STATIONARY NAVIER-STOKES EQUATIONS IN AN EXTERIOR DOMAIN, AND SOME INTEGRAL IDENTITIES FOR EULER AND NAVIER-STOKES EQUATIONS

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A dissertation submitted to the

School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Yanyan Li

And approved by

New Brunswick, New Jersey

May, 2021

ABSTRACT OF THE DISSERTATION

Stationary Navier-Stokes Equations in an Exterior Domain, and Some Integral Identities for Euler and Navier-Stokes Equations

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We study: 1) the stationary Navier-Stokes equations in a two-dimensional exterior domain Ω, 2) some integral identities for the Euler and the Navier-Stokes equations. For the first topic, we consider the non-homogenous boundary value problem in a two-dimensional exterior domain together with a prescribed condition at infinity and establish existence of a solution to the problem provided that the boundary value on $\partial\Omega$ is close to a potential flow; this assumption allows some large boundary value. Indeed, we utilize results of Galdi [24] on the Oseen equations, a linearization around a constant nonzero vector. Then we apply ideas used in Russo and Starita's work in three dimension, which is to perturb around a potential flow ([44]); in conjunction with the compactness of some linear operator related to the Oseen equations, which is a result again of Galdi [24].

For the second topic, Dobrokhotov and Shafarevich [12] in 1994 proved some integral identities for the Euler and Navier-Stokes equations. Chae [4] in 2012 proved these integral identities on a hyperplane for a weak solution with some integrability assumptions on the solution. In this thesis, we prove the integral identities on a hyperplane with some different integrability assumptions. It also furnishes a Liouville type theorem as an immediate application, providing a different approach to some of the results of Hamel and Nadirashvili [30],[31], Chae and Constantin [8].

ACKNOWLEDGEMENTS

I am deeply indebted, first and foremost, to my thesis adviser, Professor Yanyan Li, for his patience, encouragement, and support throughout my graduate study. His passion to mathematics and relentless pursuit of mathematical truth has been encouraging my research all along. He chose, for my research, a problem so important and intriguing that I have immersed myself in it, and he gave me remarkable papers to read, which have been inspiring my work all the time.

I also would like to thank Professor Zheng-Chao Han for being a mentor for me. He taught me several graduate courses and gave me a lot of help throughout my graduate study. In addition, he helped me in my application for postdoc positions. He was also on the committee member of my oral qualifying exam as well as of my thesis defense.

I also would like to thank my other committee members, Professor Dennis Kriventsov on my thesis committee as well as Professors Sagun Chanillo and Avraham Soffer on my oral exam committee. In addition, it was a great pleasure for me to take their graduate courses and learn various subjects in mathematics, which helped me to do my own research.

I am also grateful to Professor Haim Brezis for his encouragement, help, and support. I took three of his graduate courses and greatly enjoyed having a discussion with him and my colleagues. During his classes, he always asked interesting questions and encouraged us to have a discussion rather than he taught us all material, through which I learned not only mathematics but also discussion skills. I was also inspired by his interaction with other mathematicians, that he showed during seminars.

I would like to thank Professor Hector Sussmann for his help in my application for postdoc positions.

I would like to express my gratitude to Professor Dongho Chae for showing me the beauty of mathematical fluid mechanics since my undergraduate study, for helping me in my application for postdoc positions, and for his service on the committee member of my thesis defense. And this thesis is greatly inspired by his works.

I also want to thank the faculty and staff of Rutgers Mathematics Department, especially Professor Paul Feehan as a graduate director and Ms. Katie Guarino for her help throughout my graduate study.

Many thanks go to my friends who made my time at Rutgers so much fun, including but not limited to, Zhuolun Yang, Han Lu, Vernon Chan, Martin Köberl, Brooke Orgodnik, Dr. Chloe Wawrzyniak, Dr. Matthew Charnley, Dr. Jinyoung Park, and Corrine Yap.

I am also grateful to my parents and sister for their support and trust throughout my life. They have been supporting me in pursuit of mathematics all along.

Lastly, I would like to thank my lovely wife, Heejin Lee, for always being there for me and for making Navier-Stokes equation muffins for me when I made significant progress in my research. You are the love of my life.

