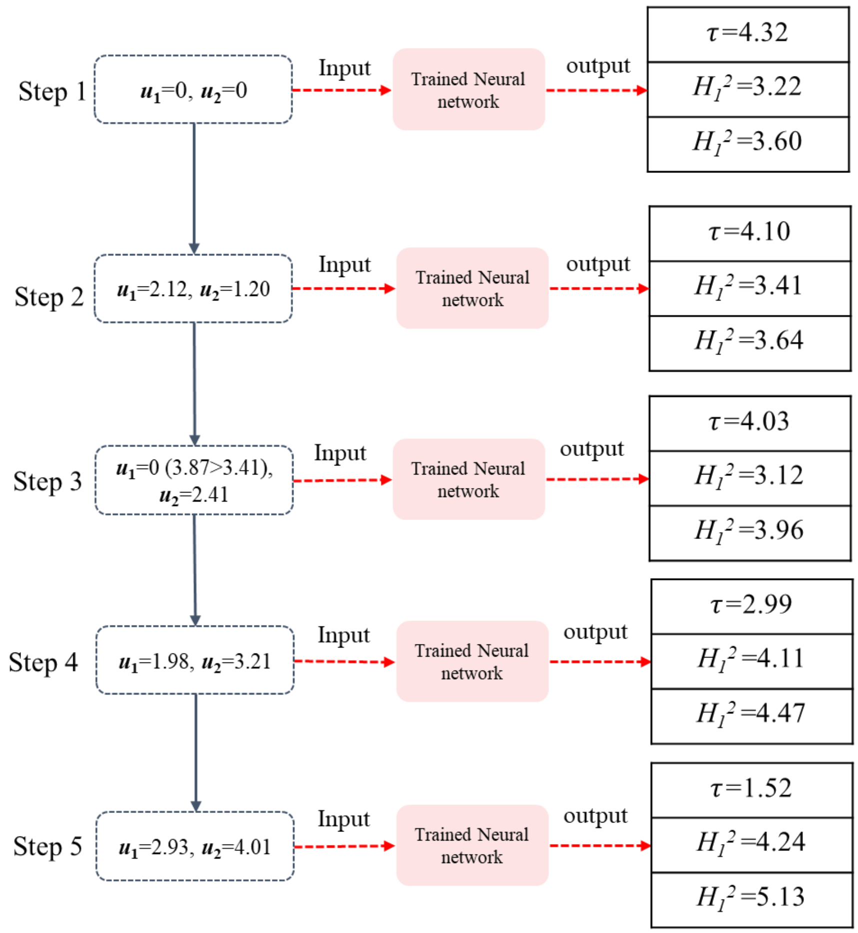


Figure 0.11 The proposed dynamic maintenance model for degradation simulation (a)

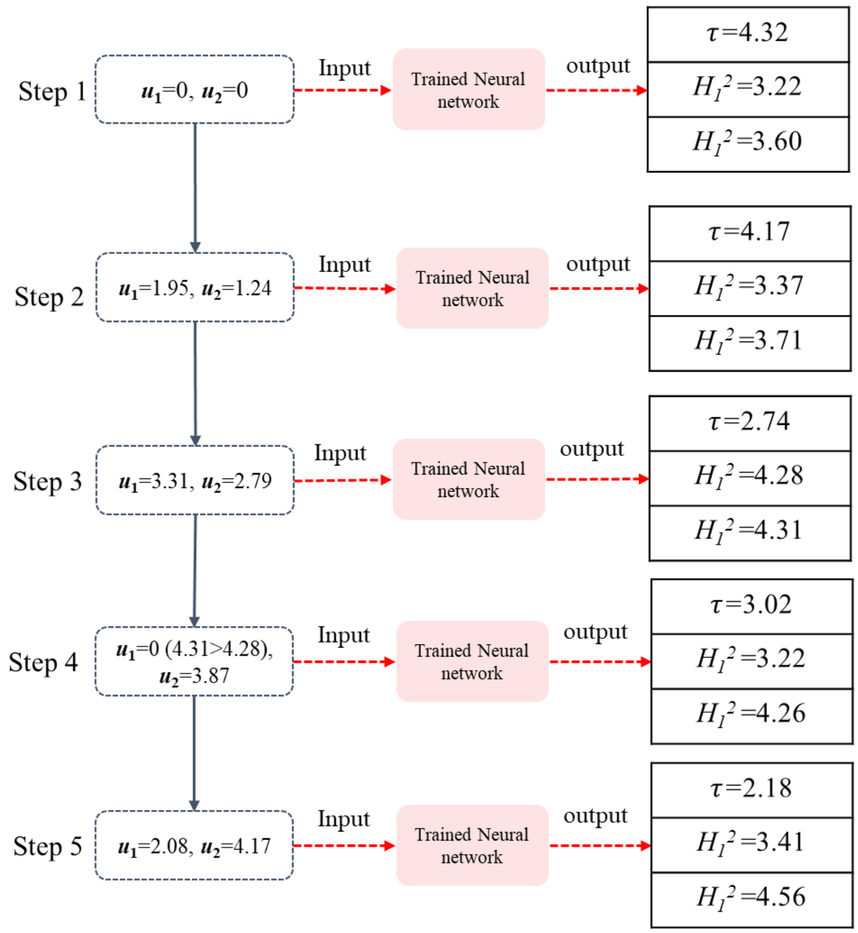


Figure 0.12 The proposed dynamic maintenance model for degradation simulation (b)

Figure 5.13 also shows the degradation path and implementation of the proposed dynamic maintenance model for the numerical example of the sliding spool system for simulation (a). The dash lines in Figure 5.13 show the maintenance thresholds for two components, and the circles show the initial degradation level of the components.

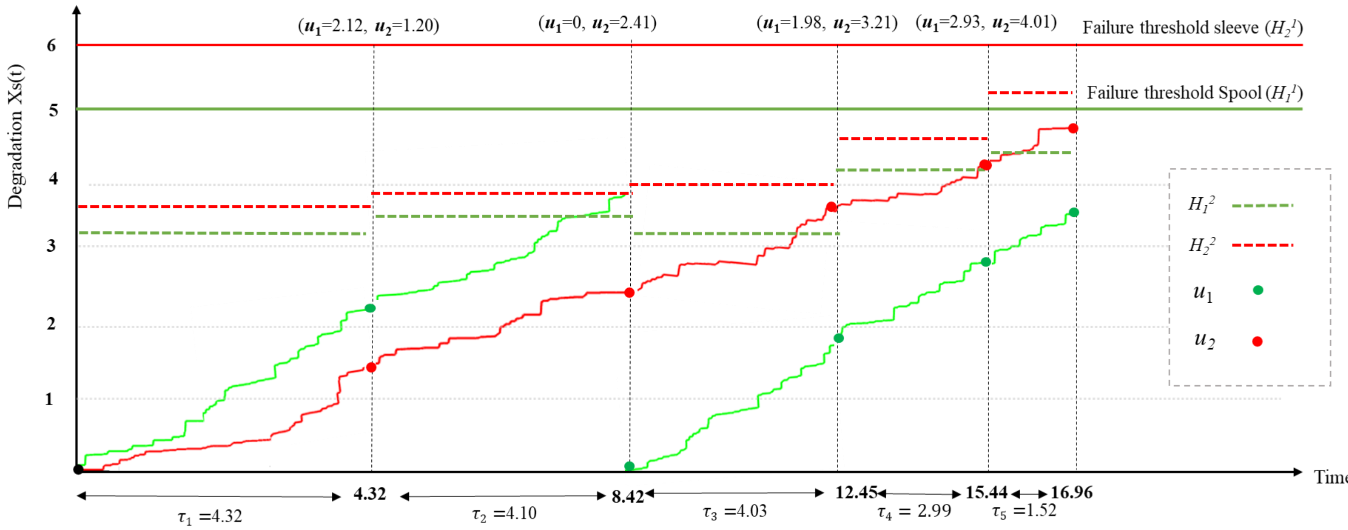


Figure 0.13 The proposed dynamic maintenance on the numerical example

Dynamic maintenance models using reinforcement learning

The goal of maintenance management is to reduce the overall maintenance cost and increase the availability and/or reliability of systems. Over the last few decades the maintenance of systems has become more complex and more critical, as maintenance costs have become a large portion of lifecycle cost. For some multi-component systems, each component can be repaired or replaced individually within the system, and there is no assumption that the system is packaged and sealed together for maintenance purposes. For such systems, it is often more cost effective to monitor the degradation of each component individually within the system and to implement the maintenance actions selectively based on its degradation level. Therefore, at any inspection interval the components which are more prone to failure should be repaired or replaced and the other components continue functioning. The problem is complicated when the degradation paths are dependent, even when they have different respective ages. Yousefi et al [100] investigated a new dynamic maintenance model for a system with individually repairable components to find the best maintenance action based on the system degradation stage.

In this Section, a dynamic maintenance policy is proposed for multi-component systems with individually repairable components. Each component of the system experiences two competing failure processes of degradation and shock arrival. Soft failure occurs when the cumulative degradation for a component is greater than a pre-defined failure threshold, and hard failure occurs when the magnitude of any shock is greater than a pre-defined hard failure threshold.

There has been a significant amount of research on maintenance optimization of complex systems which are mostly model-based methods. However, one of the main drawbacks of the model-based methods are the limitations associated with parameters and modeling assumptions. In this section, I propose a dynamic maintenance policy for a multi-component system with individually repairable components at each inspection time using the model-free method of reinforcement learning. Reinforcement learning is one type of machine learning that trains an agent to decide how to perform an action based on the system state and associated rewards. By applying the trial-and-error to maximize the reward, the agent learns how to make decisions in an uncertain, complex environment. Therefore, at any inspection interval, the trained agent can choose the best maintenance action from a set of actions based on the current age and state of the components which minimize the cost.

1.12 Dynamic maintenance for multi-component systems with finite degradation states

Degradation models can provide the relationship between product degradation and a corresponding failure time distribution. Therefore, using a proper degradation model can provide for a more precise estimation of the failure time distribution, system reliability, and the appropriate maintenance policy. Finding the degradation behavior of components and systems is a critical main step in determining a suitable maintenance plan. In this section, a multi-component system is considered where each component degrades monotonically over time. The gamma process is selected as the stochastic process to model the degradation of each component in the system.

The probability density function of degradation process for each component *i*, $X_i(t) - X_i(s)$ can be calculated using Equation (5.1). It is also assumed that random shock arrivals occur as a homogeneous Poisson process with rate λ. Therefore, the probability of *m* shocks arriving to the system by time *t* can be calculated using Equation (5.2). Each incoming shock has a damage on all components within the system; and the damage is as an additional abrupt jump Y_{ij} on the cumulative degradation path of each component *i*. The cumulative degradation of each component *i* is the summation of pure degradation by time *t*, and the cumulative damages caused by shocks by time *t* ($S_i(t) = \sum_{j=1}^{\infty} Y_{ij}$) where Y_{ij} is an i.i.d random variable for the j^{th} shock damage on component *i*. The total degradation can be accumulated as $X_{S_i}(t) = X_i(t) + S_i(t)$. It is also assumed that each component may fail due to a hard failure, which occurs when any shock magnitude is greater than a predefined hard failure threshold. W_{ij} is a *i.i.d* random variable for the j^{th} shock magnitude for component *i*. The probability that the j^{th} component dow not cause a hard failure can be calculated using Equation (5.3).

Using MDP to formulate the degradation state of systems in a maintenance problem and using a reinforcement learning algorithm,can be used to dynamically optimize maintenance planning of systems degrading over time and experiencing random shock arrivals. Reinforcement learning is a set of algorithms which are based on rewarding the desirable behaviors. The agent and environment are the two main components of reinforcement learning methods. The procedure of reinforcement learning starts by the environment detecting the system state and sending it to the agent, and based on the knowledge which is taken by the agent, it takes an action. Subsequently, based on the action chosen by the agent, the environment calculates the reward and the next state, and the agent updates the knowledge with the information returned by the environment and evaluates the last action. This procedure continues iteratively in a loop until the environment reaches a terminal state. The whole process is modeled as a MDP which can formulate a problem of learning from interactions to achieve a goal, and can describe a stochastic dynamic system. Figure 6.1 illustrates the process of reinforcement learning.

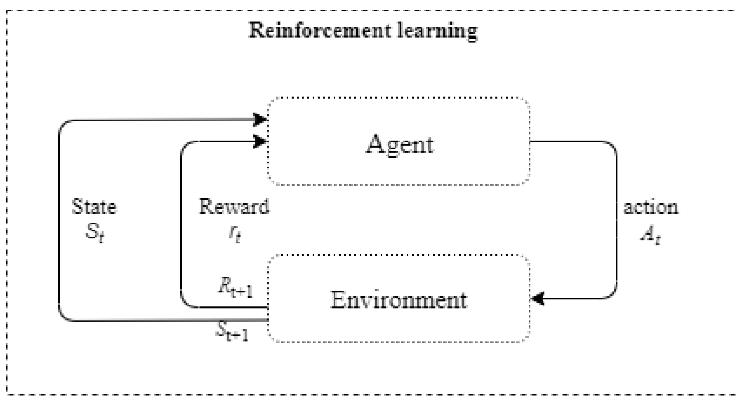


Figure 0.1 Reinforcement learning process with agent and environment [183]

For systems degrading and experiencing random shock arrivals, finding a proper maintenance policy has been a sophisticated issue which has been studied for several years. MDP can be used to model the degradation of such systems, and by using the reinforcement learning algorithm, an agent can be trained to provide an action, such as a particular maintenance action for components of systems. The agent is trained by using the interactions of environment and actions, which can be the degradation states and the available maintenance actions. At each time step which can be the inspection time, the agent detects the current degradation state of the system and chooses a maintenance action based on the system degradation knowledge. Each maintenance action is evaluated by its associated cost which can be considered as negative reward, and subsequently, based on the action taken by the agent, the system goes to the next state. The process continues until the end of the maintenance time contract or planning period.

MDP for reinforcement learning problem has the following components:

- The system has state space of S, and at each time step t, the state is $s_t \in S$.
- $A(S)$ is the set of possible actions, and the action at time t is $a \in A(S_t)$.
- $P_{ss'}^{a} = P(s_{t+1}|s_t, a)$ is the transition probability of being in state s' at time t+1, if the system was in state s at time t, and the agent chooses the action a.
- r_t is the reward at time t.
- $\pi_t(s,a)$ is the probability that action a is selected at time t given state s.

The agent policy $\pi_t(s,a)$ is the mapping from distinguished states of the environment to actions to be taken when in those states at each time step. The agent policy is the probability distribution over actions given states, and it must be true that $\sum_a \pi_t(s,a) = 1$. In other words, it is the likelihood of every action, when an agent is in a particular state. In reinforcement learning methods, R_t is the cumulative future reward which is returned to the agent at time t. The future cumulative discounted reward is used to assign greater weights to rewards occurring sooner. A discount factor $\gamma \in (0,1)$ is defined for calculating R_t, which serves to emphasize the imminent rewards rather than the future rewards. So, the equation for future cumulative discounted reward can be calculated as shown in Equation 6.1, where r_t is the reward at t.

$$R_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + ... = \sum_{k=0}^{T} \gamma^k r_{t+k+1} \tag{0.1}$$

Where T is the time horizon over which the maintenance policy is determined.

To learn the optimal policy, the value of taking action a at any state s should be defined. The action value function is the expected value of taking an action in a state following a certain policy and can be calculated as indicated in Equation 6.2:

$$Q_\pi(s,a) = E_\pi\left[R_t \mid S_t = s, A_t = a\right] = E_\pi\left[\sum_{k=0}^{T} \gamma^k r_{t+k+1} \mid S_t = s, A_t = a\right] \tag{0.2}$$

The goal of reinforcement learning using MDP is to obtain the maximum expected cumulative reward. Using Bellman equations is one of the traditional methods to solve the MDP problems. In this section, the Bellman equation is briefly explained, which is necessary to understand the reinforcement learning algorithms.

The following equations present transition probabilities and expected cumulative reward.

$$P_{ss'}{}^{a} = P(S_{t+1} = s' \mid S_t = s, A_t = a) \tag{0.3}$$

$$\Re_{ss'}{}^{a} = E_\pi[r_{t+1} \mid S_t = s, S_{t+1} = s', A_t = a] \tag{0.4}$$

Equation (6.3) is the transition probability, which is the probability of arriving at state s' at time t +1, if the process starts at state s at time t and take the action a. Equation (6.4) is the expected cumulative reward if the process starts at state s, take the action a, and move into state s'.

The first step in deriving the Bellman equations is defining the state value function, which is described as the expected return continueing from state s following policy π.

$$V^{\pi}(s) = E_{\pi}\left[\sum_{k=0}^{T}\gamma^{k} r_{t+k+1} \mid S_t = s\right] \tag{0.5}$$

The expectation can be written explicitly by summing over all possible actions and all possible returned states.

$$E_{\pi}\left[r_{t+1} \mid S_t = s\right] = \sum_{a}\pi_t(s,a)\sum_{s'}P_{ss'}^{a}\Re_{ss'}{}^{a} \tag{0.6}$$

$$E_{\pi}\left[\gamma\sum_{k=0}^{T}\gamma^{k} r_{t+k+2} \mid S_t = s\right] = \sum_{a}\pi_t(s,a)\sum_{s'}P_{ss'}^{a}\gamma E_{\pi}\left[\sum_{k=0}^{T}\gamma^{k} r_{t+k+2} \mid S_{t+1} = s'\right] \tag{0.7}$$

As it was proved in [184], by using Equations (6.2)-(6.7), the Bellman equation for the action value function can be derived as follow.

$$\begin{aligned} Q_{\pi}(s,a) &= E_{\pi}\left[r_{t+1} + \gamma\sum_{k=0}^{T}\gamma^{k} r_{t+k+2} \mid S_t = s, A_t = a\right] \\ &= \sum_{s'}P_{ss'}{}^{a}\left(\Re^{a}{}_{ss'} + \gamma E_{\pi}\left[\sum_{k=0}^{T}\gamma^{k} r_{t+k+2} \mid S_{t+1} = s'\right]\right) \\ &= \sum_{s'}P_{ss'}{}^{a}\left(\Re^{a}{}_{ss'} + \gamma\sum_{a'}E_{\pi}\left[\sum_{k=0}^{T}\gamma^{k} r_{t+k+2} \mid S_{t+1} = s', A_{t+1} = a'\right]\right) \\ &= \sum_{s'}P_{ss'}{}^{a}\left(\Re^{a}{}_{ss'} + \gamma\sum_{a'}\pi(s',a')Q_{\pi}(s',a')\right) \end{aligned} \tag{0.8}$$

a' is the next action and s' is the next state. Using the Bellman equations, the value of states can be derived as values of other states. In fact, the value of current state s_t, can be calculated knowing the value of s'. The goal of reinforcement learning is to find the optimal policy, which tells us the best action at each state to maximize the reward. By using the Bellman equations and dynamic programming, the optimal policy can be calculated. The Bellman equations help explain how the reinforcement algorithm works. In this section, one of the most common reinforcement learning algorithms called Q-learning is applied to solve the MDP.

1.12.1 Markov decision process for maintenance policy

In condition-based maintenance, the deterioration level of the system or components are used to make the decision for maintenance actions. For systems deteriorating in time, it is a more cost-effective plan to dynamically implement the maintenance action based on its deterioration level. For our model, a multi-component system is considered where each component degrades over time within the system, and can be maintained individually. For each component *i*, the degradation is classified into multiple stages based on predetermined thresholds. Figure 6.2 shows the different stages for each component *i*, used in our model. When the component is new, and its degradation level is zero ($X_i(t) = 0$), it is in stage 0. Stage 4 or failure stage is when the degradation level of the component is above the failure threshold H_i^1. When the component degradation level is between H_i^2 and H_i^3, it is in stage 2, and when it is between H_i^2 and H_i^1 , it is in stage 3.

Since the components are degrading over time, at each time step (inspection time), if the component is not maintained, it can transition from stage *j* to stage *j*+1 (or higher) or stay in the same stage *j*, but it is impossible to go back to the previous stages. Therefore, the state of the system at time *t*, is S_t= (*t*, s_{1t}, s_{2t}, …, s_{nt}), where s_{it} is the degradation stage of component *i* at time *t*, $s_{it} \in \{0,1,2,3,4\}$.