Contents

Chapter 1

Introduction

In this thesis, we study the Euler ($\nu = 0$) and the Navier-Stokes equations ($\nu > 0$):

$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla p = \nu \Delta \boldsymbol{v} + \boldsymbol{f} & \text{in } \Omega \times (0, T), \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega \times (0, T) \end{cases}$$

where Ω is a domain in $\mathbb{R}^N, N \geq 2$. The unknowns are the velocity field $\boldsymbol{v} = \boldsymbol{v}(x, t) : \Omega \times (0, T) \to \mathbb{R}^N$ and the pressure field $p = p(x, t) : \Omega \times (0, T) \to \mathbb{R}$. Here we denote by ν the viscosity and by $\boldsymbol{f}$ the external force.

The equations model incompressible inviscid ideal flow and viscous flow respectively. These equations are ones of the most important equations in Nonlinear Analysis and in Fluid Mechanics, which have many open problems attracting various researchers. Especially the topic of the stationary Navier-Stokes equations in a two-dimensional exterior domain has various long-standing open problems. This is the first topic of this thesis. We study this topic in Chapter 2.

The second topic is about some integral identities for the Euler and the Navier-Stokes equations, that was first proved by Dobrokhotov and Shafarevich in 1994 [12] and that shows each component of $\boldsymbol{v} = (v_1, \ldots, v_N)$ has the same L^2- norm and they are orthogonal with respect to the L^2 inner product. And Chae [4] in 2012 proved these integral identities on a hyperplane for a weak solution. And several researchers improved these integral identities and applied them to solve other problems. We study these integral identities in Chapter 3.

In this chapter (Introduction), we present on each topic of Chapter 2 and 3 very briefly. At the beginning of Chapter 2 and 3, detailed history of the corresponding topic is written. (For precise statements of theorems, see the introductions of Chapter 2 and 3.)

1.1 The Stationary Navier-Stokes Equations in Two-Dimensional Exterior Domains

We first briefly present on the first topic. More detailed introduction is written at the beginning of Chapter 2.

For an exterior domain Ω with non-empty boundary in $\mathbb{R}^2$, consider the stationary (incompressible) Navier-Stokes equations:

$$\begin{cases} \nu\Delta \boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

with boundary conditions on $\partial\Omega$ and at infinity:

$$\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}_* \tag{1.2}$$

$$\lim_{|x|\to\infty} \boldsymbol{v} = \boldsymbol{v}_\infty \tag{1.3}$$

Here $\boldsymbol{v}_* = \boldsymbol{v}_*(x)$ denotes a prescribed boundary function on $\partial\Omega$ and $\boldsymbol{f} = \boldsymbol{f}(x)$ a given external force and $\boldsymbol{v}_\infty$ a prescribed constant vector. The constant $\nu > 0$ is a given parameter, called the viscosity. We seek for a pair of a velocity field $\boldsymbol{v} : \Omega \to \mathbb{R}^2$ and pressure $p : \Omega \to \mathbb{R}$ that satisfies the equations with the two boundary conditions.

The boundary value problem of the stationary Navier-Stokes equations in a *two*-dimensional exterior domain has attracted so much attention from many researchers.

The study of this problem was initiated by Leray in his seminal paper [40] in 1933. Leray established the existence of a solution to the problem (1.1),(1.2) that satisfies additionally the finite Dirichlet energy condition, $\int_\Omega |\nabla \boldsymbol{v}|^2 \, dx < \infty$, under the assumption that the flux on each connected component of the boundary vanishes, that is,

$$\int_{\partial\Omega_i} \boldsymbol{v}_* \cdot \boldsymbol{n} \, d\sigma = 0 \quad \text{for all } i = 1, \cdots, M \tag{1.4}$$

where the complement of the domain Ω is written as a disjoint union of several connected components, that is, $\Omega^c = \cup_{i=1}^M \Omega_i$.

Leray was not able to show that his solution satisfies the condition (1.3) at infinity in the two-dimensional case as opposed to the three-dimensional case. A number of remarkable partial results have been made to resolve this issue of convergence at infinity in two-dimension, but still this is one of the big open problems in this area. (For more history, see Section 2.1.)

Additionally, another open question was left out by Leray. The condition that 'total flux' on the boundary $\partial\Omega$ vanishes, that is,

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} \, d\sigma = 0, \tag{1.5}$$

is a compatibility condition for bounded domains Ω due to the incompressibility condition $\text{div}\boldsymbol{v} = \boldsymbol{0}$, which is no longer a priori a compatibility condition in the case of exterior domains. In addition, condition (1.4) of Leray is even stronger than (1.5). Leray's method has been extensively studied in order to relax condition (1.4) and to remove condition (1.5) eventually. Korobkov, Pileckas, Russo [37] in 2020 made a major recent breakthrough: in two dimension, there exists a solution to (1.1), (1.2) with $\boldsymbol{f} \equiv \mathbf{0}$ under assumption (1.5). (For more details about their theorems and proofs, see Section 2.1 and Theorems 2.1.3, 2.1.4, 2.1.5.) However, it still remains open to remove even condition (1.5).

On the other hand, we can also apply perturbation methods to investigate the main problem, (1.1), (1.2), (1.3). For the case $\boldsymbol{v}_\infty = \mathbf{0}$, we can linearize equation (1.1) around $\boldsymbol{v}_\infty = \mathbf{0}$, which leads to the following linear equation:

$$\begin{cases} \nu\Delta\boldsymbol{u} - \nabla p = \boldsymbol{f} \quad \text{in } \Omega \\ \text{div}\boldsymbol{u} = 0 \quad \text{in } \Omega \end{cases} \tag{1.6}$$

These equations are called the *Stokes equations*. However, the boundary value problem of the Stokes equations (vanishing at infinity) is solvable if and only if the boundary value $\boldsymbol{u}_*$ and the external force $\boldsymbol{f}$ satisfy a certain condition. In particular, if $\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{u}_*$ is a nonzero constant vector, then the boundary value problem does not admit a solution. This phenomena is called the *Stokes paradox*. (For more details about the Stokes paradox, see Theorem 2.3.3.) And this is the case only in two dimension, which gives yet another major difficulty in the study of the two-dimensional case compared to the three-dimensional one.

However, in the case $\boldsymbol{v}_\infty \neq \mathbf{0}$, the situation is more manageable. To understand the nature of the case $\boldsymbol{v}_\infty \neq \mathbf{0}$, we first write the problem (1.1), (1.2), (1.3) in a dimensionless form. Indeed, in the case $\boldsymbol{v}_\infty \neq \mathbf{0}$, via scaling and rotation, we can normalize the vector $\boldsymbol{v}_\infty$ to the vector $\boldsymbol{e}_1 = \langle 1, 0\rangle$, which leads to the following dimensionless form:

$$\begin{cases} \Delta\boldsymbol{w} - \lambda(\boldsymbol{w}\cdot\nabla)\boldsymbol{w} - \nabla\pi = \lambda\boldsymbol{F} \quad \text{in } \Omega \\ \text{div}\,\boldsymbol{w} = 0 \quad \text{in } \Omega \end{cases} \tag{1.7}$$

with the boundary conditions

$$\boldsymbol{w}|_{\partial\Omega} = \boldsymbol{w}_* \tag{1.8}$$

$$\lim_{|x|\to\infty} \boldsymbol{w}(x) = \boldsymbol{w}_\infty = \boldsymbol{e}_1 \tag{1.9}$$

where $\lambda > 0$. Our main results of this chapter are to establish an existence theorem of a solution $(\boldsymbol{w}, \pi)$ to the problem (1.7), (1.8), (1.9).

In a series of papers [17],[49],[18] from 1965 to 1967, Finn and Smith considered the case of $\boldsymbol{v}_\infty \neq \boldsymbol{0}$ in two dimension and proved existence of a solution to (1.7), (1.8). (1.9) with $\boldsymbol{F} = \boldsymbol{0}$: if the norm of the quantity $(\boldsymbol{w}_* - \boldsymbol{w}_\infty)/|\log\lambda|$ is less than a certain constant depending on Ω, λ_0 (some upper bound of λ but independent of data $\boldsymbol{w}_*$). (See Theorem 4.1 of [18]) [1]. This is the first existence theorem of the problem (1.7), (1.8), (1.9) at least with some smallness assumption on data.

Setting $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{w}_\infty$, equation (1.7) turns into

$$\begin{cases} \Delta\boldsymbol{u} - \lambda\dfrac{\partial\boldsymbol{u}}{\partial x_1} - \nabla\pi = \lambda(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \lambda\boldsymbol{F} \quad \text{in } \Omega \\[2ex] \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega. \end{cases} \tag{1.10}$$

By formally removing the nonlinear term in (1.10), we obtain:

$$\begin{cases} \Delta\boldsymbol{u} - \lambda\dfrac{\partial\boldsymbol{u}}{\partial x_1} - \nabla\pi = \lambda\boldsymbol{F} \quad \text{in } \Omega \\[2ex] \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega. \end{cases}$$