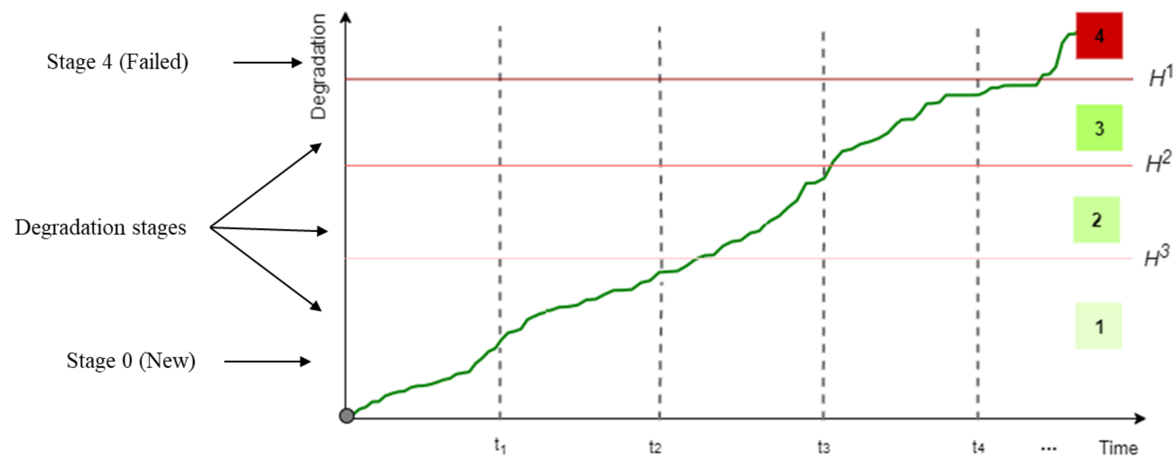


Figure 0.2 Different stages for each component

MDP can be defined as a tuple of (S, A, P, C, γ), where S is the state, which is the combination of degradation level of all the components in the system $S_t = (t, s_{1t}, s_{2t}, \ldots, s_{nt})$. A is the action set, which is defined by selecting an available maintenance action for each component. P is the transition probability of going to the next states, and C is the cost function.

The actions for reinforcement learning which are considered in this section are (1) do nothing, (2) minimal repair, (3) replace a component. Therefore, at any state j any of these actions can be chosen for any component i. If “do nothing” is selected for any component i which is in stage j, it can stay in state j or can go to any of the next stages with different transition probabilities. If I consider “minimal repair” action for any component i at state j, then the next state should be j-1 with probability of one. If “replace” is considered for component i at state j, it transitions to stage 0 with probability of one, which is “as good as new”. Hence, at each inspection, the action set for a system with n components has 3^n actions, which includes any of three actions, i.e., “do nothing, minimal repair, replace”, for all n components.

It should be noted that at each inspection time, the state information is known for the problem and the terminal state is the end of the planning time horizon for providing the maintenance policy. In fact, if it is desirable to provide a maintenance policy for T units of time and our inspection duration is τ, then there are $\kappa = \lfloor T/\tau \rfloor$ inspections in our state matrix, so t for the degradation state is between 0 and κ $(0 \le t \le \kappa)$. The agent's learning performance is significantly improved when time-awareness of the agent is introduced, by specifically incorporating a time-related space component [185, 186]

Figure 6.3 shows a Markov process for a system with one component from any time τ to τ+1, and three actions of {"do nothing: 0", "repair: 1", "replace: 2"}. Green arrows show the transition when "do nothing" action is selected at any state. Blue arrows show the transition between states when the action is "repair" and orange arrows show the "replace" transitions. The Markov chain has five states, and each of them represents the degradation stage of the component. For example, state 0 is the stage that the component is new and then with some probabilities it can transit to the next state (state 1, 2, 3, or failure), but it cannot stay in state 0. Therefore, in the next inspection time, it can move to any other states, and due to the chosen action, it can move to any of five states. For instance, $P_{12}^{\ 0}$ represents the probability that the component transitions to state 2 if action 0 is selected. $P_{10}^{\ 1}$ show the probability that component goes back to state 0 upon repair or replacement action.

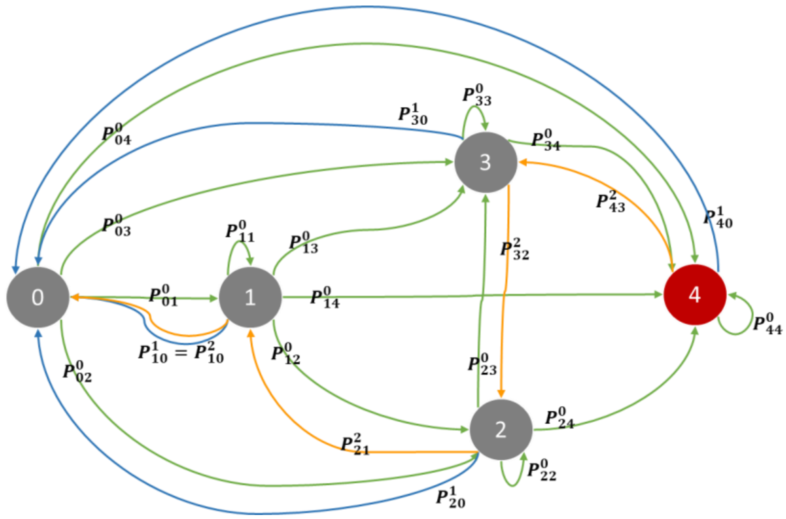


Figure 0.3 MDP for a single system

Figure 6.4 shows some example connections for a Markov process for a system with 2 components, each with 5 stages. As it is shown, if the current state is (1,1) which means both components are in their first degradation stage (stage1), then different actions make it transition to another state with a different cost (reward). If the first component is replaced but not the second component, it goes to (0,2), or (0,1), or if both components are replaced, S_{t+1} is (0,0), and so on.

It is considered that each time step is an inspection time, and there is a defined time horizon for maintenance decision-making, such as the duration for a maintenance or another convenient planning period. The combination of all component degradation at time t is the current state, and actions of "do nothing, minimal repair, replace" for each component within the system are the action set, which has 3^n possible system actions.

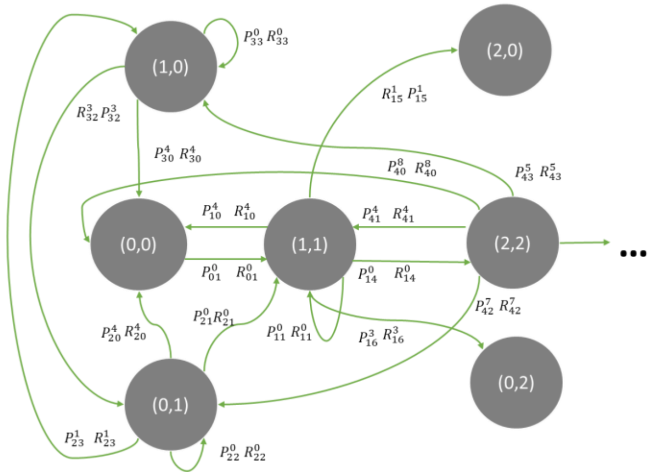


Figure 0.4 Markov process of a system with 2 components

Table 6.1 shows the action numbers for all the combination of actions for all the components. Each action has a fixed cost such as 0 for “do nothing”, C_m for “minimal repair” and C_R for “replace” in addition to a penalty cost of downtime if the system is down between two inspection time C_ρ.

Table 0.1 Action set for a system with 2 components

Component 1	Component 2	Action number
Nothing	Nothing	0
Nothing	Minimal repair	1
Nothing	Replace	2
Minimal repair	Nothing	3
Minimal repair	Minimal repair	4
Minimal repair	Replace	5
Replace	Nothing	6
Replace	Minimal repair	7
Replace	Replace	8

It the “do nothing” action is selected at any stage j for component i, it may stay in the same stage j or may go to any of the next stages with a probability. Because of increasing deterioration, it is assumed that the degradation level of the component cannot stay in stage 0. The following equations show the probability of component i, which is in stage 0 transitioning to any of stage 1, 2, 3, 4. In order to find the probability of being in stage 1, 2 or 3, the component should not experience any hard failure. In the case that hard failure occurs on any component, it transitions directly to the stage of failure. Equation (6.9), (6.10) and (6.11) indicate the probability that component *i* has not experienced any hard failure from the beginning of the inspection until the next inspection time and the degradation transitions to stage 1, 2 and 3. Respectively, Equation (6.9) shows the probability when degradation level of component *i* in next inspection is less than H_i^3.

$$\begin{aligned} P_{i,0-1}(\tau) &= \sum_{m=0}^{\infty}\Big[P(W_i < D_i)^m P\big((X_i(\tau)+S_i) < H_i^3\big) \mid N(\tau)=m\Big]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!} \\ &= \sum_{m=0}^{\infty}\left[P(W_i < D_i)^m \int_0^{H_i^3} P(X_i(\tau) < H_i^3 - y)\, f_{Y_i}^{<m>}(y)dy\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!} \\ &= \sum_{m=0}^{\infty}\left[P(W_i < D_i)^m \int_0^{H_i^3} G(H_i^3 - y;\alpha_{i0}\tau,\beta_i)\, f_{Y_i}^{<m>}(y)dy\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!} \end{aligned} \quad (0.9)$$

Where $f_{Y_i}^{<m>}(y)$ is pdf of the sum of *m* independent and identically distributed (*i.i.d.*) Y_i, and $G(x_i;\alpha_0 t,\beta)$ is the cumulative distribution function of $X_i(t)$. Equations (6.10)-(6.11) show the probability when degradation level of component *i* in next inspection is between H_i^3 and H_i^2, and when degradation level of component *i* in next inspection is between H_i^2 and H_i^1.

$$\begin{aligned} P_{i,0-2}(\tau) &= \sum_{m=0}^{\infty}\Big[P(W_i < D_i)^m P\big(H_i^3 < (X_i(\tau)+S_i) < H_i^2\big) \mid N(\tau)=m\Big]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!} \\ &= \sum_{m=0}^{\infty}\left[P(W_i < D_i)^m \int_0^{H_i^2}\big(G(H_i^2 - y;\alpha_{i0}\tau,\beta_i) - G(H_i^3 - y;\alpha_{i0}\tau,\beta_i)\big) f_{Y_i}^{<m>}(y)dy\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!} \end{aligned} \quad (0.10)$$

The probability of being in stage 3 can be calculated as follow, which is

$$\begin{aligned} P_{i,0-3}(\tau) &= \sum_{m=0}^{\infty}\Big[P(W_i < D_i)^m P\big(H_i^2 < (X_i(\tau)+S_i) < H_i^1\big) \mid N(\tau)=m\Big]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!} \\ &= \sum_{m=0}^{\infty}\left[P(W_i < D_i)^m \int_0^{H_i^1}\big(G(H_i^1 - y;\alpha_{i0}\tau,\beta_i) - G(H_i^2 - y;\alpha_{i0}\tau,\beta_i)\big) f_{Y_i}^{<m>}(y)dy\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!} \end{aligned} \quad (0.11)$$

When the component *i* fails due to hard failure or its total degradation exceed failure threshold H_i^1, it will be in stage 4 or failure stage with the following probability.

$$\begin{aligned} P_{i,0-4}(\tau) &= 1-\sum_{m=0}^{\infty}\Big[P(W_i<D_i)^m P\Big((X_i(\tau)+S_i)<H_i^1\Big)\,|\,N(\tau)=m\Big]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!} \\ &= 1-\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m\int_0^{H_i^1}G(H_i^1-y;\alpha_{i0}\tau,\beta_i)\,f_{Y_i}^{<m>}(y)dy\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!} \end{aligned} \tag{0.12}$$

For calculating the transition probability from stage 1 to 2, 3 and 4 a random variable U_{i1} is needed as the initial degradation of the component's degradation in stage 1, which is between 0 and H_i^3 , and follows uniform distribution between 0 and H_i^3. It is also assumed that if any component *i* is in stage $j \neq 0$, it can stay in the same stage *j* in the next time step with probability of P_{j-j}. In the same way, the transition probabilities can be calculated from stage 2 to 3 or 4, or staying in stage 2 considering U_{i2} as the initial degradation of component in stage 2. All the transition probabilities for stage 3 also can be calculated in the same way using U_{i3} as the initial degradation of components in stage 3. Table 6.2 shows all the transition probabilities. Where $f_{U_i}(u)$ is the probability density function of initial random degradation for component *i* which follows uniform distribution between the corresponding thresholds for each specific degradation stage.

The presence of shocks creates some form of dependency. However, to compute transition probabilities, it is assumed that component transitions are independent. Therefore, the total probability that the system transits from any state to other states can be calculated by multiplying the transition probability of each component within the system. For example for a system with two components, if component 1 is in stage 2 at time *t* and component 2 is in stage 3 at time *t*, the state for whole system is (2,3,*t*) and the probability that the next state will be (3,3,*t*+1) can be calculated using the transition probabilities of these two components, $P(S_{t+1} = (3,3,t+1) \mid S_t = (2,3,t)) = P_{1,2-3} \times P_{2,3-3}$.

Table 0.2 Transition probabilities for all the states

Transition	Probability
1 to 1	$P_{i,1-1}{}^{0}(\tau)=\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m P\left((X_i(\tau)+S_i+U_{i1})<H_i^3\right) \mid N(\tau)=m\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$ (0.13) $=\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m\int_0^{H_i^3}\int_0^{H_i^3-u_{i1}}P(X_i(\tau)<H_i^3 - y-u_{i1})\,f_{Y_i}{}^{<m>}(y)\,f_{U_i}(u_{i1})dydu \mid N(\tau)=m\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$ $=\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m\int_0^{H_i^3}\int_0^{H_i^3-u_{i1}}G(H_i^3-y;\alpha_{i0}\tau,\beta_i)\,f_{Y_i}{}^{<m>}(y)\,f_{U_i}(u_{i1})dydu \mid N(\tau)=m\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$
1 to 2	$P_{i,1-2}{}^{0}(\tau)=\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m P\left(H_i^3<(X_i(\tau)+S_i+U_{i1})<H_i^2\right) \mid N(\tau)=m\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$ (0.14) $=\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m\int_0^{H_i^3}\int_0^{H_i^2-u_{i1}}\left(G(H_i^2-y-u_{i1};\alpha_{i0}\tau,\beta_i)-G(H_i^3-y-u_{i1};\alpha_{i0}\tau,\beta_i)\right)\right.$ $\left.\times f_{Y_i}{}^{<m>}(y)\,f_{U_i}(u_{i1})dydu \mid N(\tau)=m\right]\times\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$
1 to 3	$P_{i,1-3}{}^{0}(\tau)=\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m P\left(H_i^2<(X_i(\tau)+S_i+U_{i1})<H_i^1\right) \mid N(\tau)=m\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$ (0.15) $=\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m\int_0^{H_i^3}\int_0^{H_i^1-u_{i1}}\left(G(H_i^1-y-u_{i1};\alpha_{i0}\tau,\beta_i)-G(H_i^2-y-u_{i1};\alpha_{i0}\tau,\beta_i)\right)\right.$ $\left.\times f_{Y_i}{}^{<m>}(y)\,f_{U_i}(u_{i1})dydu \mid N(\tau)=m\right]\times\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$
1 to failure	$P_{i,1-4}{}^{0}(\tau)=1-\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m P\left((X_i(\tau)+S_i+U_{i1})<H_i^1\right) \mid N(\tau)=m\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$ (0.16) $=1-\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m\int_0^{H_i^3}\int_0^{H_i^1-u_{i1}}G(H_i^1-y-u_{i1};\alpha_{i0}\tau,\beta_i)\,f_{Y_i}{}^{<m>}(y)\,f_{U_i}(u_{i1})dydu \mid N(\tau)=m\right]\times\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$

2 to 2	$$P_{i,2-2}{}^{0}(\tau)=\sum_{m=0}^{\infty}\Big[P(W_i<D_i)^m P\Big((X_i(\tau)+S_i+U_{i2})<H_i^2\Big)\,\vert\, N(\tau)=m\Big]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$$ $$=\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m\int_{H_i^3}^{H_i^2}\int_{0}^{H_i^2-u_{i2}}G(H_i^2-y-u_{i2};\alpha_{i0}\tau,\beta_i)\,f_{Y_i}{}^{<m>}(y)\,f_{U_i}(u_{i2})dydu\,\vert\, N(\tau)=m\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$$ (0.17)
2 to 3	$$P_{i,2-3}{}^{0}(\tau)=\sum_{m=0}^{\infty}\Big[P(W_i<D_i)^m P\Big(H_i^2<(X_i(\tau)+S_i+U_{i2})<H_i^1\Big)\,\vert\, N(\tau)=m\Big]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$$ $$=\sum_{m=0}^{\infty}[P(W_i<D_i)^m\int_{H_i^3}^{H_i^2}\int_{0}^{H_i^1-u_{i2}}\Big(G(H_i^1-y-u_{i2};\alpha_{i0}\tau,\beta_i)-G(H_i^2-y-u_{i2};\alpha_{i0}\tau,\beta_i)\Big)$$ $$\times f_{Y_i}{}^{<m>}(y)\,f_{U_i}(u_{i2})dydu\,\vert\, N(\tau)=m\Big]\times\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$$ (0.18)
2 to failure	$$P_{i,2-4}{}^{0}(\tau)=1-\sum_{m=0}^{\infty}\Big[P(W_i<D_i)^m P\Big((X_i(\tau)+S_i+U_{i2})<H_i^1\Big)\,\vert\, N(\tau)=m\Big]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$$ $$=1-\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m\int_{H_i^3}^{H_i^2}\int_{0}^{H_i^1-u_{i2}}G(H_i^1-y-u_{i2};\alpha_{i0}\tau,\beta_i)\,f_{Y_i}{}^{<m>}(y)\,f_{U_i}(u_{i2})dydu\,\vert\, N(\tau)=m\right]\times\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$$ (0.19)
3 to 3	$$P_{i,3-3}{}^{0}(\tau)=\sum_{m=0}^{\infty}\Big[P(W_i<D_i)^m P\Big((X_i(\tau)+S_i+U_{i3})<H_i^1\Big)\,\vert\, N(\tau)=m\Big]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$$ $$=\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m\int_{H_i^2}^{H_i^1}\int_{0}^{H_i^1-u_{i3}}G(H_i^1-y-u_{i3};\alpha_{i0}\tau,\beta_i)\,f_{Y_i}{}^{<m>}(y)\,f_{U_i}(u_{i3})dydu\,\vert\, N(\tau)=m\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$$ (0.20)
3 to failure	$$P_{i,3-4}{}^{0}(\tau)=1-\sum_{m=0}^{\infty}\Big[P(W_i<D_i)^m P\Big((X_i(\tau)+S_i+U_{i3})<H_i^1\Big)\,\vert\, N(\tau)=m\Big]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$$ $$=1-\sum_{m=0}^{\infty}\left[P(W_i<D_i)^m\int_{H_i^2}^{H_i^1}\int_{0}^{H_i^1-u_{i3}}G(H_i^1-y-u_{i3};\alpha_{i0}\tau,\beta_i)\,f_{Y_i}{}^{<m>}(y)\,f_{U_i}(u_{i3})dydu\,\vert\, N(\tau)=m\right]\times\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!}$$ (0.21)

Finally, the cost function/reward function can be calculated using the associated cost for each action on the components, and the penalty cost for system downtime. In this study, the reward function r_t is defined as negative value of total cost ($r_t = -C_t$). At each inspection time t, each action $A_t = a$, is the combination of maintenance actions of all the components $A_t = a = (\sigma_{d0}, \sigma_{d1}, ..., \sigma_{dn})$, and has a specific cost associated with actions for all the components. . In the tuple of maintenance actions $a = (\sigma_{d0}, \sigma_{d1}, ..., \sigma_{dn})$, each $\sigma_{d\ i}$ for component i, is a binary variable for actions of {"do nothing: σ_{0i}", "repair: σ_{1i}", "replace: σ_{2i}"}, where $\sigma_{0i} + \sigma_{1i} + \sigma_{2i} = 1$. So, for each component i, the total cost can be calculated as follow:

$$C_t = \sum_{i=1}^{n} \left(C_m \sigma_{1i} + C_R \sigma_{2i} \right) + C_\rho \omega \tag{0.22}$$

C_m is the cost for minimal repair, C_R is the replacement cost and C_ρ is the penalty cost for downtime. ω is a binary variable indicating the failure states. If the system fails, a penalty cost of C_ρ should be added to the cost function. For each maintenance problem, failure states should be defined based on the system configuration and the system structure function. For a series system, if any component fails within the system, the system fails. Therefore, the failure states are defined as all states that any component fails i.e., it is in stage 4 (failure stage). For a parallel system, the system fails if all the components are failed, so there is just one failure state. Therefore, for each specific system configuration, the failure states are defined in advance.

The goal of using reinforcement learning and MDP for a maintenance problem is providing the best maintenance policy, which has the minimum total cost. The optimal maintenance policy is determined by finding the best Q-value for each state of the system, which suggests what maintenance actions should be implemented for each degradation state at each inspection time. There are different algorithms to solve the MDP, while a particularly effective one is Q-learning, which is explained in the next subsection.

1.12.2 Dynamic programming for reinforcement learning

Dynamic programming is an algorithm used to solve complex problems. The problem is solved in distinct stages using recursive functions. The solution of each stage or sub-problem is stored and reused to find the overall optimal solution of the problem. In this section, dynamic programming is used to find the best policy of Markov decision processes in reinforcement learning. The Bellman equation decomposes the overall optimal value into the optimal policy of each step and optimal value of remaining steps. The value function can be used to restore and retrieve the solution of each sub-problem.

Q-learning is a well-known algorithm, as a method of dynamic programming, to solve the reinforcement learning problems, which is proposed by Watkins [187]. In the Q-learning method, the agent takes one action at any particular state and evaluates its consequences, and by trying actions in all the possible states it learns what are the best actions which have the best long run rewards. Q-learning is an off-policy learning algorithm which has the following rule for updating the Q-values.

$$Q(s,a) \leftarrow Q(s,a) + \theta\left[r + \gamma \max_a Q(s',a) - Q(s,a)\right] \qquad (0.23)$$

Where θ is the learning rate which can have a value between 0 and 1, where 0 means the algorithm is never updated and θ =1 means the learning occurs quickly, and γ is the discount factor which can have a value between 0 and 1.

The procedural form of the Q-learning algorithm is presented as Table 6.3. $Q(s,a)$ gives the value of an action a in state s at time t. r_t is the reward at time t for moving from state s to s' for action a. An episode of the algorithm ends when s is the terminal state. A terminal state can be defined several ways, such as the time that required to provide a maintenance policy.

Table 0.3 Q-learning algorithm

Algorithm 1: Q-learning [187]
Initialize $Q(s,a)$ arbitrarily Repeat (for each episode): Initialize s Repeat (for each step of episode): Choose a from s using policy derived from Q (e.g., ε greedy) Take action a, observe r, s' $Q(s,a) \leftarrow Q(s,a) + \theta\left[r + \gamma \max_a Q(s',a) - Q(s,a)\right]$ $s \leftarrow s'$ Until s is terminal

When Q-learning is performed, a matrix with states, and actions is created which is defined as a Q-table. After each episode, the Q-value which is stored in each cell of the Q-table is updated. There are two ways to select the actions at each step, the agent can use the value of the Q-table as a reference and selects the best action which has the maximum value, which is called exploiting, or it can selects an action randomly which can let the agent explore and discover new states, where this is called exploring. To have a balance between exploration and exploitation, a parameter ε is used to balance how often the agent uses exploring vs. exploiting. After a numerous number of episodes, the optimal Q-table is found which represents the recommended actions for each state which results in the best long run reward. Figure 6.5 shows the flowchart of algorithm for updating Q-table for each episode.

For a maintenance problem, using the Q-learning method provides an algorithm to find the best agent policy for implementing maintenance actions based on the system degradation states which has the minimum cost. At each episode, the value of maintenance actions for all the specific degradation states are calculated and the episode terminated when it reaches the terminal states. The terminal state is the degradation state for the time that is the end of our proposed maintenance policy.

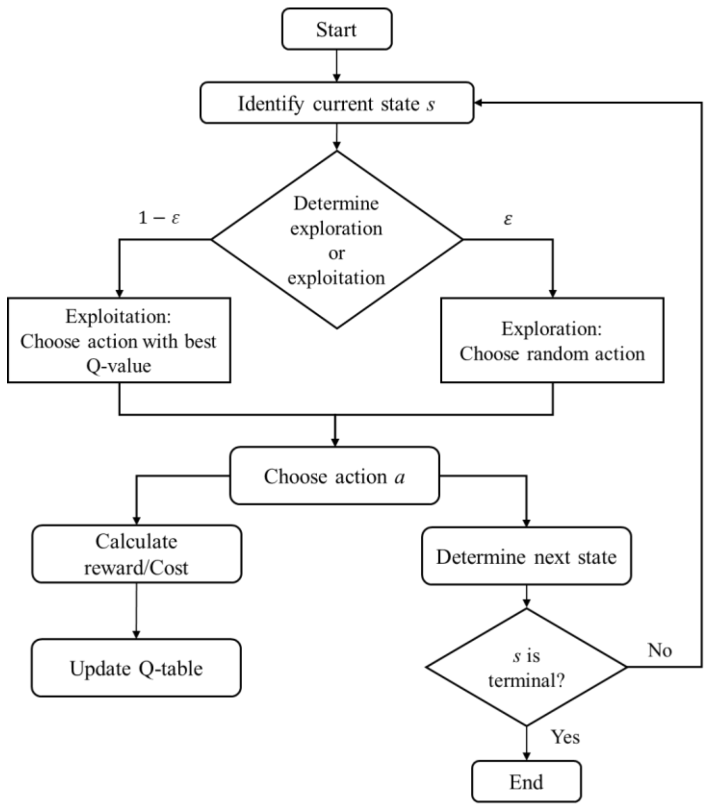


Figure 0.5 Updating algorithm for each episode

1.12.3 Numerical example for a dynamic maintenance plan using reinforcement learning

Three different system configurations of series, parallel, and combination of series-parallel are used to demonstrate the proposed method. In the series system with n components, if any component i fails, the whole system is failed; while in the parallel system, all n components must fail to make the system fail. Therefore, these three examples are used to demonstrate the proposed method. Each system with three components is considered where each component degrades over time and can be replaced/repaired individually within the system. It is also considered that there is a fixed inspection interval of 2 time units (e.g., months, weeks, …), and the objective is to find the optimal maintenance actions for a time horizon of 50 time units. Table 6.4

shows the parameters which are considered for the proposed model, where cost of minimal repair C_m is \$100, cost of replacement C_R is \$300 and penalty cost for system downtime is C_ρ=1000. The discount factor is $\gamma = 0.9$, learning rate is $\theta = 1$ and $\varepsilon = 0.02$

Table 0.4 Example system parameters

Parameters	Component 1	Component 2	Component 3
H_i^1	17	20	26
H_i^2	10	15	18
H_i^3	5	6	7
D_i	40	42	35
α_i	0.6	0.2	0.2
β	1.2	1.4	1
λ_0	2.5×10^{-5}		
W_{ij}	$W_{ij} \sim N(10, 5^2)$	$W_{ij} \sim N(14, 3^2)$	$W_{ij} \sim N(12, 2^2)$
Y_{ij}	$Y_{ij} \sim N(0.5, 0.1^2)$	$Y_{ij} \sim N(0.55, 0.1^2)$	$Y_{ij} \sim N(0.6, 0.1^2)$

Figure 6.6 shows the degradation behavior of each of the components within the system, and the dash lines shows their failure threshold. As shown in Figure 6.6, component 1 degrades faster compared to component 2 and 3, and since its failure threshold is lower than others the probability that it fails is higher than the other components.

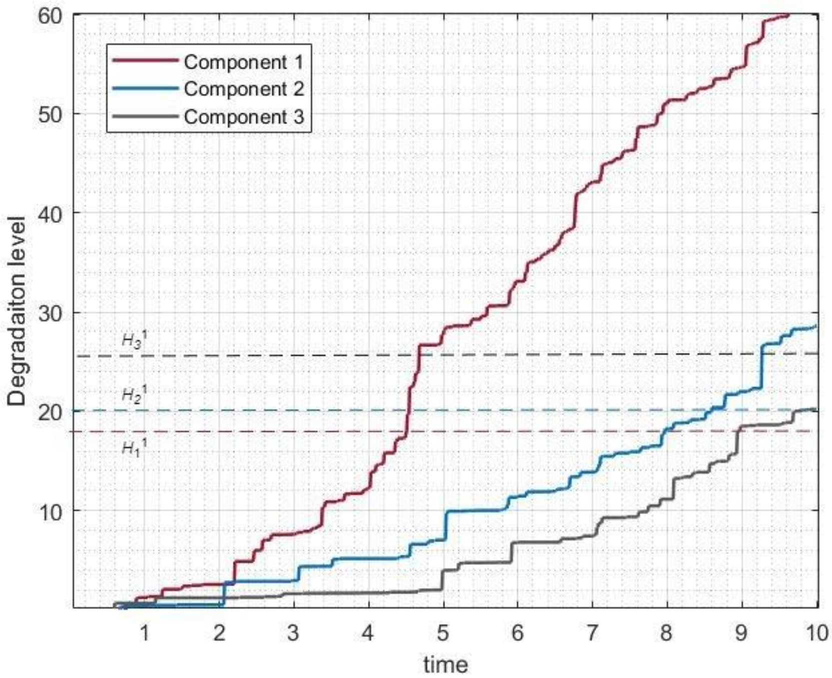


Figure 0.6 Degradation of components of a system

There are three actions which are “do nothing, minimal repair, and replace” and the cost of each action at each state is determined. The transition probability of moving from each state s to s' can be calculated using the equations in Table 6.4. Using reinforcement learning, the agent is trained with interaction with the environment, and the final Q-table is determined. Using the final optimal Q-table the maintenance team can select the best action for each level of system degradation.

The three examples for three different configurations are studied for this system with three components. The first configuration is when all the three components are configured as series. The second configuration is when they are configured as parallel and the third configuration is when component 2 and 3 are in parallel, and they are in series with component 1 which is shown in Figure 6.7.

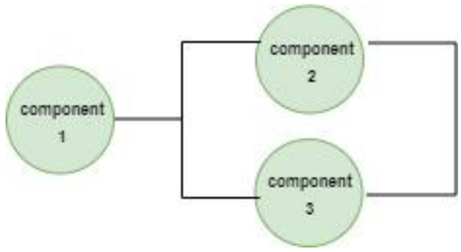


Figure 0.7 Third configuration which is studied in this study

Based on the action set for the proposed method, all the action scenarios and their corresponding numbers can be determined. The action set has three maintenance actions for each component {doing nothing: 0, minimal repair: 1, replace: 2}. For a system with three components there are 27 maintenance actions which are shown on Table 6.5.

Using the proposed method, the optimal maintenance action is found for each state for each component. The best maintenance actions for different scenarios are shown on Tables 6.6, 6.7, and 6.8 to illustrate the performance of the Q-learning method in obtaining dynamic maintenance policies for different configurations. The action sets for all three components can be found in Table 6.5.

Table 0.5 Action sets

Action number	Action scenario	Component1	Component 2	Component 3
1	(0,0,0)	Nothing	Nothing	Nothing
2	(0,0,1)	Nothing	Nothing	Minimal repair
3	(0,0,2)	Nothing	Nothing	Replace
4	(0,1,0)	Nothing	Minimal repair	Nothing
5	(0,1,1)	Nothing	Minimal repair	Minimal repair
6	(0,1,2)	Nothing	Minimal repair	Replace
7	(0,2,0)	Nothing	Replace	Nothing
8	(0,2,1)	Nothing	Replace	Minimal repair
9	(0,2,2)	Nothing	Replace	Replace
10	(1,0,0)	Minimal repair	Nothing	Nothing
11	(1,0,1)	Minimal repair	Nothing	Minimal repair
12	(1,0,2)	Minimal repair	Nothing	Replace
13	(1,1,0)	Minimal repair	Minimal repair	Nothing
14	(1,1,1)	Minimal repair	Minimal repair	Minimal repair
15	(1,1,2)	Minimal repair	Minimal repair	Replace

16	(1,2,0)	Minimal repair	Replace	Nothing
17	(1,2,1)	Minimal repair	Replace	Minimal repair
18	(1,2,2)	Minimal repair	Replace	Replace
19	(2,0,0)	Replace	Nothing	Nothing
20	(2,0,1)	Replace	Nothing	Minimal repair
21	(2,0,2)	Replace	Nothing	Replace
22	(2,1,0)	Replace	Minimal repair	Nothing
23	(2,1,1)	Replace	Minimal repair	Minimal repair
24	(2,1,2)	Replace	Minimal repair	Replace
25	(2,2,0)	Replace	Replace	Nothing
26	(2,2,1)	Replace	Replace	Minimal repair
27	(2,2,2)	Replace	Replace	Replace

Table 6.6 shows the optimal maintenance action for different system states, which represent a combination of component degradation, for a series system. The optimal action numbers can be described using Table 6.5. The column of state numbers in Table 6.6 shows the combination of degradation stages for all the components in the system. For example, (2,3,1) indicates that component 1 is in degradation stage 2, component 2 is in degradation stage 3, and component 3 is in stage1. Using the proposed method, it is found that the best maintenance action for this state is action number 1 which is described in Table 6.5 and suggests "do nothing" for all the components.

Table 0.6 Optimal maintenance actions for different scenario of series system

Scenario number	State number	Optimal action number	Scenario number	State number	Optimal action number
1	(1,1,1)	1 (0,0,0)	8	(2,2,2)	14 (1,1,1)
2	(2,3,1)	25 (2,2,0)	9	(1,1,2)	2 (0,0,1)
3	(3,2,1)	22 (2,1,0)	10	(1,2,3)	6 (0,1,2)
4	(3,1,1)	19 (2,0,0)	11	(2,1,2)	11 (1,0,1)
5	(1,3,3)	9 (0,2,2)	12	(3,3,1)	25 (2,2,0)
6	(2,2,3)	17 (2,1,2)	13	(3,2,2)	23 (2,1,1)
7	(3,3,3)	18 (2,2,2)	14	(1,3,1)	7 (0,2,0)

In a series system, if any component fails the whole system fails, so there are different failure states, but in all of them, at least one of the components is in its failure stage. As it is shown in Table 6.6, based on the degradation stage of all the components, the best maintenance action that should be implemented on each component is found. Table 6.6 shows different possible component combinations, such as state (1,1,1) where all the components are in their first stage, the optimal maintenance action is “do nothing” for all of them. For some cases, when any component is close to failure or in stage 3, such as (3,1,1) the component which is close to failure should be replaced while other maintenance action should be implemented on other components based on their degradation stages.

Two cases of (2,3,1) and (3,2,1) seem similar, while the maintenance actions are different. For both cases, nothing should be done for the component which is in stage 1 and replace it when it is in stage 3; however, when component 1 is in stage 2, the optimal maintenance action is found as “replace” while when component 2 is in stage 2 the optimal maintenance action is “repair”. It can be interpreted that since component 1 degrades faster and may have a lower failure threshold, it would have higher failure probability, so the optimal maintenance action is found as “replace” to prevent the system failure.

In scenarios 6 (2,2,3) and 13 (3,2,2), the system state also seems similar. In both of them, the component which is in stage 3 and it is close to failure state should be replaced. Component 2, which is in stage 2 in both scenarios, should be repaired. The difference of the optimal maintenance action with these two scenarios is that when component 1 is in stage 2, it should be replaced while component 3, which is also in stage 2, should be repaired. The difference between relative degradation rate of component 1 and component 3 makes this difference in optimal maintenance of these two scenarios.

Table 6.7 shows the optimal maintenance policy for different states of a system with parallel configuration. In a parallel system, the system fails if all the components fail, so the only failure state is when all the components are in their failure stages.

Table 0.7 Optimal maintenance actions for different scenario of parallel system

Scenario number	State number	Optimal action number	Scenario number	State number	Optimal action number
1	(1,1,1)	1 (0,0,0)	8	(2,2,2)	1 (0,0,0)
2	(2,3,1)	4 (0,1,0)	9	(1,1,2)	1 (0,0,0)
3	(3,2,1)	10 (1,0,0)	10	(1,2,3)	2 (0,0,1)
4	(3,1,1)	10 (1,0,0)	11	(2,1,2)	10 (1,0,0)
5	(1,3,3)	5 (0,1,1)	12	(3,3,1)	13 (1,1,0)
6	(2,2,3)	11 (1,0,1)	13	(3,2,2)	19 (2,0,0)
7	(3,3,3)	14 (1,1,1)	14	(1,3,1)	4 (0,1,0)

As it is shown in Table 6.7, for cases that any component is in stage 3, which is close to failure, the optimal maintenance action is “repair”, and there are very few “replace” actions in the optimal Q-table. For series system the optimal maintenance action for state (3,3,3) is to replace all the components while for this state in the parallel system, the optimal maintenance actions are found as “repair, repair, and repair”. Moreover, the optimal maintenance actions for two cases of (3,2,1) and (3,1,1) are the same, and the reason is, since the system fails if all the components fail, as long as one of the component is relatively new and in its first stage, it is better to not replace the other components, even if they are in their stage 3 which is close to failure. In this way, the maintenance cost is reduced by preventing unnecessary replace or repair cost.

In this example, since the system is configured as parallel it is very robust to failure, and compared to the previous example, which is a series system, action 2 which is replacement is rarely suggested for this system states. Among all the scenarios shown in Table 6.7, replacement is only suggested for Scenario 13 (3,2,2) where component 1 is in stage 3 and the rest are in their stage 2. Since component 1 degrades faster and it has a higher probability of failure it is suggested to be replaced, and nothing should be done for the rest. Comparing Scenario 13 (3,2,2) with the similar Scenario 4 (3,1,1), although component 1 is in its third stage in both of them the optimal maintenance action for this component is different. In scenario 13, it is suggested to be replaced while in scenario 4, it should be repaired. The reason for this difference can be explained by considering the degradation stages of other components. In Scenario 4, although component 1 is very close to its failure, component 2 and 3 are still far from their failure stages and their degradation rate are slower than component 1, and since they are configured as parallel, failure of component 1 may not cause any failure. While in Scenario 13, all the components are close to their failure, so it is suggested to replace component 1 and do nothing on other components.

Table 6.8 shows the optimal policy for the third configuration which is shown in Figure 6.7. In this configuration, the system fails if component 1 fails or component 2 and 3 fail together. In other words, if component 2 or 3 fails but the other components work, the system works.

Table 0.8 Optimal maintenance actions for different scenario of series-parallel system

Scenario number	State number	Optimal action number	Scenario number	State number	Optimal action number
1	(1,1,1)	1 (0,0,0)	8	(2,2,2)	10 (1,0,0)
2	(2,3,1)	19 (2,0,0)	9	(1,1,2)	1 (0,0,0)
3	(3,2,1)	19 (2,0,0)	10	(1,2,3)	5 (0,1,1)
4	(3,1,1)	19 (2,0,0)	11	(2,1,2)	4 (1,0,1)
5	(1,3,3)	5 (0,1,1)	12	(3,3,1)	19 (2,0,0)
6	(2,2,3)	4 (1,0,1)	13	(3,2,2)	20 (2,0,1)
7	(3,3,3)	23 (2,1,1)	14	(1,3,1)	4 (0,1,0)

As it is shown in Table 6.8, three states of (2,3,1), (3,2,1), (3,1,1) and (3,3,1) have the same optimal policy, where component 1 should be replaced if it is in stage 2 or 3 because of its relatively larger degradation behavior, and since component 2 and 3 are configured as parallel, there should not be any maintenance actions implemented on component 2 and 3, as long as one of them is in its first stage. For state (3,3,3) the optimal maintenance policy is found as "replace, repair, and do nothing" while for a series system, it is "replace, replace, replace" and for the parallel system, it is "repair, repair, repair". The reason is, failure of component 1 causes the system failure, so it should be replaced, while the parallel configuration of component 2 and 3 makes the system more robust to failure and the optimal maintenance action is found as "repair" for both of them.