These linear equations are called the *Oseen equations*. The non-homogeneous boundary value problem of the Oseen equations are solvable even in two dimension in contrast to the Stokes problem. Finn and Smith achieved their existence theorem by using the Oseen equations. (For more details, see Section 2.4)

[1] This statement might appear to be different from Finn and Smith's, but they are actually equivalent because here we have assumed $\boldsymbol{w}_\infty = \langle 1, 0\rangle$ without loss of generality

On the other hand, Galdi investigated the same problem of proving existence of a solution $(\boldsymbol{w}, \pi)$ to (1.7) , (1.8), (1.9) by using the Oseen equations as Finn and Smith did, but Galdi investigated the Oseen problem in a way totally different from Finn and Smith's and used Sobolev type function spaces to prove that: if $|\log \lambda|^{-1}\|\boldsymbol{w}_* - \boldsymbol{w}_\infty\|_{W^{2-1/q,q}(\partial\Omega)} + \lambda^{2/q-1}\|\boldsymbol{F}\|_{L^q(\Omega)}$ is less than some constant that depends on Ω, q, λ_0, then there exists a solution $\boldsymbol{w}, p$ to (1.7) , (1.8), (1.9) such that $\boldsymbol{w} - \boldsymbol{w}_\infty$ and p are in some Sobolev type function spaces, $X^{2,q}(\Omega), D^{1,q}(\Omega)$. See Theorem 2.0.1 for a statement of this result. In Chapter 2, we mainly make use of these results of Galdi on the Oseen equations.

It is worth to notice that Finn and Smith's results as well as Galdi's do not impose any assumption on flux as like (1.5).

However, both Galdi's result and Finn and Smith's have some limitation: with λ fixed, data $\boldsymbol{w}_*$ must be chosen close to $\boldsymbol{w}_\infty$ to ensure existence of a solution.

A mian objective of Chapter 2 is to overcome this limitation. The key idea toward this end comes from A. Russo and Starita [44] in 2008. They proved existence of a solution vanishing at infinity ($\boldsymbol{v}_\infty = \boldsymbol{0}$) to the stationary Navier-Stokes equations in *three* dimensional exterior domain with a boundary condition which allows some large data on the boundary. Their main idea is to perturb around a potential flow (see below for a definition) rather than the constant vector $\boldsymbol{v}_\infty$. (See Theorem 2.0.2 for a statement of Russo and Starita's work)

To incorporate this idea into our main problem (1.7), (1.8), (1.9) in the case $\boldsymbol{v}_\infty \neq \boldsymbol{0}$ and in *two* dimension, we fix a harmonic function β in Ω such that $\nabla\beta$ decays at infinity, (this type of flow $\nabla\beta$ is called a *potential flow*) and we derive another form of the stationary Navier-Stokes equations rather than (1.10) by setting $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{w}_\infty - \mu\nabla\beta$ where μ is a scalar parameter when $\boldsymbol{w}$ is a solution to (1.7), (1.8), (1.9) with $\boldsymbol{w}_\infty = e_1$. Then for $\boldsymbol{u}_* = \boldsymbol{w}_* - \boldsymbol{w}_\infty - \mu\nabla\beta$ on $\partial\Omega$ and $p = \pi + \frac{\lambda\mu^2}{2}|\nabla\beta|^2 + \lambda\mu\frac{\partial\beta}{\partial x_1}$, equations (1.7) turn into

$$
\begin{cases}
\Delta \boldsymbol{u} - \lambda \dfrac{\partial \boldsymbol{u}}{\partial x_1} - \nabla p = \lambda (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \lambda\mu(\nabla\beta\cdot\nabla)\boldsymbol{u} + \lambda\mu(\boldsymbol{u}\cdot\nabla)\nabla\beta + \lambda \boldsymbol{F} \quad \text{in } \Omega \\
\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega
\end{cases}
\tag{1.11}
$$

Deriving this form, we have used the fact that $\nabla\beta$ is the gradient of a harmonic function.

In this form (1.11), we control the contribution of the additional terms on the right hand side involving $\nabla\beta$ by a compactness result of an operator associated to these additional terms, that was proved by Galdi, in order to establish existence theorems that allow some large boundary value $\boldsymbol{w}_*$.

Main result. One of our main results in Chapter 2 is to prove that, with a harmonic function β fixed, for $1 < q < 6/5$, $\mu \in \mathbb{R} \setminus G$ where G is some countable set, if $\|\boldsymbol{w}_* - \boldsymbol{w}_\infty - \mu\nabla\beta\|_{W^{2-1/q,q}(\partial\Omega)} + \|\boldsymbol{F}\|_{L^q(\Omega)}$ is less than a certain constant that only depends on $\Omega, q, \lambda, \mu, \nabla\beta$, then there exists a solution $\boldsymbol{w}, \pi$ to the main problem (1.7), (1.8), (1.9). (See Theorems 2.0.3 for a precise statement, and see also Theorem 2.0.8 for another main result of this topic.)

As G is countable, μ can be chosen large allowing some big boundary value $\boldsymbol{w}_*$.

1.2 Some Integral Identities of the Euler and the Navier-Stokes Equations

Let Ω be a domain in $\mathbb{R}^N, N \geq 2$. For this topic, we mainly consider the Euler ($\nu = 0$) or Navier-Stokes ($\nu > 0$) equations:

$$\begin{aligned} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla p = \nu \Delta \boldsymbol{v} \qquad &\text{in } \Omega \times (0, T) \\ \operatorname{div} \boldsymbol{v} = 0 \qquad &\text{in } \Omega \times (0, T) \end{aligned} \tag{1.12}$$

where $\boldsymbol{v}(x,t) = (v_1(x,t), \cdots, v_N(x,t)) : \Omega \times (0,T) \to \mathbb{R}^N$ is a velocity field, $p : \Omega \times (0,T) \to \mathbb{R}$ is a pressure field. The parameter $\nu \geq 0$ is called viscosity.

Dobrokhotov and Shafarevich [12] in 1994 proved some integral identities for the Navier Stokes and the Euler equations for a classical solution $(\boldsymbol{v}, p)$ in three dimension: if $\boldsymbol{v}$ and its derivatives $\partial \boldsymbol{v}/\partial t, \partial \boldsymbol{v}/\partial x_j, j = 1, 2, 3$ decay faster than $|x|^{-4}$ at some time t, then

$$\int_{\mathbb{R}^3} (v_j(x,t) v_k(x,t) + \delta_{jk} p(x,t))\, dx = 0 \qquad \text{for all } j, k = 1, 2, 3. \tag{1.13}$$

Even though the integral identities are applicable to various problems, not only the integral identities (1.13) but also this paper [12] does not seem to be well-known.

For this topic, we will focus on the study of the integral identities (1.13) of the Euler and Navier-Stokes equations (1.12) in terms of the form of the integral identities and assumptions on $(\boldsymbol{v}, p)$ that are needed, and then we will also study Liouville type theorems of the stationary Euler equations as an immediate corollary of the integral identities.

We first present on history of the integral identities (1.13) and then on one of Liouville type theorems of the stationary Euler equations.

Some Integral Identities for the Euler and the Navier-Stokes equations

Dongho Chae in 2011 [3] proved the integral identities (1.13) in $\mathbb{R}^N$ for a weak solution: if $(\boldsymbol{v}, p) \in L^1(0,T; L^2_{loc,\sigma}(\mathbb{R}^N)) \times L^1(0,T; \mathcal{S}'(\mathbb{R}^N))$ is a weak solution to the Euler or the Navier-Stokes equations (1.12) in $\mathbb{R}^N$ such that $(\boldsymbol{v}, p) \in L^1(0,T; L^2(\mathbb{R}^N)) \times L^1(0,T; L^1(\mathbb{R}^N))$, then the integral identities (1.13) in $\mathbb{R}^N$ holds for almost every $t \in (0,T)$.

Chae [4] in 2012 proved that if $(\boldsymbol{v}, p) \in L^1(0,T; L^2_{loc,\sigma}(\mathbb{R}^N)) \times L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ is a weak solution to the Euler or the Navier-Stokes equations (1.12) such that $(\boldsymbol{v}(\cdot,t), p(\cdot,t)) \in L^2(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$ for some $t \in (0,T)$, then some integral identities on a hyperplane holds; precisely, for all $k \in \{1, \cdots, N\}$,

$$\int_{\mathbb{R}^{N-1}} \left(v_k^2(x,t) + p(x,t) \right) \, dx'_k = 0 \qquad \text{for almost every } x_k \in \mathbb{R}. \tag{1.14}$$

(See Theorem 3.0.1 for a statement.) Here $x'_k = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_N)$ and $dx'_k = dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_N$. The integral identities (1.13) involve two indices j, k whereas these integral identities (1.14) only one index. We will call a type of integral identities involving two indices like (1.13) a *matrix form* whereas the other type, that involves one index, *a vector form.*

We derive integral identities on a hyperplane with assumptions slightly different from Chae's in [4].