Comparing scenario 13 in the three examples of series, parallel and series-parallel, it illustrates the different optimal maintenance policies of the same state but different system configuration. The optimal action for a series system for this state is number 23 (2,1,1) while for a parallel system is 19 (2,0,0) and for a series-parallel system is 20 (2,0,1). Since the failure of the system is different based on its configuration, the optimal maintenance is different. In a series system it is suggested to replace component 1 and repair the other components, while in a parallel system, the only maintenance action is replacing component 1. In the series-parallel system, failure of component 1 causes the system failure, so it is suggested to replace this component, but component 2 and 3 are configured as parallel, so it is suggested to repair only component 3. Another interesting point about repairing component 3, not component 2, while both of them are in stage 2. Comparing the degradation speed and failure threshold of components 2 and 3, it can be concluded that component 3 degrades at a slower rate, and its failure time is longer than component 2, so by repairing component

3 instead of component 2, the system may fail in a longer time.

To show how the optimal maintenance policy can be different based on the maintenance cost, the proposed method is applied for the same series system with three components using parameters in Table 6.4, but the maintenance costs are now different. In the previous example, the minimal repair cost is C_m = \$100, the cost of replacement C_R is \$300, and the penalty cost for system downtime is C_ρ=1000. By changing the minimal repair cost and making it \$300, there will be the same repair and replacement cost. Table 6.9 shows the difference in optimal maintenance policy of different scenarios for the previous example and this example.

Table 0.9 Different maintenance policy for examples with different maintenance cost

State number	Optimal action number For system with different cost (C_m = \$100, C_R = \$300)	Optimal action number for system with same maintenance cost (C_m = \$300, C_R = \$300)
(1,1,1)	1 (0,0,0)	1 (0,0,0)
(2,2,2)	14 (1,1,1)	27 (2,2,2)
(1,2,3)	6 (0,1,2)	25 (2,2,0)
(3,2,2)	23 (2,1,1)	19 (2,0,0)
(1,3,3)	9 (0,2,2)	9 (0,2,2)
(2,2,3)	17 (2,1,2)	27 (2,2,2)
(1,1,2)	2 (0,0,1)	27 (2,2,2)

As it is shown in Table 6.9, in the optimal maintenance policies of the different scenarios for the same system, there is no minimal repair (maintenance action 1). With the same maintenance cost, all the repair actions are changed to replacement. By replacing the component instead of repairing, it returns to the new stage, with the same cost.

To show how the proposed method can provide a maintenance action dynamically based on the degradation state of the systems, different scenarios, shown in Figure 6.8, that shows the different maintenance actions based on the optimal Q-table for a series-parallel system. The configuration of the series-parallel system is shown on Figure 6.8 with the parameters of Table 6.4. Seven different possible scenarios are shown for a series-parallel system for different degradation of components. Based on the optimal Q-table, the best maintenance action can be chosen for each scenario. In the series-parallel system, component 1 is series with two components of two and three, which are configured as parallel. Failure of component 1 causes the whole system to fail, while components 2 and 3 must fail at the same time to make the system fail. Figure 6.8 shows how the degradation of components are different, and for each degradation state, the optimal maintenance policy is different.

For example, in Scenario 1, in time step 1, all the components are new, and in the next time step all of them degrade and transition to their first stages. The optimal maintenance action at time step 2 based on the optimal Q-table is (0,0,0) which indicates to do nothing for all the components. In the next time step, component 1 and 2 stays in the same stage while component 3 degrades faster and transitions to its stage 3. The suggested maintenance action for this state is (0,0,1), which is repairing only component 3. In time step 4, components 1 and 2 go to stage 2, and component 3 stays in its stage 2, which has been after repairing. Subsequently, the optimal maintenance action is (1,1,0), which is repairing only component 1.

In scenario 6, all the components are new at time step 1. Component 1 degrades faster and transitions to its stage 2, while component 2 and 3 are in their stage 1, the optimal action is found as (1,0,0) which is repairing only component 1. In the next step component 1 stays in the stage 1 and component 2 transit to its stage 2, while component 3 degrades faster and transitions to its stage 3. The optimal maintenance action for this time step is (0,1,1) which is repairing both component 2 and 3. In time step 4, component 1 goes to stage 2, component 2 and 3 stays in the stages which they have been after repairing, so the optimal action is (1,0,0) which indicates repairing of component 1.

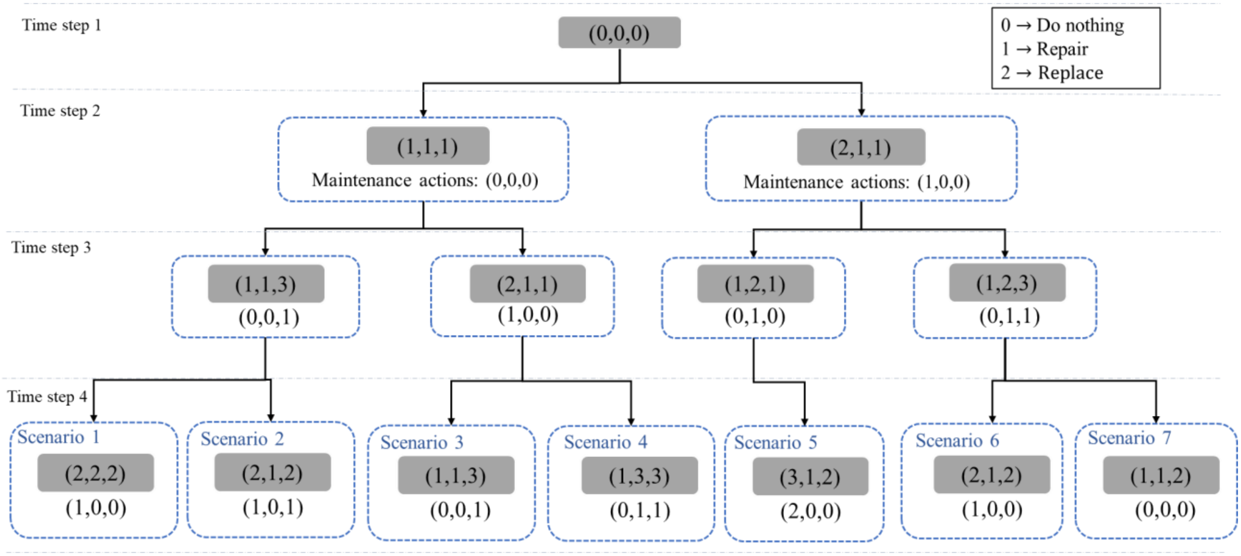


Figure 0.8 Different optimal maintenance policy based on different scenarios

All other scenarios can be described in the same way. Generally, Figure 6.8 shows how the maintenance actions differ based on the degradation behavior of all the components in the system. The numbers of maintenance actions can be described using Table 6.5.

The number of episodes can influence the performance of reinforcement learning algorithms. Finding the best number of episodes is a challenge in reinforcement learning algorithms which can be investigated by trying different numbers of episodes and observing its performance for each scenario. Figure 6.9 shows the running experience for different numbers of episodes for the series system, and Figure 6.10 shows the procedure for the different number of episodes for the series-parallel system. The horizontal axis on both plots shows the different percentiles of the total number of episodes, and the average total reward for each batch of episodes is shown on the vertical axis. As it is shown on both Figure 6.9 and 6.10, the well-suited number of episodes for this study is 10^7 because it has the best performance compared to other number of episodes. Moreover, comparing the result of 10^6 and 10^7, it can be concluded that the difference is negligible and increasing the numbers of episodes after 10^7 is an inefficient use of computational time and resources.

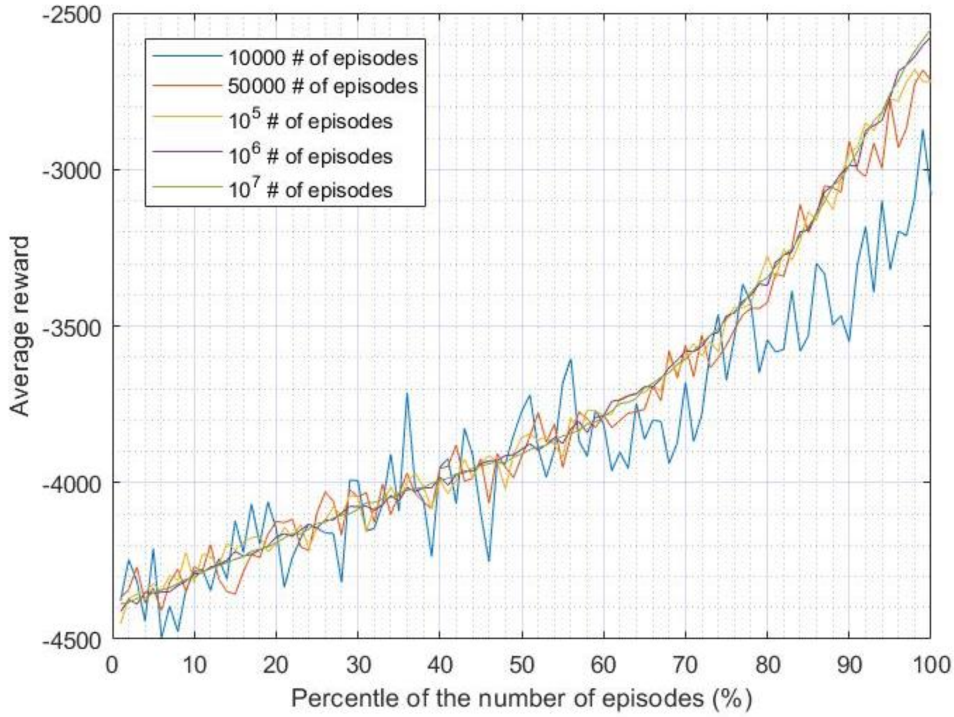


Figure 0.9 Convergence check for the required number of episodes for the series system

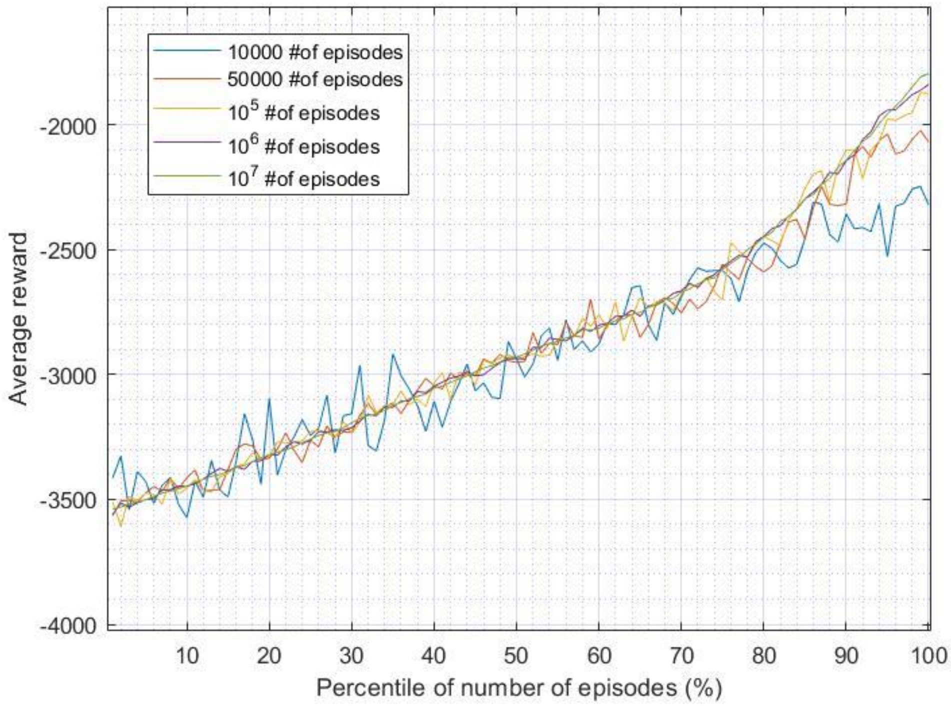


Figure 0.10 Convergence check for the required number of episodes for the series-parallel system

1.13 Dynamic maintenance for a multi-component system with an infinite degradation state

Reinforcement learning is one type of machine learning technique that trains an agent to decide how to perform an action based on the system state and associated rewards. By applying trial-and-error to maximize the reward, the agent learns how to make decisions in an uncertain, complex environment [100]. Therefore, at any inspection interval, the trained agent can choose the best maintenance action from a set of available actions based on the system status to minimize the cost. However, when the number of the system states or the actions increases, reinforcement learning may not be able to capture the best pattern. Hence, a combination of reinforcement learning and deep learning called deep reinforcement learning is used to train the model accurately. Yousefi et al [188] developed a dynamic condition-based maintenance model for a multi-component system considering infinite numbers of degradation states. Yousefi et al [188] used a deep Q-learning algorithm to solve the MDP maintenance problem to make the maintenance decisions dynamically, based on the degradation level of all the components in the system.

Deep reinforcement learning is the combination of reinforcement learning and deep learning, which is useful for problems with a large number of states or actions. In this study, deep reinforcement learning is used to find the best maintenance policy based on the system degradation level. The degradation process of the system is modeled using the gamma process, and at each inspection time, the best action can be suggested using the proposed maintenance model to minimize the maintenance cost for the duration of the maintenance contract.

Degradation is a major reason of failure for most of the industrial, manufacturing and technical systems. Due to difficulty of collecting failure information of systems, degradation modelling techniques have been widely used in the fields of reliability analysis and maintenance modeling. Using a proper degradation model is the main step in formulating the maintenance problem. In this study, it is assumed that for each component *i*, the degradation process between two time intervals *t* and *s*, follows gamma distribution with shape parameter of $\alpha_i(t)-\alpha_i(s)$ and scale parameter of β_i. It is also assumed that random shock occurs as a homogeneous Poisson process with rate λ. Each shock has results in damage on the degradation path of the components, which is an additional incremental damage on the system's degradation path. If total degradation, containing both pure degradation and the sum of additional incremental shock damage, is greater than a defined soft failure threshold level, then soft failure occurs. The damage from shocks is considered as an additional abrupt jump Y_{ij} on the cumulative degradation path of the component *i*. For any component *i*, the cumulative degradation is the summation of pure degradation by time *t*, and the cumulative damages caused by shocks by time t ($\Omega_i(t) = \sum_{j=1}^{\infty} Y_{ij}$) where Y_{ij} is an i.i.d random variable for the the j^{th} shock damage on i^{th} component. The total degradation can be accumulated as $X_{S_i}(t) = X_i(t) + \Omega_i(t)$.

1.13.1 Deep Q-learning algorithm

Q-learning is a powerful algorithm that creates an exact matrix for states and actions including Q-values in each cell, and the best action can be selected by referring to this optimal matrix. Q-learning algorithms are very effective when the number of actions and states is small. As the number of states and actions increases, Q-learning loses its feasibility and efficiency. An alternative approach for systems with a large number of states and actions is using an approximation for the Q-values. Deep Q-learning is an alternative algorithm to solve a problem with huge state and action spaces or when the state or action spaces is continuous. Deep Q-learning is a combination of Q-learning and deep learning. In deep Q-learning, the Q-values are approximated by using a neural network. The deep Q learning method tries to recognize patterns instead of mapping every state to its best action. The difference between the Q-learning and deep Q learning can be illustrated in Figure 6.11.

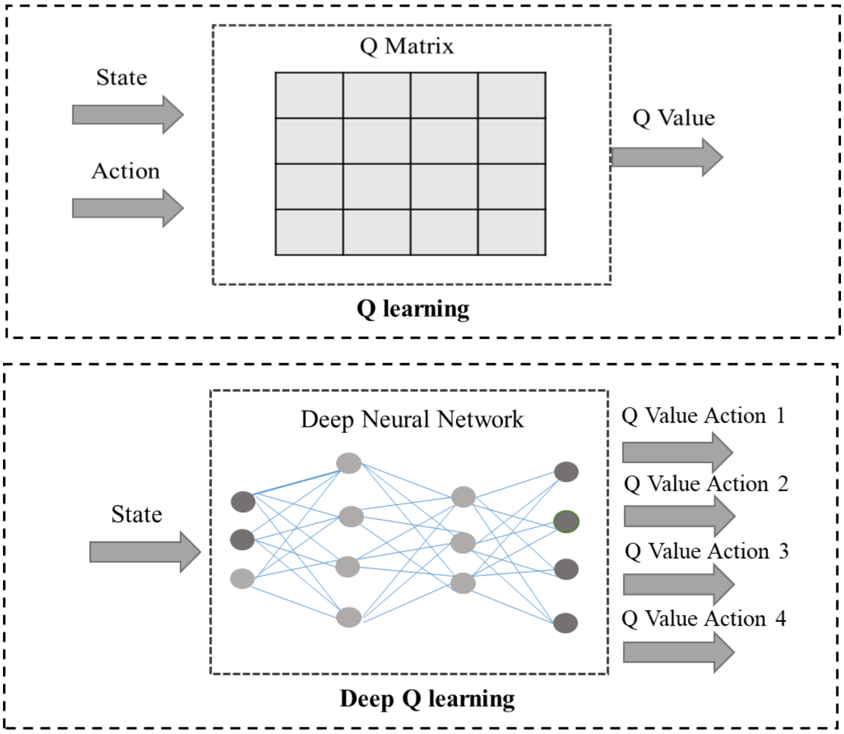


Figure 0.11 Q-learning VS deep Q-learning

The input for the neural network is the state, and the outputs which are the Q-values for all possible actions. The maximum value of the outputs is the next action. The main key points of deep Q-learning are as follows:

- All the past experiences are stored in a memory buffer.
- A batch sample of previous experiences is fed into the neural network.
- The next action can be determined by selecting the maximum output of the neural network
- The network is trained by using a loss function, which is the mean squared error of the predicted Q-value and the target Q-value. It is shown in Equation (6.24)

$$Loss = (\text{predicted}_{Q\text{-value}} \text{-target}_{Q\text{-value}})^2 \tag{0.24}$$

The predicted Q-value is the maximum output of the neural network and target Q-value can be estimated as in Equation 6.25.:

$$\text{target}_{Q\text{-value}} = R_{t+1} + \gamma \max_{a'} Q(s', a') \tag{0.25}$$

Where R_{t+1} is the immediate reward, and $\max_{a'} Q(s',a')$ represents the maximum output of the neural network for Q-values. The neural network can be updated through backpropagation. Figure 6.12 shows the process of training.

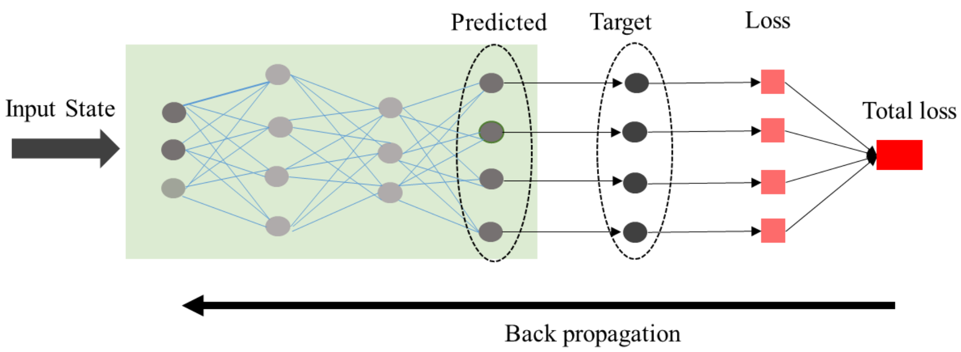


Figure 0.12 Training process in deep Q-learning

Therefore, the Q-value is updated according to its self-consistent equation, where the Q-value is regressed toward the target value, which depends on itself. However, since the target value changes automatically as the network parameters are updated in each iteration, it is not stable. Since both the target value and the predicted value are calculated by the same neural network, this learning process becomes unstable due to dynamical changes in the target, and in the worst case, the Q function diverges [184].

In deep Q-learning algorithm, it is suggested to have a separate network as a target network in deep Q-learning methods. The main training neural network (Q network) is used to update the parameters of the network, and the target network (Q' network) can be used to calculate the target value which is the same as the original training network, but all the parameters are frozen and fixed. After some specific iterations z (parameter of update frequency), the parameters of the first network can be copied to the target network. The goal of this process is to fix the Q-value targets temporarily, so there is a stable target. Figure 6.13 shows two networks of the deep Q-learning algorithm.

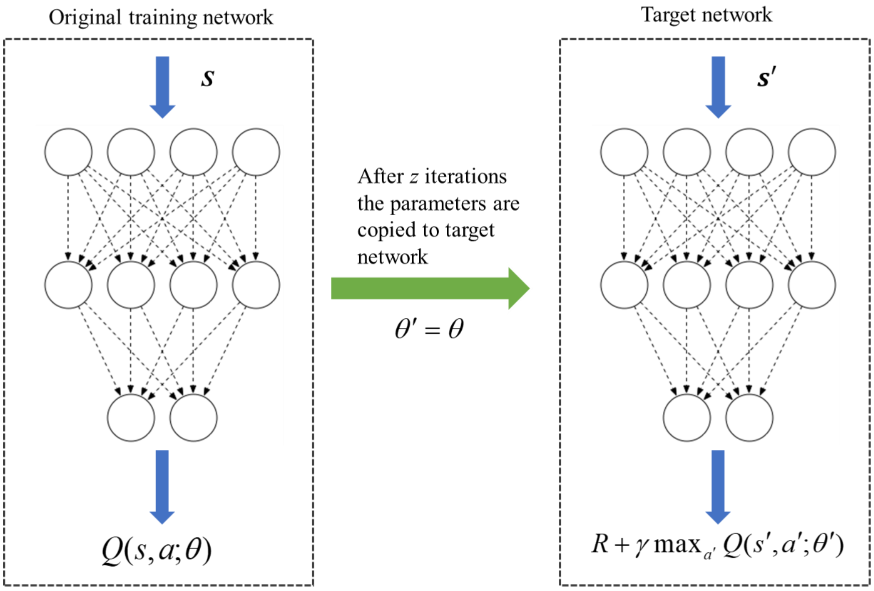


Figure 0.13 Two network of deep Q-learning algorithm

The approximation function for Q-value in a deep Q learning network is denoted as $Q(s,a;\theta)$, where θ represents the trainable parameters of the main network and θ' is the parameters for the target network. These parameters are updated according to the gradient method, as shown in Equation 6.26. Table 6.10 shows the algorithm of deep Q-learning.

$$\theta \leftarrow \theta + \alpha(Q(s,a;\theta) - \text{target}_{Q-value})\nabla_{\theta} Q(s,a;\theta) \qquad (0.26)$$

Table 0.10 Deep Q learning algorithm with two networks

Algorithm 2: Deep Q-learning
Initialize the memory for buffer
Initialize the original training network with random parameters θ
Initialize the target network with random parameters $\theta' = \theta$
For episode 1 to ***N***:
Initialize s
For t =1 to T:

Action, a = $\begin{cases} \text{random action} & \text{with } \varepsilon \text{ probability} \\ \arg\max_a Q(s,a;\theta) & \text{otherwise} \end{cases}$
Take action a, observe R, $\boldsymbol{s}'$
Set $\boldsymbol{s}' = \boldsymbol{s}$
Store the experience (s, a, R, $\boldsymbol{s}'$) in the memory
Sample random mini batch of experiences (s, a, R, $\boldsymbol{s}'$) from memory
Estimate the target y for the experiences
$y = \begin{cases} -1 & \text{if terminated} \\ r + \max_{a'} Q'(s',a';\theta') & \text{otherwise} \end{cases}$
Perform an optimization algorithm on $(y - Q(s,a;\theta))^2$ with respect to θ
Every z step reset $\theta' = \theta$
End for
End for

1.13.2 Dynamic Maintenance model for a system with an infinite number of degradation states

For most industrial systems, having an appropriate decision-making tool can lead to a substantial amount of savings. A dynamic maintenance plan as a decision-making tool for the maintenance team can suggest different actions based on the degradation level of the system and prevent failure or unnecessary maintenance action, which wastes the useful life of the components. Using artificial intelligence for maintenance can help industries to have a more efficient and cost-effective plan for their systems rather than using a fixed maintenance plan for the whole period of the maintenance contract. Having appropriate dynamic maintenance can suggest the best maintenance action for the system dynamically based on the system status.

Yousefi et al [188] proposed a dynamic maintenance plan for the degrading multi-component systems using a deep reinforcement learning method. By formulating the degradation process of the system as an MDP and using deep reinforcement learning, an agent can be trained to provide a particular maintenance action for each component within the system based on the current deterioration status of the system. The agent is trained by using the interactions of environment and actions, which can be the degradation states and the available maintenance actions. The system should be inspected periodically, and at each inspection time, the agent detects the current degradation state of each component and chooses a maintenance action based on the degradation knowledge. Each maintenance action is evaluated by its associated cost, which can be considered as a negative reward, and subsequently, based on the action taken by the agent, the current state goes to the next state. The process continues until the end of the maintenance time contract or planning period.

It should be noted that at each inspection time, the state information is known for the problem and the terminal state is the end of the planning time horizon for providing the maintenance policy. In fact, if it is required to provide a maintenance policy for T units of time and our inspection duration is τ, then there are $\kappa = \lfloor T/\tau \rfloor$ inspections in our state matrix, so t for the degradation state is between 0 and κ ($0 \le t \le \kappa$). The agent's learning performance is significantly improved when time-awareness of the agent is introduced, by specifically incorporating a time-related space component [189, 190] .

The goal of using deep reinforcement learning for maintenance problems is finding the best maintenance action at each inspection time, based on the degradation level of all the components that minimizes the expected maintenance cost. By using deep Q learning, the outputs of the neural network are the value of taking the maintenance actions which are obtained for the current degradation level. Since the goal of the proposed problem is minimizing the cost, the R_t can be interpreted as the cost function which is the negative value of the reward ($R_t = -\mathrm{C}_t$).

The first step for modeling a maintenance reinforcement learning problem is formulating an MDP which can be defined as a tuple of (S, A, R, γ), where S is the state, which is the combination of degradation levels of all the components in the system, S_t=

(s_{1t},s_{2t}, …, s_{nt}). A is the action set, which is defined by selecting an available maintenance action for each component. R is the reward/cost function and γ is the discount factor which has a value between 0 and 1.

In this study, it is assumed that the system is degrading based on the gamma process, which is a stochastic process with the monotonic increasing property. In the previous study, Yousefi et al. [100] formulated the states of MDP problem such that the degradation of systems is divided into different regions based on some predetermined thresholds. However, defining some thresholds for degradation is not a realistic assumption and approach for the systems which are degrading over time. Moreover, specifying the region thresholds is a big challenge, which needs some professional knowledge about system degradation. In this study, these assumptions are relaxed, and there are no predefined thresholds for determining the states of the MDP problem. It is assumed that at each inspection time, the combination of the degradation level of all the components can be considered as the system states of the MDP. In this case, there will be an infinite number of states, and it is necessary to use deep reinforcement learning to solve the problem. At the beginning, for each component, the degradation level is 0, which indicates the component is new and based on the selected maintenance action and its effect on the degradation level, its next state can be calculated.

In this study, it is assumed that each component fails due to soft failure or hard failure. If the total degradation level of the component i is greater than a failure threshold (H_i), the component is failed, and it should be replaced. The probability that the component is failed (P_{Fi}) in an interval τ can be calculated as follow:

$$\begin{aligned} P_{F_i}(\tau) &= 1 - \sum_{m=0}^{\infty}\Big[P\Big((X_i(\tau) + x_{t_i} + \Omega_i(\tau)) < H_i\Big) \mid N(\tau) = m\Big]P(N(\tau) = m) \\ &= 1 - \sum_{m=0}^{\infty}\left[\int_0^H G(H_i - y - x_{t_i}; \alpha_i\tau, \beta_i) f_Y^{<m>}(y)dy\right]\frac{\exp(-\lambda\tau)(\lambda\tau)}{m!} \end{aligned} \quad (0.27)$$

Where $f_Y^{<m>}(y)$ is pdf of the sum of m independent and identically distributed (i.i.d) Y_i, and $G(x;\alpha_i t,\beta_i)$ is the cumulative distribution function of $X_i(t)$, and x_{t_i} is the current degradation level of component i at the beginning of the inspection interval.

In this study, 5 different maintenance actions are considered for each component. Table 6.11 shows the description of each action in this study. The last column of Table 6.11 shows what is the next state of each component based on each action.

Table 0.11 The maintenance actions

Action	Description	Description of action Effects	Effects on the degradation level (X_t)
0	Do nothing	The system degrades more based on the gamma process	Based on Equation (9)
1	Imperfect repair	The system is repaired but the repair was not perfect	Based on Equation (10) and $\upsilon_2 = 0.6$
2	repair	The system is repaired	Based on Equation (10) and $\upsilon_2 = 0.5$
3	Imperfect replacement	The system becomes very close to new, but the degradation level is not zero	Based on Equation (10) and $\upsilon_2 = 0.3$
4	Replacement	The system is as good as new and the degradation level goes to zero.	$X_{t+1} = 0$

If action 0 is selected at any point for component i, it may fail with a probability of having degradation level greater than its failure threshold or hard failure. If it does not fail, it degrades more, and the next state can be calculated, but if it fails, it should be replaced with a new one and the next state should be 0. Therefore, for action 0, the next state for component i can be calculated as follow:

$$X_{t+1_i} = \begin{cases} X_{t_i} + \dfrac{\beta_i^{\alpha_i(\tau)} x^{\alpha_i(\tau)-1} \exp(-\beta_i X_{t_i})}{\Gamma(\alpha_i(\tau))} + \Omega_i(\tau) & \text{with probability } 1 - P_{F_i}(\tau) \\ 0 & \text{with probability } P_{F_i}(\tau) \end{cases} \tag{0.28}$$

Moreover, if action 1, 2, or 3 is selected the next state can be calculated as follow where the coefficient υ_k is corresponding to the degradation improvement based on action *k*.

$$X_{t+1_i} = \begin{cases} \left(\upsilon_k * X_{t_i}\right) + \dfrac{\beta_i^{\alpha_i(\tau)} x^{\alpha_i(\tau)-1} \exp(-\beta_i X_{t_i})}{\Gamma(\alpha_i(\tau))} + \Omega_i(\tau) & \text{with probability } 1 - P_{F_i}(\tau) \\ 0 & \text{with probability } P_{F_i}(\tau) \end{cases}$$

(0.29)

At each time, the state of each component is determined and the next state can be calculated based on different actions. Each action has a fixed cost for all the components, such as 0 for action 0 which indicates "do nothing", c_{m1} for action 1 (imperfect repair), c_{m2} for action 2 (repair), c_{R1} for action 3 (imperfect replacement) and c_{R2} for action 4 (replacement). In addition, there is a penalty cost of downtime if the system fails C_ρ. This penalty cost is used to describe the loss production of the system when it is down and failed.

At each inspection time *t*, each action $A_t = a$, is the combination of maintenance actions of all the components *n*, $A_t = a = (\sigma_{d0}, \sigma_{d1}, ..., \sigma_{dn})$, and there is a specific cost associated with each action in this tuple. In the tuple of maintenance actions $a = (\sigma_{d0}, \sigma_{d1}, ..., \sigma_{dn})$, each $\sigma_{d\,i}$ for component *i*, is a binary variable for actions of {"do nothing: σ_{0i}", "imperfect repair: σ_{1i}", "repair: σ_{2i}", "imperfect replacement: σ_{3i}", "replacement: σ_{4i}"}, where $\sigma_{0i} + \sigma_{1i} + \sigma_{2i} + \sigma_{3i} + \sigma_{4i} = 1$. So, the total cost can be calculated as follow:

$$C_t = \sum_{i=1}^{n} \left(C_{m1}\sigma_{1i} + C_{m2}\sigma_{2i} + C_{R1}\sigma_{3i} + C_{R2}\sigma_{4i}\right) + C_\rho \omega \qquad (0.30)$$

ω is a binary variable indicating the system failure. If the system fails, a penalty cost of C_ρ is added to the cost function. For each maintenance problem, failure states should be defined based on the system configuration and the system structure function. For a series system, if any component fails within the system, the system fails. So, at each time, the degradation level of all the components should be compared to their own failure threshold, and if any component *i*, has a degradation level greater than the failure threshold H_i, the system is detected as failure, $\omega = 1$ and the penalty cost is considered in the cost function. For a parallel system, the system fails if all the components are failed. Therefore, at each time, if the degradation level of all the components are greater than their failure threshold the system is failed and $\omega = 1$.

Since there is an infinite number of states for the MDP problem, a deep Q learning algorithm can be used as it is described in the previous subsection. By using the deep Q learning algorithm, a neural network is trained and used as a decision-making tool for the maintenance team to predict what is the best action based on the current state of the system. At the time of prediction, the current state should be given to the network as input and the output of the network is the Q values for all the actions and the maximum of Q values should be selected as the best maintenance action to be implemented.

1.13.3 Numerical example for dynamic maintenance using deep Q-learning

To present the performance of the proposed dynamic maintenance model, a degrading multi-component system is considered in this section. Each component in the system is subject to degradation and shocks. It is assumed that there is a fixed inspection interval of 2 time units (e.g., months, weeks, …), and the objective is to find the optimal dynamic maintenance actions for a time horizon of 50 time units. Table 6.12 shows the parameter for failure processes of the system.

Table 0.12 System parameters for dynamic maintenance using deep Q-learning

Parameters	Component 1	Component 2	Component 3
H_i	30	33	26
α_i	0.3	0.2	0.6
β_i	1.2	1.1	1.1
λ_0	2.5×10^{-5}		
Y_{ij}	$Y_{ij} \sim N(0.5, 0.1^2)$	$Y_{ij} \sim N(0.55, 0.1^2)$	$Y_{ij} \sim N(0.6, 0.1^2)$

Figure 6.14 shows the degradation behavior of each component within the system, and the dash lines show their failure thresholds. As it is shown in Figure 6.14, component 3 degrades faster compared to component 1 and 2, and since its failure threshold is lower than others the probability that it fails is higher than the other components. In fact, component 3 is more prone to failure than other components.

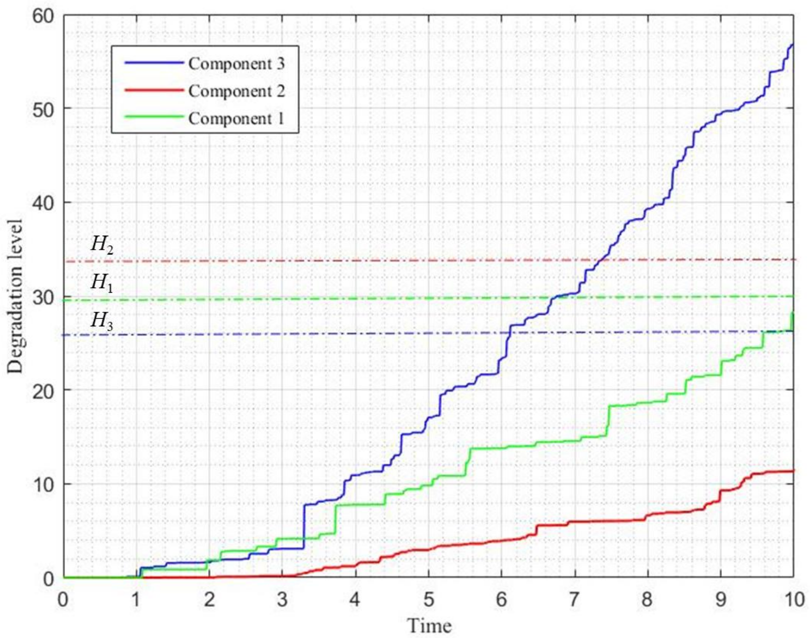


Figure 0.14 Degradation of components of a system

At each inspection time, a maintenance action should be selected from Table 6.13. Each maintenance action has a cost which should be calculated based on the parameters in Table 28. The discount factor is $\gamma = 0.9$, learning rate is $\theta = 1$ and $\varepsilon = 0.01$.The neural networks (main and target) have a total of 2 hidden layers. The hidden units in each layer are 128. The target update frequency is every 5,000 episodes (*z*=5000).

Table 0.13 Cost of maintenance actions

Imperfect repair (c_{m1})	Repair (c_{m2})	imperfect replacement (c_{R1})	Replacement (c_{R2})	Downtime cost (C_{ρ})
40	100	200	300	1000

Using the proposed method, the optimal maintenance action is found for each state of each component. It is assumed that there are three components in the system, and for each component there are 5 actions. Therefore, the action at each time step is a combination of actions for each component. $a_t = (a_1, a_2, a_3)$. For example, action (1,2,4) indicates that action 1 (‘imperfect repair”) should be implemented on component 1, action 2 (“repair”) on component 2 and for component 3, action 4 (“replacement”) should be implemented. Table 6.14 shows some of the actions and the action combination scenarios. In total there are 5^3 =125 number of actions for this paper, where each action indicates a combination of actions for components in the system.

Table 0.14 Action set for dynamic maintenance problem using deep Q-learning

Action scenario	Component1	Component 2	Component 3
(0,0,0)	Nothing	Nothing	Nothing
(0,0,1)	Nothing	Nothing	Imperfect repair
(0,0,2)	Nothing	Nothing	Repair
(0,0,3)	Nothing	Nothing	Imperfect replacement
(0,0,4)	Nothing	Nothing	Replacement
(0,1,0)	Nothing	Imperfect repair	Nothing
(0,2,0)	Nothing	Repair	Nothing
(0,3,0)	Nothing	Imperfect replacement	Nothing
…	…		
(2,3,4)	Repair	Imperfect replacement	Replacement
(3,3,4)	Imperfect replacement	Imperfect replacement	Replacement
(4,1,2)	Replacement	Imperfect repair	Repair
(1,4,2)	Imperfect repair	Replacement	Repair
(4,4,3)	Replacement	Replacement	Imperfect replacement
(4,4,4)	Replacement	Replacement	Replacement

By using a deep Q learning algorithm, a neural network is trained on the past experiences and can provide the best action for each state of the system. The input of the neural network can be a combination of all the component's degradation levels and the output is the best action which is a combination of actions for all the components. This neural network can be a dynamic decision-making tool for the maintenance team to find what is the best maintenance action at each state.

In this study, two system configurations of series and parallel are considered and the best maintenance actions for different states of both configurations are computed using the proposed maintenance model. For a series system, if any component fails in the system, the whole system is failed, but in the parallel system all the components must fail to make the whole system fail.

Table 6.15 shows how the proposed model provides the best actions for the components of a series system based on the degradation of all the components. Some different combinations of the degradation level of components are simulated and given to the proposed model as inputs and the best maintenance actions for all the components are shown for each state.

Table 0.15 Optimal maintenance actions for different scenario of a series system

Scenario number	Degradation state	Action description		
		Component 1	Component 2	Component 3
1	(1,1,1)	Nothing	Nothing	Nothing
2	(5,5,10)	Nothing	Nothing	Imperfect repair
3	(0,10,0)	Nothing	Nothing	Nothing
4	(0,20,0)	Nothing	Nothing	Nothing
5	(20,0,0)	Repair	Nothing	Nothing
6	(0,0,20)	Nothing	Nothing	Replace
7	(28,10,10)	Replace	Nothing	Imperfect repair
8	(10,28,10)	Nothing	Imperfect replace	Imperfect repair
9	(15,15,15)	Imperfect repair	Nothing	Repair
10	(20,20,15)	Repair	Nothing	Repair
11	(25,25,10)	Replace	Repair	Imperfect repair
12	(23,27,25)	Replace	Imperfect replace	Replace
13	(25,30,30)	Replace	Replace	Replace

As it is shown in Table 6.15, based on the degradation stage of all the components, the best maintenance action that should be implemented on each component is found. For Scenario 1, which shows all the components are new, the best action for all of them is “do nothing”. For Scenario 2, all the components are degraded, but component 1 and 2 are still not degraded enough, so the best action for them is “do nothing”, however component 3 needs action 1 “imperfect repair” because it is degraded more and its probability of failure is more than other components.