Main Result. One of our main results is to prove that if $(\boldsymbol{v}, p) \in L^1(0,T; L^2_{loc}(\mathbb{R}^N)) \times L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ is a weak solution to the Euler or the Navier-Stokes equations such that for some $t \in (0,T)$, and for all $k \in \{1,\dots,N\}$, [2]

$$\left.\begin{aligned} |\boldsymbol{v}(x,t)|^2 + |p(x,t)| \in L^1(\mathbb{R}^{N-1};x_k') \quad \text{for a.e. } x_k \in \mathbb{R}, \\ \big\| |\boldsymbol{v}(x,t)|^2 + |p(x,t)| \big\|_{L^1(\mathbb{R}^{N-1};x_k')} \in L^1_{loc}(\mathbb{R};x_k), \\ \liminf_{|x_k|\to\infty} \big\| |\boldsymbol{v}(x,t)|^2 + |p(x,t)| \big\|_{L^1(\mathbb{R}^{N-1};x_k')} = 0 \end{aligned}\right\} \tag{1.15}$$

then the vector form of the integral identities (1.14) on a hyperplane holds in general whereas the matrix form of the integral identities on a hyperplane holds for the stationary Euler equations, that is, for all $j,k \in \{1,\dots,N\}$,

$$\int_{\mathbb{R}^{N-1}} (v_j(x)v_k(x) + p(x)\delta_{jk})dx_k' = 0 \quad \text{for a.e. } x_k \in \mathbb{R}.$$

(See Theorem 3.0.2 for a statement or Section 3.3 for more details.)

To compare this main result to the previous result [4], if we only consider $\boldsymbol{v}, p$ satisfying

$$|\boldsymbol{v}(x)|^2 + |p(x)| \le \frac{C}{1+|x|^\alpha}, \quad \alpha > 0,$$

then the assumption $|\boldsymbol{v}|^2 + |p| \in L^1(\mathbb{R}^N)$ of [4] is satisfied if $\alpha > N$ whereas assumptions (1.15) are satisfied if $\alpha > N-1$, which is less restrictive. And it does not seem clear whether the assumption of [4] along with the Euler or the Navier-Stokes equations imply better decay of $\boldsymbol{v}, p$, to the best of my knowledge. (See also Remark 3.0.4.) In addition, in this main result, we prove the matrix form of the integral identities on a hyperplane for the stationary Euler equations (in addition to the vector form for the evolutionary Euler or Navier-Stokes equations) whereas the vector form was proved in [4] (for the evolutionary Euler or Navier-Stokes equations.)

For the main ideas of our proof of this result, see the introduction of Chapter 3

[2] What we mean by the notation in the first assumption in (1.15) is that for almost every fixed x_k, the map $x_k' \in \mathbb{R}^{N-1} \to |\boldsymbol{v}(x,t)|^2 + |p(x,t)|$ is in $L^1(\mathbb{R}^{N-1})$. Similarly for the second and third assumption in (1.15), we treat the norm $\big\| |\boldsymbol{v}(x,t)|^2 + |p(x,t)| \big\|_{L^1(\mathbb{R}^{N-1};x_k')}$ as a function of x_k.

or Section 3.3.

We also prove the integral identities (1.14) on a section in a domain with boundary for the stationary Euler equations and establish, as an immediate application of the integral identities, Liouville type theorems, which provide a different approach to some of the results of Hamel and Nadirashvili [30], [31] and Chae and Constantin [8].

Liouville type theorems of the stationary Euler equations

We now present biref history of Liouville type theorems of the stationary Euler equations.

First of all, in Chae [5], a Liouville type theorem is established in a straightforward way by using the integral identities (1.14) : for a continuous weak solution $(\boldsymbol{v}, p) \in L^1(0,T; L^2_{loc,\sigma}(\mathbb{R}^N)) \times L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ to the Euler and the Navier-Stokes equations (1.12) in $\mathbb{R}^N$ satisfying $(\boldsymbol{v}(\cdot,t), p(\cdot,t)) \in L^2(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$ for some time $t \in (0,T)$, if $p(\cdot,t) \geq 0$ on almost every hyperplane of $\mathbb{R}^N$, then $\boldsymbol{v}(\cdot,t) = 0$ in $\mathbb{R}^N$.