By comparing Scenario 4, 5, and 6, it can be concluded that the degradation rate and the failure threshold have an impact on the optimal maintenance actions. In Scenario 4, the system state is (0,20,0), which indicates, the first and third components are new, but component 2 has degradation level 20. However, the best action is found (0,0,0), which means "do nothing" on all the components. Although the degradation level of component 2 is high but based on the degradation parameter of Table 6.12 and Figure 6.14, it can be concluded that component 2 has a very slow degradation speed, and since its failure threshold is 33, it is far from failure, and no maintenance action is required to implement. However, in scenario 5, the state is (20,0,0), which indicates components 2 and 3 are new, but component 1 is degraded and has a degradation level of 20. Based on the output of the neural network, the best action is (2,0,0) which suggests, "do nothing" for components 2 and 3, and "repair" should be implemented on component 1. In Scenario 6, the state is (0,0,20), components 1 and 2 are new, and component 3 has a degradation level of 20. The best action is "do nothing" for component 1 and 2, and "replace" for component 3. Although the degradation level of the most degraded component in Scenario 4, 5 and 6 are the same and equal to 20, the optimal action for different components with the same degradation level is totally different. The reason for this difference in optimal action is the difference in degradation rate and failure thresholds of the different components, which is taken into account in the proposed maintenance model.

Moreover, Scenario 12 and 13 also show how the low degradation rate of component 2 influences the best maintenance action for this component. In Scenario 13, all the components are very close to their failure thresholds, and the best maintenance action is "replace" them. In Scenario 12, all the components are degraded, but their degradation level is a little bit better than their degradation level in Scenario 13. The best action for components 1 and 3 is "replace", which is the same as Scenario

13. However, for component 2, the best action changes from “replace” to “imperfect replace”, which can be interpreted based on the degradation rate and failure probability of component 2 compared to other components.

Table 6.16 shows how the proposed model provides the best actions for the components of a parallel system based on the degradation of all the components. The degradation parameters of the components are the same as the series system, which is shown in Table 6.12. To show that the proposed maintenance model is appropriate for different system configuration, and how it is different based on the system configuration, the same scenarios as the series system are given into the neural network for parallel systems.

Table 0.16 Optimal maintenance actions for different scenario of a parallel system

Scenario number	Degradation state	Action description		
		Component 1	Component 2	Component 3
1	(1,1,1)	Nothing	Nothing	Nothing
2	(5,5,10)	Nothing	Nothing	Nothing
3	(0,10,0)	Nothing	Nothing	Nothing
4	(0,20,0)	Nothing	Nothing	Nothing
5	(20,0,0)	Nothing	Nothing	Nothing
6	(0,0,20)	Nothing	Nothing	Nothing
7	(28,10,10)	Imperfect repair	Nothing	Nothing
8	(10,28,10)	Nothing	Imperfect repair	Nothing
9	(15,15,15)	Nothing	Nothing	Nothing
10	(20,20,15)	Nothing	Imperfect repair	Nothing
11	(25,25,10)	Nothing	Imperfect repair	Nothing
12	(23,27,25)	Imperfect repair	Imperfect repair	Imperfect repair
13	(25,30,30)	Imperfect repair	Imperfect repair	Imperfect repair

In a parallel system, the system fails if all the components fail, and if one component is new in the system, the whole system can work no matter if the other components are new or degraded. As it is shown in Table 6.16, when at least one component is new in the combination of components degradation, the best actions for most of them are "do nothing". Moreover, there are no "replace" actions in the system. Since the cost of different actions are different, and the lowest maintenance cost is for "imperfect repair", most of the time, the best action is suggested as "nothing" or "imperfect repair". Moreover, based on all the optimal actions for different scenarios in Table 6.16, it can be concluded that the agent tried to implement the maintenance action somehow that it keeps one of the components new and far from its failure. In fact, the proposed method provides an intelligent decision-making tool, which can suggest the best maintenance action based on the degradation of components and their effect on the system failure. In different system configurations, the system fails in different ways, and the proposed method can find the optimal actions on all the components to avoid the system failure and increase the useful life of the system.

By comparing Tables 6.15 and 6.16, it can be concluded that since component 2 has the lowest degradation rate, it fails in a longer time than other components, and in a series system, most of the time, there is no need to implement any action on component 2. However, in a parallel system, it is suggested to implement an action on components 2, instead of other components when all of them are degraded. The reason is, the costs of maintenance actions for all the components are the same, so the agent tried to implement the action on a component that has the slowest degradation rate and fails in a longer time. It is more illustrated in Scenario 10 of both Tables 6.15 and 6.16. In Scenario 10, the combination of components degradation is (20,20,15), and for a series system, the best action is (2,0,2), which indicates, components 1 and 3 need a repair action, but no maintenance action is needed for component 2. However, for a parallel system, the best action for the same system state (20,20,15), is (0,1,0), which is doing nothing on components 1 and 3 and implementing “imperfect repair” on component 2.

Comparing Scenarios 11 and 13 in Table 6.16, component 1 has the same degradation level of 25 in both scenarios. However, the best action for component 1 is different. In Scenario 11, the best action is “nothing” while in Scenario 13 it is “imperfect repair”. Therefore, it can be deduced that the best action for each component depends on the degradation level of other components and the system configuration. In Scenario 13, component 2 and 3 are more degraded than Scenario 11, so the probability of system failure is different, and the actions for component 1 with the same degradation level is different.

In Scenario 13 (25,30,30), all the components are degraded and close to their failure threshold. For the series system, all of them need a replacement to prevent system failure. However, for the parallel system, the best action for all of them is “imperfect repair”. Since the system failure happens when all the components fail, it is not beneficial to implement a replacement on a component even if it is close to its failure threshold.

The number of neurons in each layer of the neural network can influence the performance of the agent. To find the best number of neurons in each layer of the main and target network, the average maintenance cost (reward) for different numbers of neurons is calculated using the proposed model. Figure 6.17 shows the best number of neurons for each layer of the neural network is 128 neurons, because it provides a better performance compare to models with other number of neurons.

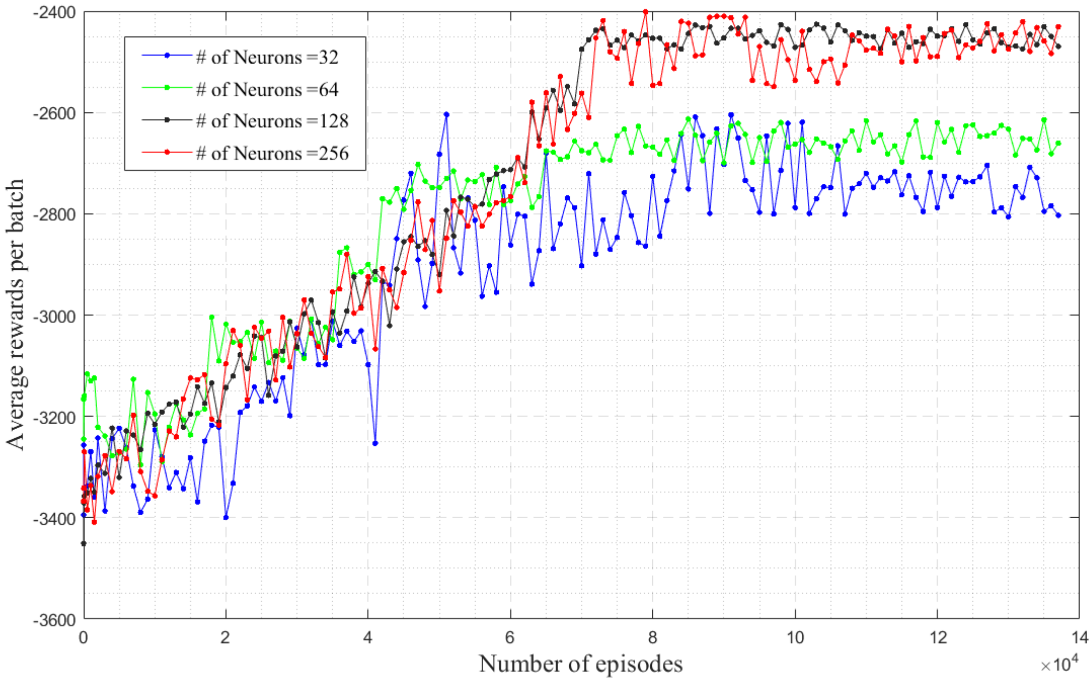


Figure 0.15 Average reward for different number of neurons

If the networks have 256 neurons, the results will be similar to 128, so between these two models, the simplest one should be selected. Moreover, the network with 128 neurons is more stable than 256 neurons, and the reason might be due to the overfitting of the network.

To find the effect of the update frequency of the target network, the average reward (cost) is calculated for different values of parameter *z*, after running for 10^5 number of episodes. Figure 6.16 shows the effect of *z* (update frequency for target network). As it is shown in Figure 6.16, the average reward does not converge when the update frequency of the target network is z=1 or z=10, and it does not progress well. However, for large enough update parameter such as z=1000 or z=5000 the deep Q network converges after 10^5 number of episodes.

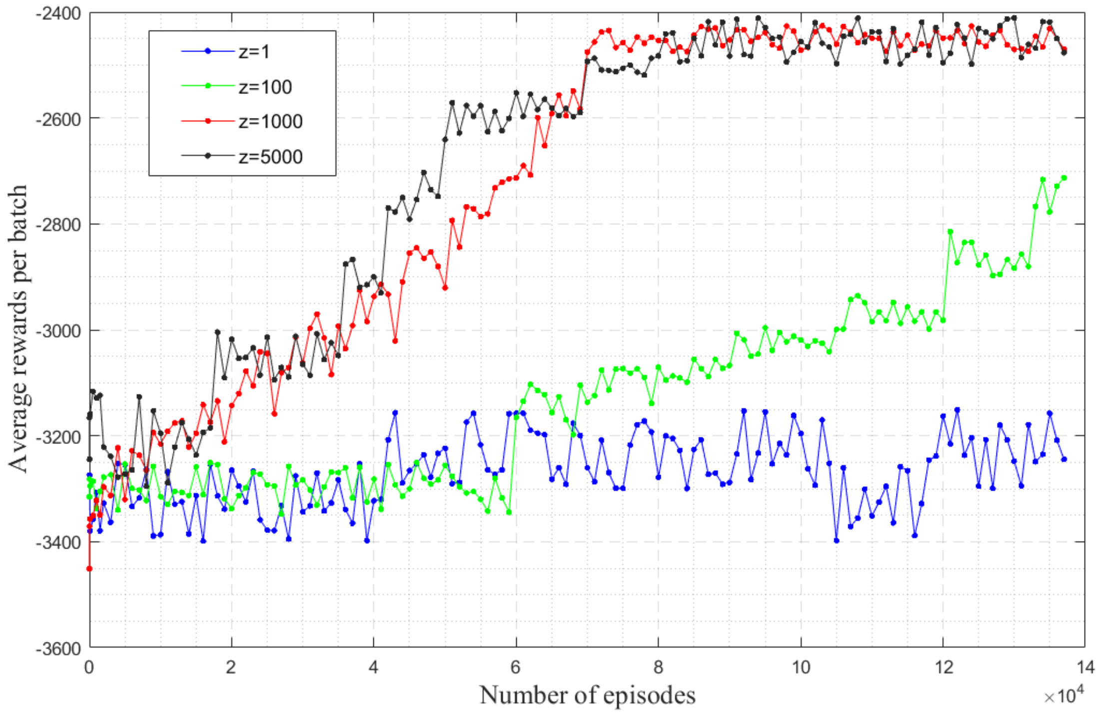


Figure 0.16 Effect of update frequency of target network

1.14 Dynamic Maintenance Policy for Multi-State Production Systems

In production systems, maintenance activity is very important for sustaining the normal work and efficient operation of production machines [191]. Having an appropriate maintenance model can reduce the overall maintenance cost and increase the availability and reliability of production systems. By restoring the system to a better functioning state, maintenance activities can prevent the system failure and improve the production process. However, implementing the maintenance actions frequently reduce the productivity of a production system. Therefore, production and maintenance plans are often suboptimal with respect to the objective of minimizing the combined maintenance and production cost [192]. Providing a maintenance policy that has a balance between the frequency of maintenance actions and the production state of the machine is the goal of many recent researches. At any time, that maintenance actions should be implemented, spare parts are needed for some actions, and the maintenance activities will be delayed if there are not enough spare parts available. Delay in maintenance may cause sudden failure due to the system deterioration and consequently, an expensive penalty cost due to loss of production. On the other hand, holding too many spare parts has inventory costs. Also, spare parts may become useless due to chemical interaction with the environment and on-shelf deterioration during long-term storage.

In this study, a dynamic condition-based maintenance model is proposed for a degrading production system considering the spare parts and their stochastic arrival times. At each inspection time, the degradation state of the system is determined, and maintenance actions should be selected based on the current state. The degradation state contains the estimated machine age, the virtual age, and the state of spare parts. It is assumed that the machine state can be estimated by quality information or other measurable indicators. There may be some error during the estimation process, which makes the obtained machine state possibly inaccurate. To compensate for the possible error, the virtual machine age is simultaneously considered as part of the maintenance decision and production cost calculation. Different maintenance actions that are considered in this study are replacement and some imperfect repairs. Spare parts ordering and the uncertainty of the order arrivals are also considered in this study.

Yousefi et al [193] developed a dynamic maintenance problem of a production system considering different maintenance actions, spares part ordering, delay in spare parts arrival and uncertainty of maintenance implementation. They formulated the problem as a time-discrete Markov decision process, and it is solved by using the Q-learning algorithm. Using an artificial intelligent method for maintenance problems, a more time-efficient and cost-efficient decision-making tool is developed compared to traditional optimization solutions. Having a dynamic maintenance model reduces the maintenance costs by removing some unnecessary maintenance implementations and using the effective useful life of a system.

1.14.1 Production System description and maintenance model

In this study, a degrading multi-state production system is considered, and a dynamic maintenance policy is proposed to make a decision on spare parts ordering and maintenance action based on the current state of the system. The problem is formulated as a Markov decision process (MDP), and it is solved by using a reinforcement learning algorithm. In using reinforcement learning for maintenance MDP problems, the agent is trained by using the interactions with the environment and actions, which can be the degradation states and the available maintenance actions. At each time step which can be the inspection time, the agent detects the current degradation state of the system and chooses a maintenance action based on the system degradation knowledge. Each maintenance action is evaluated by its associated cost which can be considered as negative reward, and subsequently, based on the action taken by the agent the system goes to the next state. The process continues until the end of the maintenance time contract.

MDP can be defined as a tuple of (S, A, P, C, γ), where S is the state of the system. A is the action set, which is defined by selecting an available maintenance action and spare part ordering. P is the transition probability of going to the next states, and C is the cost function which is the negative value of system reward ($C = -r$). It should be noted that at each inspection time, the state information is known for the problem and the terminal state is the failure state of the system.

(1) System states

In this study, the machine state can be estimated at discrete time epochs based on some inspections and analysis of quality information or other measurable indicators.

The estimated machine states are classified into N levels, denoted by 0, 1, …, N - 1. The state '0' means the new or as-new state, and the state 'N - 1' means failure. The other states are intermediate conditions, at which the machine is deteriorated, but still operates with increasing production cost, and with larger probability to fail. In this study, if the machine state is found to be at state $i\ (0 \le i \le N-1)$ at a time epoch, the state $j\ (i < j \le N-1)$ can be reached at the next time epoch with known transition probability p_{ij}. In fact, the transition probability is zero between two states i and j when $j < i$, which means that the machine cannot improve on its own, and any state j can be reached from any state i where $i < j$. In this study, for simplicity, the word 'machine state' refers to 'estimated machine state' in the remainder of this paper. The machine virtual age is also used to consider the possible error of state estimation.

The system state is denoted by $S = (\tau, t, i, o)$, which contains the machine virtual age t, the machine state i, and the spare part state o and τ, which is the inspection time step . The machine's virtual age t is described by discrete numbers in the range $[0, T_{\max}]$. $T_{\max}$ is set as the maximum value of the machine's virtual age. When $T_{\max}$ is reached, a replacement is required, similar to a system failure (being at the state $N-1$). The possible value of machine state i could be a state from $\{0,1,...,N\text{-}1\}$. The spare part state o may be 0, 1, or 2. '0' represents that the action 'ordering' has not been selected. '1' indicates that the spare part is ordered but has not arrived yet. '2' represents the situation where the ordered spare part has arrived.

It should be noted that at each inspection time, the state information is used for the problem formulation and the terminal state is the end of the planning time horizon. In fact, if it is required to provide a maintenance policy for T_{Main} units of time and our inspection duration is l, then there are $\kappa = \left\lfloor T_{Main} / l \right\rfloor$ inspections in our state matrix, so τ

is between 0 and κ $(0 \le \tau \le \kappa)$.

The virtual age of the machine is a positive function of its real age. It is assumed that a maintenance activity can restore the machine, and subsequently, it reduces the failure intensity or machine virtual age [194, 195]. In this paper, different levels of imperfect maintenance can result in different reductions of the virtual age. Because the virtual age can also reflect the machine situation depending on past maintenance, it is a part of the proposed policy to compensate for possible errors during the system state estimation.

(2) Maintenance actions

At the beginning of each time epoch, a decision about the maintenance actions are made based on the current system state. The optional action a is chosen from five possible actions, $\{0,1,2,3,4\}$, which successively represent “no action”, “maintenance level 1”, “maintenance level 2”, “replacement”, and “ordering”. When “no action” is selected, there are no maintenance activities carried out and the system continues degrading. Then, at the next time epoch, the system state possibly changes to a more degraded state. The actions “maintenance level 1” and “maintenance level2” both represent implementing an imperfect maintenance action immediately which takes one time unit. The imperfect maintenance actions reduce the virtual age at different degrees, and simultaneously improve the machine state, or make it remain at the current state with known probability. “maintenance level 1” and “maintenance level 2” can improve the machine state and virtual age. For example, when the machine is in state *i*, with virtual age of *t*, by implementing the “maintenance level 1”, it transitions to state j_1 (where $0 < j_1 \le i$), with probability of $p_{ij_1}^1$ and its virtual age becomes $\upsilon_1 t$ (where $0 < \upsilon_1 < 1$). Similarly, the machine state and virtual age would be respectively changed to be j_2 (with probability $p_{ij_2}^2$) and $\upsilon_2 t$ after performing a 'maintenance level 2' ($0 < j_2 \le i$, $0 < \upsilon_2 < 1$). It is assumed that υ_1 is larger than υ_2 because implementing "maintenance level 2" can improve the virtual age more than "maintenance level 1"

When action 0 ("no action") is selected, the system goes to the next state with some known probability. The action 4 ("ordering") is to order spare parts immediately. It is assumed that only one spare part is ordered each time, and the lead time follows a known geometric distribution with parameter λ_s. As soon as the ordered spare part arrives, the action "replacement" is selected, and the perfect maintenance action is implemented immediately. It is assumed that the time duration of the replacement follows a known geometric distribution, and its success rate during a time unit is λ_r.

(3) System's cost function

The cost function/reward function can be calculated using the associated cost for each action on the components, and all the other costs in the system such as penalty cost of failure, shortage cost, production cost and spare part ordering cost. In this paper, the reward function r is defined as negative value of total cost ($r = -C$). The considered system cost includes the maintenance cost, the production cost, the shortage cost, and the spare parts ordering cost. It is assumed that implementing imperfect maintenance takes one time unit and the costs are c_{pm1} and c_{pm2} for "maintenance level 1" and "maintenance level 2". The cost of implementing "replacement" is c_{cm}, but it may also be larger than c_{cm} due to it's the success rate λ_r of replacement action.

It is assumed that the production cost depends on the system state and the virtual age. To consider the effect of both system state and virtual age, Equation (6.31) is used for production cost, $c_{P(t,i)}$, where i represents the system state and virtual age is t.

Using Equation (6.31), the higher increase in machine state and virtual age can result in a more expensive production cost. This implies the requirement of the proposed policy to keep the two factors at a low level. However, any other type of model or different value set can be considered to estimate the effect of system state and virtual age.

$$c_{P(t,i)} = \max\{0.6(i+2), 0.6t\} \tag{0.31}$$

When the machine undergoes maintenance or arrives at the failure state, a shortage cost c_r is incurred during that time unit. Implementing imperfect maintenance ("maintenance level 1" and "maintenance level 1") takes only one time unit, but because of the success rate of replacement λ_r, a delay may occur for implementing the replacement, so it may have more than one shortage cost c_r. The cost of ordering a spare part per time is c_s. Therefore, c_s is considered as soon as the action 'ordering' is selected. Moreover, when the system is failed, it cannot perform its function and there is a penalty cost due to failure c_f which is similar to shortage cost, but larger than shortage cost. However, by defining the penalty cost c_f, the loss of production and penalty cost are different for implementing maintenance versus for system failure.

At any inspection time (τ), based on the current state of the system $S = (\tau, t, i, o)$, the machine state is the $S = (t, i, o)$ and the expected cost from the current time epoch is denoted by $C(t, i, o)$. At any inspection time (τ), when system state is $S = (t, i, 0)$ for any $0 \le t < T_{\max}, 0 \le i < N-1$, the action "replacement" cannot be selected, because there are no spare parts available (*o* = 0). When the current state is $S = (t, i, 1)$, the spare part is ordered but has not yet arrived, so the only action that can be selected is "no action," and wait until it arrives. Therefore the cost for current state based on the possible actions are as follow:

Table 0.17 Cost function based on current state and possible actions

Current state	Action	Associated Cost
Any state other than failure states, where there is no spare part available $S(t,i,0)$ $(0 \le t < T_{\max}, 0 \le i < N-1)$	$a=0$	$C(t,i,0)=C_p(t,i)$ (0.32)
	$a=1$	$C(t,i,0)=C_{pm1}+C_r$ (0.33)
	$a=2$	$C(t,i,0)=C_{pm2}+C_r$ (0.34)
	$a=4$	$C(t,i,0)=C_p(t,i)+C_s$ (0.35)
Any state other than failure states, where spare part is ordered but has not arrived $S(t,i,1)$ $(0 \le t < T_{\max}, 0 \le i < N-1)$	$a=0$	$C(t,i,1)=C_p(t,i)$ (0.36)
Any states, where spare part has arrived $S(t,i,2)$	$a=3$	$C(t,i,2)=C_{cm}+C_r$ (0.37)
For failure state, where there is no spare part available $S(t,N-1,0)$ $(0 \le t < T_{\max})$ or $S(T_{\max},i,0)$ $(0 \le i < N-1)$	$a=4$	$C(t,N-1,0)=C_f+C_s$ (0.38) $C(T_{\max},i,0)=C_p(T_{\max},i)+C_s$ (0.39)
For failure state, where spare part is ordered but has not arrived $S(t,N-1,1)$ $(0 \le t < T_{\max})$ or $S(T_{\max},i,1)$ $(0 \le i < N-1)$	$a=0$	$C(t,N-1,1)=C_f$ (0.40) $C(T_{\max},i,1)=C_P(T_{\max},i)+C_s$ (0.41)

Equation (6.40) and (16.41), are associated with failure cost, however, when system is in state N - 1 it cannot perform its designated function due to the failure, so there is a penalty cost of C_f for Equation (6.40), but when the system is in a state with virtual age of T_{max}, it can be considered as a failure and it should be replaced, but it can still work with a large amount of production cost. Therefore, in Equation (6.41) there is no C_f .

(4) Transition probability

In this study, there is a degrading multi-component system. At each inspection time, the current degradation state of the system is detected, and an appropriate maintenance action is selected. Based on the selected maintenance action, the system state transitions to the other state. If at any inspection time, action 0 (“no action”) is selected, the state transitions to any of the next states based on a known probability matrix P, where each P_{ij} in matrix P, describes the transition probability from state *i* to *j*. and the virtual age changes from *t* to *t* +1.

When action 1 “maintenance level 1” or 2 “maintenance level 2” is selected the state *i* transits to state *j* with corresponding probability matrices of P^1 and P^2. For example, when the machine is in state *i*, with virtual age of *t*, by implementing the “maintenance level 1”, it transitions to state j_1 (where $0 < j_1 \leq i$), with probability of $p^1_{ij_1}$ and its virtual age becomes $\upsilon_1 t$ and it would be changed to be j_2 (with probability $p^2_{ij_2}$) and $\upsilon_2 t$ after performing a ‘maintenance level 2’. To reduce the complexity of the mathematical analysis, the virtual age is adjusted to be integer-valued after imperfect maintenance by using a rounding-off method. Figure 6.17 shows a part of Markov decision process for actions 0, 1 and 2 for some of the machine states. At any time epoch, at any state, if “no action” is selected it transitions to any of the next states with a known probability, and if any of the imperfect maintenance actions (“maintenance level 1” and “maintenance level 2”) are selected, the system state transitions to any of the previous states with a probability for that action.

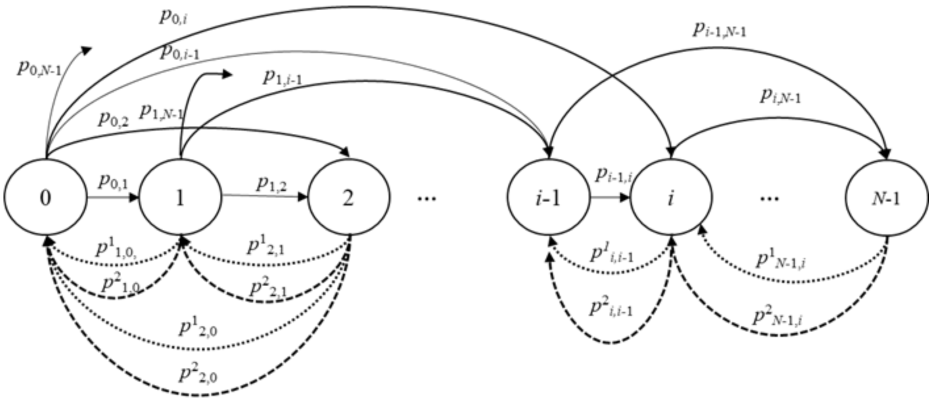


Figure 0.17 Markov decision process for different actions

The action "ordering" is to order spare parts immediately. It is assumed that only one spare part is ordered each time, and the lead time follows a known geometric distribution with parameter λ_s. This means that, until the beginning of the next unit time, the arrival rate of ordered spare part is λ_s. As soon as the ordered spare part arrives, the action "replacement" is selected, and the perfect maintenance action is implemented immediately. It is assumed that the time duration of the replacement follows a known geometric distribution, and its success rate during a time unit is λ_r. This means that after one time unit the machine state and virtual age are changed to be initial values (i.e., the as-new state) with successful rate λ_r, or remain unchanged with unsuccessful rate $(1-\lambda_r)$. Therefore, for any system state with no spare part available, if action 4 "ordering" is selected, the state changes as follows:

$$S(t,i,0)=\begin{cases} S(t+1,\, j,2) & \text{with probability } \lambda_s \times p_{ij} \\ S(t+1,\, j,1) & \text{with probability } (1\text{-}\lambda_s)\times p_{ij} \end{cases} \tag{0.42}$$

When the current state is $S=(t,i,1)$, the spare part is ordered but has not been arrived, so the only action can be selected is "no action" and wait until it arrives. The next state can be calculated as in Equation 6.43:

$$S(t,i,1)=\begin{cases} S(t+1,j,2) & \text{with probability } \lambda_s \times p_{ij} \\ S(t+1,j,1) & \text{with probability } (1\text{-}\lambda_s)\times p_{ij} \end{cases} \tag{0.43}$$

Moreover, since it is assumed that replacement has a success rate λ_r, after selecting action 3 “replacement”, the next state can be calculated as shown in Equation 6.44:

$$S(t,i,2)=\begin{cases} S(0,0,0) & \text{with probability } \lambda_r \\ S(t,i,2) & \text{with probability } (1\text{-}\lambda_r) \end{cases} \tag{0.44}$$

For a maintenance problem, using the Q-learning method provides an algorithm to find the best agent policy for implementing maintenance actions based on the system states which has the minimum cost. At each episode, the value of maintenance actions for all the specific degradation states are calculated and the episode terminates when it reaches the terminal states. The terminal state is the state for the time that is the end of our proposed maintenance policy.

In the Q-learning method, the agent takes one action at any particular state and evaluates its consequences, and by trying actions in all the possible states it learns what is the best action which has the best long run rewards. In Q-learning the value function is updated based on the Bellman equation which is shown in Equation (6.23). $Q(s,a)$ is the expected value of taking action a in state s. Where θ is the learning rate which can have a value between 0 and 1, where 0 means the algorithm is never updated and θ = 1 means the learning occurs quickly. r is the reward (-cost) at time t for moving from state s to s' for action a. γ is the discount factor which can have a value between 0 and 1, it is used to balance immediate and future reward. $\max_a Q(s',a)$ represents the maximum expected future reward. In this study, $Q(s,a)$ is the expected value of taking a maintenance action a, in system state s. The system state is $S=(\tau,t,i,o)$, which contains the machine virtual age t, the machine state i, and the spare part state o and τ, which is the inspection time step. r is the immediate cost of taking a maintenance action, which can be calculated using Equations (6.32) - (6.41) in Table 6.17.

1.14.2 Numerical example of a production system

To demonstrate the performance of the proposed maintenance policy in this study, a conceptual production system is considered which has multiple degradation states. It is assumed that the system has 6 different degradation states, where state 0 indicates the system is new and state 5 is the failure state. The considered maximum value of machine's virtual age is assumed to be 10 time-units ($T_{\max} = 10$). The system is inspected periodically at every one time unit. The maintenance contract duration (T_{Main}) is assumed to be 25 time-units. The success rate of the replacement during a time unit is $\lambda_r = 0.9$ and the arrival rate of the ordered spare part during a time unit is $\lambda_s = 0.6$. The reducing degree of the machine virtual age after completing a maintenance level 1 $\upsilon_1 = 0.6$, and for maintenance level 2 is $\upsilon_2 = 0.4$. The cost of production is calculated as $c_{P(t,i)} = \max\{0.6(i+2), 0.6t\}$, and the other cost parameters are shown on Table 6.18.

Table 0.18 Maintenance cost parameters

Shortage cost	Ordering cost	Cost of maintenance level 1	Cost of maintenance level 2	Cost of replacement	Penalty cost for failure
$c_r = 4$	$c_s = 1$	$c_{pm1} = 1$	$c_{pm2} = 3$	$c_{cm} = 5$	$c_f = 8$

The transition probability matrices for different maintenance actions are shown in Table 6.19. The element $P(i+1, j+1)$ in Matrix P describes the value of p_{ij}, i.e., the transition probability from the machine state i to state j. Similarly, the element $P^1(i+1, j+1)$ and $P^2(i+1, j+1)$ in Matrix P^1 and P^2 describe the values of p_{ij}^1 and

p_{ij}^2 which is the probability of transiting from state i to j after implementing maintenance level 1 and maintenance level 2.

Table 0.19 Transition matrices for maintenance actions

Maintenance action	Transition probabilities
No action	$P = \begin{bmatrix} 0.2 & 0.35 & 0.23 & 0.15 & 0.06 & 0.01 \\ 0 & 0.2 & 0.36 & 0.3 & 0.12 & 0.02 \\ 0 & 0 & 0.18 & 0.5 & 0.26 & 0.06 \\ 0 & 0 & 0 & 0.14 & 0.66 & 0.2 \\ 0 & 0 & 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
Maintenance level 1	$P^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.33 & 0.67 & 0 & 0 & 0 & 0 \\ 0.05 & 0.2 & 0.75 & 0 & 0 & 0 \\ 0 & 0.05 & 0.10 & 0.85 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.95 & 0 \end{bmatrix}$
Maintenance level 2	$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.53 & 0.47 & 0 & 0 & 0 & 0 \\ 0.10 & 0.35 & 0.55 & 0 & 0 & 0 \\ 0 & 0.15 & 0.20 & 0.65 & 0 & 0 \\ 0 & 0 & 0.10 & 0.15 & 0.75 & 0 \end{bmatrix}$

The number of episodes can influence the performance of reinforcement learning algorithms. Finding the best number of episodes is a challenge in reinforcement learning algorithms which can be investigated by trying different numbers of episodes and observing its performance for each scenario. Figure 6.18 shows the average obtained rewards for different numbers of episodes for the conceptual production system. The horizontal axis on this plot is the different percentiles of the total number of episodes, and the average total reward for each batch of episodes is shown on the vertical axis. As it is shown on Figure 6.18, the well-suited number of episodes for this study is 10^5 because it has the best performance compared to other numbers of episodes. Moreover, comparing the result of 10^4 and 10^5, it can be concluded that difference is negligible and increasing the number of episodes after 10^5 is an inefficient use of computational time and resources.

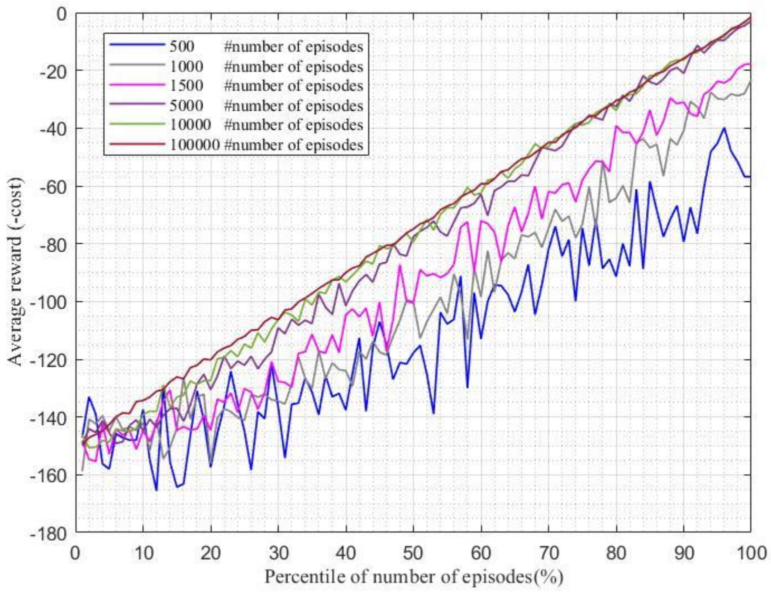


Figure 0.18 Convergence check for the required number of episodes for the system

Using the proposed method, the optimal maintenance action is found for each state of the system. The best maintenance actions for different scenarios are shown in Table 6.20 to illustrate the performance of the Q-learning method in obtaining dynamic maintenance policies. Each state in Table 6.20 is a combination of virtual age *t*, machine state *i*, and the spare part status *o*, $S(t,i,o)$ at each inspection time. Replacement (action 3) is the only action for all states where spare parts arrived $S(t,i,2)$. Doing nothing (action 0) is the only action which can be selected for states such as $S(t,i,1)$ that spare part is ordered but it has not arrived yet. For other states, the optimal action is found using Q table.

Table 0.20 Optimal maintenance actions for different scenario

Scenario number	System state	Optimal action number	Scenario number	System state	Optimal action number
1	(0,0,0)	0	8	(7,2,2)	3
2	(1,1,0)	0	9	(7,4,0)	2
3	(2,2,0)	1	10	(7,3,0)	4
4	(3,1,0)	0	11	(8,2,0)	4
5	(5,0,0)	1	12	(9,3,0)	2
6	(5,5,0)	4	13	(9,4,0)	2
7	(5,4,0)	2	14	(10,2,0)	4

As it is shown in Table 6.20, for states close to failure states, the best action is action 2 "maintenance level 2" which is restoring the system to a better state with higher probability than action 1 'maintenance level 1 ", Scenario number 7, 9,12, and 13 are examples of these states close to failure states. For states which are very close to the new state, the best action is action 0, "do nothing". In fact, it suggests not to implement any action when the system is new and time to failure is long. When the system is degraded enough, but it still has time to fail, the best action is ordering a spare part (action 4), such as scenario number 10 and 11, where the virtual age of the system is high, but it is not very close to failure, so it is a good time to order a spare part and avoid wasting time for waiting for spare parts delivery.

In Scenario number 5, although the machine state is 0, which is a new state, the virtual age is 5, and the best maintenance action is action 1 "maintenance level 1". The reason can be the production cost. As the virtual age of the system increases, the production system cannot perform as well as a new system, and there is a production cost associated with the virtual age and machine state. Therefore, although the system will not fail soon, it is suggested to repair the system to reduce the virtual age and avoid paying more production costs.

To show how the proposed method can provide a maintenance action dynamically based on the degradation states of the system, different scenarios are shown in Figure 6.19. It shows the different maintenance actions based on the optimal Q-table for the production system. Seven different scenarios are shown in Figure 6.19. Based on the optimal Q-table the best action for each state is shown on the right of each state.

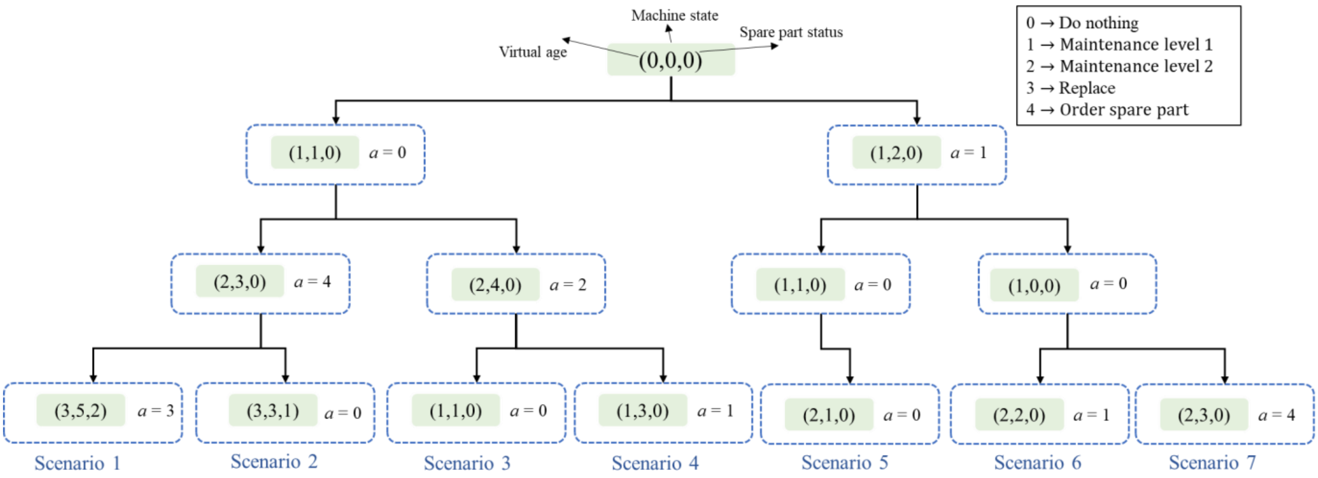


Figure 0.19 Different optimal maintenance policy for different scenarios

As it is shown when the system is almost new, such as states with low virtual age and low machine state numbers, the best action is doing nothing. As the system degrades and it transits to a higher machine state, the best action changes to repair and to order the spare part. For example, for Scenario 2, at the first time step the virtual age and machine state of the system increased by one unit, and the best action is "do nothing", and then it transits to state (2,3,0) which indicates the machine state is close to failure, but the virtual age is still low. However, the best action for this state based on the optimal Q-table is action 4, which is ordering a spare part. Since there is an uncertainty in order arrival, in the next state, the spare part has not arrived, and the system state is (3,3,1); the only action that can be done is waiting for the arrival and "do nothing'. Moreover, in Scenario 1, everything is the same as Scenario 2, except after the spare part is ordered, in the next state is (3,5,2), which indicates the spare part has arrived, and the machine state is in the machine state 5 which is the failure state, and since the spare part inventory is not considered the only action is replacement, action 3, which is the replacement, but since the system is failed there is a penalty cost incurred.

A sensitivity analysis is done on the reduction parameter of the machine virtual age after completing imperfect maintenance. The parameter of maintenance level 2 υ_2 is changed from 0.1 to 0.75 in increments of 0.05, and for each value of υ_2, the parameter of maintenance level 1 υ_1 is set as $(\upsilon_2+0.2)$. The other system parameters as the same as the basic system parameter settings. For each set of (υ_1,υ_2), the average cost is calculated using the Q -learning algorithm and it is shown in Figure 6.20.