In addition, for Liouville type theorems of the stationary Euler equations, several researchers especially studied Liouville type properties of a Beltrami solution in $\mathbb{R}^3$. (See Definition 3.5.1 for a definition of a Beltrami solution.)

Chae and Constantin [8] in 2015 provided two different sets of sufficient conditions for a Beltrami solution $\boldsymbol{v}$ to be trivial: 1) if $\boldsymbol{v} \in L^2(\mathbb{R}^3)$, then $\boldsymbol{v} \equiv \mathbf{0}$; 2) if $\boldsymbol{v} \in L^\infty_{loc}(\mathbb{R}^3)$ satisfies either $\boldsymbol{v} \in L^q(\mathbb{R}^3)$ for some $q \in [2,3)$ or there exists $\varepsilon > 0$ such that $|\boldsymbol{v}(x)| = O(1/|x|^{1+\varepsilon})$ as $|x| \to \infty$, then $\boldsymbol{v} \equiv 0$. [3] (For a statement, see Theorem 3.0.5)

For the first sufficient condition of Chae and Constantin's result, they proved it by using the integral identities (1.13) in the entire space. On the other hand, they argued differently without using the integral identities for the second set of sufficient conditions. (For these proofs, see page 110.)

[3]The first part of Theorem 3.0.5 is just part of Theorem 1.2 of [8]. In the omitted part, they provided an alternative assumption that includes λ (the function λ from the definition of Beltrami solutions).

However, we can still prove a statement with conditions similar to the second set of sufficient conditions of Chae and Constantin [8] by using the integral identities (1.14) on a hyperplane under our assumptions (1.15).

Main Result. One of our main result is to prove by using the integral identities (Theorem 3.0.2) that for a weak Beltrami solution $\boldsymbol{v}$, if $\boldsymbol{v}(x)$ satisfies (1.15) (without assumptions on p), or if $\boldsymbol{v} \in L^\infty_{loc}((\mathbb{R}^3)$ satisfies $|\boldsymbol{v}(x)| = O(1/|x|^{1+\varepsilon})$ as $|x| \to \infty$ for some $\varepsilon > 0$, then $\boldsymbol{v} \equiv \mathbf{0}$. (For a statement of this result, see Theorem 3.0.7.)

On the other hand, Hamel and Nadirashvili in 2017, 2019 [30], [31] provide a new approach to Liouville type theorems of the stationary Euler equations in various special domains in $\mathbb{R}^2$. Their theorems pertain to a shear flow, which is a flow parallel to a vector everywhere. (See (3.21) for a definition.) Special examples of a shear flow include the trivial solution $\boldsymbol{v} \equiv 0$ and $\boldsymbol{v}(x) = (v_1(x_2), 0)$. Instead of asking what assumptions lead to the trivial solution $\boldsymbol{v} \equiv 0$, they ask what assumptions lead to a shear flow.

For example, for $\Omega = (0,1) \times \mathbb{R}$, they proved that if $\boldsymbol{v} \in C^2(\overline{\Omega})$ and $\boldsymbol{v}$ is tangential on the boundary such that $\inf_\Omega |\boldsymbol{v}| > 0$, then $\boldsymbol{v}(x) = (0, v_2(x_1))$ for all $x \in \Omega$. [4] (For a statement of this result, see Theorem 3.0.9.)

As for the other domains, $\Omega = \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}^2$, that they considered, their statements are slightly different, but they assumed $\inf_\Omega |\boldsymbol{v}| > 0$ for all the cases.

Their theorems have a limitation as pointed out by them: the assumption, $\inf_\Omega |\boldsymbol{v}| > 0$, is not equivalent to being a shear flow because, for instance, $\boldsymbol{v} = (v_1(x_2), 0)$ is a shear flow but v_1 may vanish at many points, which does not satisfy the assumption $\inf_\Omega |\boldsymbol{v}| > 0$.

However, we can also derive integral identities on a section of a domain with boundary and, as an immediate corollary of that, we can establish assumptions on a

[4] As a matter of fact, Hamel and Nadirashvili investigated the problem in $\mathbb{R} \times (0,1)$, not $(0,1) \times \mathbb{R}$. But this is just a matter of rotation. And here we have just stated their theorem in $(0,1) \times \mathbb{R}$ for the sake of comparison to our results.

solution $(\boldsymbol{v}, p)$ to the stationary Euler equations with some boundary condition, that are not only sufficient but also necessary to ensure that the solution is in fact a shear flow.