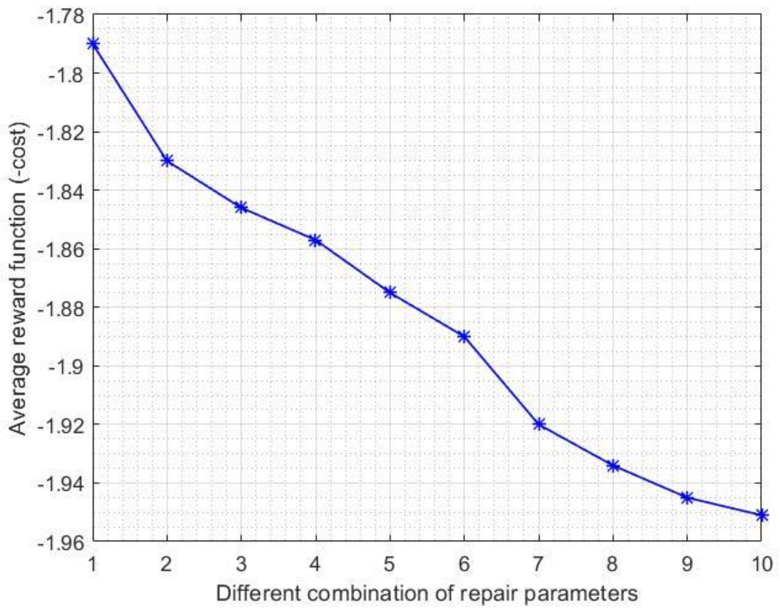


Figure 0.20 Maintenance cost rate sensitivity analysis on (υ_1,υ_2)

As it is shown in Figure 6.20, when the reduction parameters for " maintenance level 1 " and "maintenance level 2" increase, the average cost increases. In fact, by increasing the reduction parameters, these maintenance repairs are not effective and are not able to improve the virtual age and machine state very much, due to the systems cost, especially production cost, the average maintenance cost also increases.

1.14.3 Model validation for dynamic maintenance of a production system

To demonstrate the advantages of the proposed method, the maintenance cost of this numerical example is compared with traditional methods in the literature such as replace-on-failure maintenance and on-condition maintenance. For replace-on-failure, each component is replaced only if the system is detected as failed, and there is no maintenance action before failure. For on-condition maintenance models, different thresholds are considered at each state and by comparing the virtual age and these thresholds at each machine state, a maintenance activity is selected. To compare the maintenance cost of the proposed method and the traditional maintenance models, the same parameters are considered. By running a simulation model for 10^5 replication runs, the maintenance cost of replace-on-failure and on-condition maintenance are found and shown in Table 6.21. As it is shown on Table 6.21 the maintenance cost for the proposed model is lower than the traditional methods.

Table 0.21 Comparison of the proposed method and the traditional methods

Model	Proposed method	On-condition maintenance	Replace-on-failure
Average cost of the system	1.85	4.57	7.34

Model validation

The current research intends to develop new reliability and condition-based maintenance models for multi-component systems with individually repairable components subject to dependent competing failure processes. However, the preference for the proposed maintenance models should be presented by comparing the results to models that are already existing. Model validation is one of the crucial steps to present the preference of the newly proposed models. In this study, the proposed maintenance policies are compared to the existing maintenance models, such as time-based maintenance and replace-on-failure maintenance. In replace-on-failure models, failure is detected by inspection, and if failures are not detected promptly, there is costly downtime. In time-based maintenance models, the maintenance actions are implemented at some specified inspection times, and there are no other maintenance thresholds or conditions for implementing the maintenance actions. By setting some model parameters, or using the simulation, the total maintenance cost of the time-based maintenance and replace-on-failure models are computed. The preference of the proposed models is shown by comparing the total maintenance cost of the proposed maintenance models and two models of time-based maintenance and replace-on-failure. It can be concluded that the maintenance models considering the degradation level of the components in the system can provide a more cost-effective maintenance plan.

For the proposed dynamic maintenance models, the maintenance costs of the proposed models are compared to the simulation results of the static maintenance models, such as replace-on-failure and on-condition maintenance model. The proposed dynamic maintenance models suggest a maintenance action or the next inspection time considering the degradation level of all the components within the system. Therefore, it can be concluded that by using the proposed dynamic maintenance models, the maintenance actions are only implemented when they are necessary, and it would help to avoid the unnecessary actions that may waste the useful life of the components.

Conclusions

This research develops a reliability model for multi-component systems with individually repairable components where each component is subject to two failure processes of degradation and random shock arrival. For systems functioning for a very long time, each component is repaired/replaced several times, so their age is different at each inspection time. Considering the initial degradation of each component, the conditional probability of failure and conditional system reliability can be calculated using the proposed method in this study. Condition-based maintenance models are one of the ways to increase the reliability and availability of the systems and minimize the total cost by preventing the event of failure. For systems with a high downtime cost, the system failure can be prevented, by replacing or repairing the components which are prone to failure. In this study, different condition-based maintenance models are studied for such systems.

The proposed static maintenance models provide some maintenance thresholds for each component to be maintained before a failure occurs. Low maintenance thresholds can be inefficient because they waste component's life, and high thresholds are risky because the components are prone to costly failure. Therefore, new optimization models are formulated and solved to find the optimal maintenance thresholds and inspection times simultaneously.

In this research work, by considering the initial age of all the components, dynamic maintenance policies are developed and solved to suggest maintenance actions and inspection intervals based on the current information of components degradation. Two different solution methods are used for the dynamic maintenance models such as optimization and reinforcement learning.

In the first dynamic models, the optimal decision variables such as maintenance thresholds and inspection intervals are found by solving an optimization method. At the beginning of each inspection, the next inspection time and preventive maintenance thresholds are found dynamically based on the initial age of the components. A neural network model is also proposed as an alternative to solving the optimization problem at each inspection. In the second dynamic models, the problem is formulated as a Markov decision process model and reinforcement learning algorithms are used to solve the dynamic maintenance problem to find the optimal maintenance actions for each degradation state of the system. Therefore, using the proposed models, a decision-making tool can be provided to the maintenance team to find the best maintenance action based on the degradation of the components.

1.15 Research Extensions

Towards this idea of never-ending research, it would be interesting to present some potential research extensions of the current work. The first research extension can be related to the shock models. There are different types of environmental shock in real life, which can be fatal or non-fatal and can have different effects on the degradation path of the systems or components. Moreover, the environment is often actually changing dynamically; thus, the shock process should be modeled differently to consider the uncertainty of the environments and its effect on the system's degradation path. Conducting some research studies on different shock models and formulating the system reliability and maintenance models based on the new shock models can provide a more realistic maintenance problem.

The second research extension can be related to reinforcement learning algorithms. In this study, the main focus is using Q-learning, and its neural network-based versions. However, it is imperative that other RL techniques are also examined and compared with the ones already implemented. There is recently increasing attention to actor-critic methods in RL, such as Deep Deterministic Policy Gradient (DDPG). Actor-critic methods can simultaneously estimate value functions (critic) and update the policy distribution (actor) accordingly. Moreover, Monte Carlo Tree Search (MCTS) is also one of the algorithms that are studied recently for game-based reinforcement learning problems. The basic idea of MCTS is to build a tree on all the possible scenarios of the simulation but explore only those that are the most promising ones.

The last research extension refers to the inventory level of spare parts. Excessive inventory results in a high holding cost for the spares, while insufficient inventory causes inadequate preventive maintenance. Considering the inventory control of spare parts in the dynamic condition-based maintenance models would lead to a more realistic maintenance problem. The number of spare parts can be one part of the system state in the reinforcement learning problem. The dynamic maintenance action at each inspection time will be made based on the inventory cost and the available number of spare parts until the end of the maintenance contract. Furthermore, the other constraints of available resources such as personnel and materials or special maintenance requirements, such as space limits and conflict between maintenance actions, can be considered in future studies for proposing an appropriate maintenance policy.

The ultimate goal of this research is to provide an appropriate maintenance model for systems with different repairable components subject to aging and degradation and environmental shocks. Using the proposed maintenance models in this study, the maintenance team can have a decision-making tool for implementing the maintenance actions on each component. Having an appropriate maintenance model can prevent the system failure and its huge cost and can increase the reliability and availability of systems.

Appendix A

Parameter estimation for systems considering clusters of dependent degrading components

To obtain explicit reliability predictions, it is necessary to know or estimate the parameters in the reliability equations. In this section, a method is proposed to estimate parameters for the reliability models given available data. This model is based on degradation data, and not failure data, and therefore numerous data records can be collected on a component prior to failure. There are different ways to estimate the parameters required for this model, including probability plotting, least square methods, rank methods for censored data, maximum likelihood estimation (MLE), Bayesian parameter estimation methods, etc. Some parameters in the model might be already available, e.g., the coefficients c_i and b_i in $v_i(t) = c_i t^{b_i}$ that can be obtained from material properties or related studies.

If shocks are present, the increment of degradation in a time interval is equal to pure degradation plus cumulative damage contribution from shocks. To demonstrate, I develop the MLE for the special case when the shock damage is normally distributed and the pure degradation for a time interval is distributed as a gamma distribution. Considering a case with one cluster (k = 1), the shape parameter for the gamma distribution is the difference of $v_i(t) = c_i t^{b_i}$ for two time points, and the scale parameter $\theta(i)$ is a function of a single random variable θ_1. In this example, it is assumed that W_{ij} follows a normal distribution $W_{ij} \sim N(\mu_{W_i}, \sigma_{W_i}^2)$, Y_{ij} follows a normal distribution $Y_{ij} \sim N(\mu_{Y_i}, \sigma_{Y_i}^2)$, and $\alpha_{0,i} = 0$.

Consider data collected from multiple systems, l = 1, 2, …, L, of the same type with components i = 1, 2, …, n, and one cluster, k = 1. For each system l, data is collected at periods τ = 1, 2, … T_l. To determine the distribution of θ_1, it is necessary to obtain specific estimates of θ_1 for each individual system l included in the data set, or $\tilde{\theta}_{1,l}$. If $\tilde{\theta}_{1,l}$ are all approximately the same, then there is not meaningful dependence among components within a system, and the degradation paths are not clustered.

Define $\Delta_{i,l,\tau}$ as the increment in the pure degradation, and $S_{i,l}(t_{i,l,\tau})$ as the sum of shock damages for the i^{th} component in the l^{th} system during the τ^{th} data collection period,

$$\Delta_{i,l,\tau} = X_{i,l}(t_{i,l,\tau}) - X_{i,l}(t_{i,l,\tau-1}) \sim Ga(x; c_i, b_i, \alpha_{1,i}, \tilde{\theta}_l).$$

$$S_{i,l}(t_{i,l,\tau}) - S_{i,l}(t_{i,l,\tau-1}) \sim N(x; (m_{i,l,\tau} - m_{i,l,\tau-1})\mu_{Y_i}, (m_{i,l,\tau} - m_{i,l,\tau-1})\sigma_{Y_i}^2).$$

where $m_{i,l,\tau} - m_{i,l,\tau-1}$ is the number of shocks observed between the τ and τ-1$^{\text{th}}$ observation periods.

Data is collected from L units of systems with n different components. Degradation and failure data are collected at T_l different time periods for each system l. The data collection periods can be of any duration, but to estimate all model parameters, it is necessary to collect data often to provide a diverse and plentiful data set. Two types of data are collected including data for a system with pure degradation and data for a system subject to both degradation and the shock process. If there are no shocks, then the following estimates can be simplified.

The following data is available:

1) For any failure, it is known specifically if it is a hard or soft failure, and the total number of shocks the component was exposed to until failure. The total degradation is known at failure whether it is a hard or soft failure. If the system is repaired and returned to operation, it is considered as a new system.

2) Degradation data is collected at T_l different time intervals prior to a failure. For all components, total degradation is recorded at each time, and total number of shocks and shocks since last data record are is known and recorded. It is not known how much of the total degradation is due to the pure degradation and how much is due to the cumulative shock damages.

Likelihood function for hard failure

Considering systems subject to both degradation and shock process, there are three types of data that can be used: (*i*) the recorded time and shocks until hard failure, (*ii*) the recorded time and shocks until soft failure, (*iii*) and the recorded time and shocks for degradation data prior to any failure. Define sets *H*, *S*, *P* for (i, l, τ) belonging to the three types of data. The parameters for each component can be solved separately. The likelihood and log-likelihood function are given by,

For $i = 1, \ldots, n$,

$$L_i(\mu_{W_i}, \sigma_{W_i}; m_{i,l,\tau}) = \prod_{(i,l,\tau)\in H} F_{W_i}(D_i)^{m_{i,l,\tau}-1}(1 - F_{W_i}(D_i)) \prod_{(i,l,\tau)\in S} F_{W_i}(D_i)^{m_{i,l,\tau}} \prod_{(i,l,T_l)\in P} F_{W_i}(D_i)^{m_{i,l,\tau}}.$$

$$\begin{aligned} l_i(\mu_{W_i}, \sigma_{W_i}; m_{i,l,\tau}) = &\sum_{(i,l,\tau)\in H} (m_{i,l,\tau} - 1)\ln(F_{W_i}(D_i)) + \sum_{(i,l,\tau)\in H} \ln(1 - F_{W_i}(D_i)) + \sum_{(i,l,\tau)\in S} m_{i,l,\tau} \ln(F_{W_i}(D_i)) \\ &+ \sum_{(i,l,T_l)\in P} m_{i,l,\tau} \ln(F_{W_i}(D_i)). \end{aligned}$$

where *H* = set of data for hard failure, *S* = set of data for soft failure, and *P* = set of data prior to failure. $F_{Wi}(\cdot)$ can be any parametric distribution but W_{ij} is often assumed to be normally distributed.

Likelihood function for soft failure/component degradation

1) ***Likelihood function without shocks***

For observed incremental degradation $\delta_{i,l,\tau} = x_{i,l,\tau} - x_{i,l,\tau-1}$ for time $t_{i,l,\tau}$ to $t_{i,l,\tau-1}$ with no shocks, the likelihood function is the product of the conditional probability density functions of $\Delta_{i,l,\tau} = X_{i,l}(t_{i,l,\tau}) - X_{i,l}(t_{i,l,\tau-1})$. Considering $k = 1$, θ_1 is defined as a random variable with a distribution, so the plan is to obtain specific estimates of θ_1 for each individual system $l, \tilde{\theta}_{1,l}$. The distribution of θ_1 can then be inferred based on estimated $\tilde{\theta}_{1,l}$. The $\tilde{\theta}_{1,l}$ for each system are then analyzed to determine an appropriate distribution (e.g., normal, gamma) and distribution parameters. It can be very beneficial is a distributional assumption can be made, e.g., normal. If k = 2, then estimates are needed for $\theta_{1,l}$ and $\theta_{2,l}$, which would require more data.

With $\delta_{i,l,\tau} = x_{i,l,\tau} - x_{i,l,\tau-1}$, the probability density function (pdf) is then given as:

$$f_{\Delta_{i,l}}(\delta) = gamma(\delta_{i,l}; c_i, b_i, \alpha_{1,i}, \tilde{\theta}_{1,l})$$

The likelihood function is given by:

$$L(c_i, b_i, \alpha_{1,i}, \tilde{\theta}_{1,l}; \delta_{i,l,\tau}, t_{i,l,\tau}) = \prod_{l=1}^{L}\prod_{i=1}^{n}\prod_{\tau=1}^{T_l} f_{\Delta_{i,l}}(\delta_{i,l,\tau}; c_i, b_i, \alpha_{1,i}, \tilde{\theta}_{1,l}).$$

Occasionally, b_i and c_i (and sometimes even $\alpha_{0,i}$ and $\alpha_{1,i}$) can be estimated based on previous engineering studies or physical properties. In this case, the likelihood function reduces to:

$$L(\alpha_{1,i}, \tilde{\theta}_{1,l}; \delta_{i,l,\tau}, t_{i,l,\tau}) = \prod_{l=1}^{L}\prod_{i=1}^{n}\prod_{\tau=1}^{T_l} f_{\Delta_{i,l}}(\delta_{i,l,\tau}; c_i, b_i, \alpha_{1,i}, \tilde{\theta}_{1,l}).$$

2) Likelihood function with shocks

For observed incremental degradation $\delta_{i,l,\tau} = x_{i,l,\tau} - x_{i,l,\tau-1}$ for time $t_{i,l,\tau}$ to $t_{i,l,\tau-1}$ with $m_{i,l,\tau} - m_{i,l,\tau-1}$ shocks, the likelihood function is the product of the conditional probability density functions of $\Delta_{i,l,\tau} = X_{i,l}(t_{i,l,\tau}) - X_{i,l}(t_{i,l,\tau-1})$.

With $\delta_{i,l,\tau} = x_{i,l,\tau} - x_{i,l,\tau-1}$, the probability density function (pdf) can be derived as:

$$\begin{aligned} f_{\Delta_{i,l}}(\delta) &= \Pr\{\Delta_{i,l,\tau} = \delta\} = \Pr\{\Delta X_{i,l}(t_{i,l,\tau}) + \Delta S_{i,l}(t_{i,l,\tau}) = \delta\} \\ &= \int_0^{\delta} \Pr\{\Delta X_{i,l}(t_{i,l,\tau}) = \delta - \Delta S_{i,l}(t_{i,l,\tau}) \,|\, \Delta S_{i,l}(t_{i,l,\tau}) = q\} f_{\Delta S_{i,l}}(q)dq \\ &= \int_0^{\delta} gamma\left(\delta - q; t_{i,l,\tau}, t_{i,l,\tau-1}, c_i, b_i, \alpha_{0,i}, \alpha_{1,i}, \tilde{\theta}_{1,l}\right)\phi\left(\frac{q - (m_{i,l,\tau} - m_{i,l,\tau-1})\mu_{Y_i}}{\sigma_{Y_i}\sqrt{m_{i,l,\tau} - m_{i,l,\tau-1}}}\right)dq. \end{aligned}$$

The likelihood function is given by:

$$L(c_i, b_i, \alpha_{0,i}, \alpha_{1,i}, \tilde{\theta}_{1,l}, \mu_{Y_i}, \sigma_{Y_i}; \delta_{i,l,\tau}, t_{i,l,\tau}, m_{i,l,\tau}) = \prod_{l=1}^{L}\prod_{i=1}^{n}\prod_{\tau=1}^{T_l} f_{\Delta_{i,l}}(\delta_{i,l,\tau}; c_i, b_i, \alpha_{0,i}, \alpha_{1,i}, \tilde{\theta}_{1,l}, \mu_{Y_i}, \sigma_{Y_i})$$

If b_i and c_i can be estimated by other sources, the likelihood function reduces to:

$$L(\alpha_{0,i}, \alpha_{1,i}, \tilde{\theta}_{1,l}, \mu_{Y_i}, \sigma_{Y_i}; \delta_{i,l,\tau}, t_{i,l,\tau}, m_{i,l,\tau}) = \prod_{l=1}^{L}\prod_{i=1}^{n}\prod_{\tau=1}^{T_l} f_{\Delta_{i,l}}(\delta_{i,l,\tau}; c_i, b_i, \alpha_{0,i}, \alpha_{1,i}, \tilde{\theta}_{1,l}, \mu_{Y_i}, \sigma_{Y_i})$$

$$\begin{aligned} f_{\Delta_{i,l}}(\delta_{i,l,\tau}) &= \int_0^{\delta} gamma\{\delta - q; t_{i,l,\tau}, \alpha_{0,i}, \alpha_{1,i}, \tilde{\theta}_{1,l}\}\phi\left(\frac{q-(m_{i,l,\tau}-m_{i,l,\tau-1})\mu_{Y_i}}{\sigma_{Y_i}\sqrt{m_{i,l,\tau}-m_{i,l,\tau-1}}}\right)dq \\ &= \frac{(\delta-q)^{(c_i\left(t_{1,l,\tau}^{b_1}-t_{1,l,\tau-1}^{b_1}\right))-1}\exp(-(\delta-q)/\alpha_{0,i}+\alpha_{1,i}\tilde{\theta}_{1,l})}{\Gamma\left(c_i\left(t_{1,l,\tau}^{b_1}-t_{1,l,\tau-1}^{b_1}\right)\right)(\alpha_{0,i}+\alpha_{1,i}\tilde{\theta}_{1,l})^{(c_i\left(t_{1,l,\tau}^{b_1}-t_{1,l,\tau-1}^{b_1}\right))}}\phi\left(\frac{q-(m_{i,l,\tau}-m_{i,l,\tau-1})\mu_{Y_i}}{\sigma_{Y_i}\sqrt{m_{i,l,\tau}-m_{i,l,\tau-1}}}\right)dq. \end{aligned}$$

Estimation of all parameters can require significant data, and this does represent a limitation of the model. However, if the clustering is actually present, then it is necessary to collect the requisite data to provide the most appropriate models.

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