Main Result. One of our main results is to show that, for example, for $\Omega = (0,1)\times\mathbb{R}$, if $(\boldsymbol{v}, p)$ is a $C^1(\overline{\Omega})$ solution to the stationary Euler equations satisfying $v_2|_{\partial\Omega} = 0$ and

$$\liminf_{|x_2|\to\infty} \int_0^1 \big(v_2^2(x) + p(x)\big)\, dx_1 = 0, \qquad \int_0^1 p(x) dx_1 \geq 0 \quad \text{for all } x_2 \in \mathbb{R}, \tag{1.16}$$

then $\boldsymbol{v}$ is a shear flow, that is, $\boldsymbol{v}(x) = (v_1(x_2), 0)$ for all $x \in \Omega$. (Our main theorems of this topic for various types of domains can be found in Theorems 3.0.10, 3.4.2, 3.4.3, 3.4.4, 3.4.9.)

The assumption on the integral of the pressure p is equivalent to to being a shear flow. See also Remark 3.4.5.

Regarding the second assumption of (1.16), we cannot simply remove this assumption because there are counterexamples, the example defined in (3.20).

One downside of our result is that we are making a non-standard boundary condition. But example (3.20) even satisfy both the non-standard boundary condition (3.23) and the standard boundary condition $\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$, that is, $v_1|_{\partial\Omega} = 0$. See also Remarks 3.4.6, 3.4.7

Chapter 2

Stationary Navier-Stokes Equations in Exterior Domains

Let Ω be an exterior domain with non-empty boundary in $\mathbb{R}^2$, that is, the complement of a non-empty compact subset of $\mathbb{R}^2$. (In this chapter, we will mainly consider the case of two dimension unless stated otherwise.) Consider the stationary (incompressible) Navier-Stokes equations:

$$\begin{cases} \nu\Delta\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

with boundary conditions on $\partial\Omega$ and at infinity:

$$\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}_* \tag{2.2}$$

$$\lim_{|x|\to\infty} \boldsymbol{v} = \boldsymbol{v}_\infty \tag{2.3}$$

where $\boldsymbol{v}_* = \boldsymbol{v}_*(x)$ is a prescribed boundary function on $\partial\Omega$, and $\boldsymbol{f} = \boldsymbol{f}(x)$ is a given external force, and $\boldsymbol{v}_\infty$ is a prescribed constant vector. The constant $\nu > 0$ is a given parameter, called viscosity. The unknowns are a pair of a velocity field $\boldsymbol{v} : \Omega \to \mathbb{R}^2$

and pressure $p : \Omega \to \mathbb{R}$. This system models three-dimensional flow past a infinite straight cylindrical body when it flows in the direction orthogonal to the axis of the cylinder. For more details on the physical meaning of this system, see Galdi (Section I.2 of [21]). The boundary value problem of the stationary Navier-Stokes equations in a two-dimensional exterior domain has been undoubtedly one of the most intriguing topics in fluid mechanics.

In his seminal paper [40] of 1933, Leray initiated the study of this problem with a major breakthrough. Among many things, Leray proved the existence of a solution to the problem (2.1),(2.2) (the first boundary condition only) that satisfies additionally the finite Dirichlet energy condition

$$\int_{\Omega} |\nabla \boldsymbol{v}|^2 \, dx < \infty \tag{2.4}$$

under the assumption that the flux on each connected component of the boundary vanishes, that is,

$$\int_{\partial\Omega_i} \boldsymbol{v}_* \cdot \boldsymbol{n} \, d\sigma = 0 \quad \text{for all } i = 1, \cdots, M \tag{2.5}$$

where the complement of the domain Ω is written as a disjoint union of several connected components, that is, $\Omega^c = \cup_{i=1}^{M} \Omega_i$. A solution to the problem (2.1), (2.2) with finite Dirichlet energy (2.4) is called a Dirichlet solution or D-solution.

The main idea of Leray's method is to first find a sequence of solutions $\boldsymbol{v}_k$ to analogous problems of bounded domains $\Omega_k = \Omega \cap B_{R_k}$, that is,

$$\left.\begin{aligned} \nu\Delta\boldsymbol{v}_k - (\boldsymbol{v}_k\cdot\nabla)\boldsymbol{v}_k - \nabla p_k = \mathbf{0} \quad &\text{in } \Omega_k \\ \operatorname{div}\boldsymbol{v}_k = 0 \quad &\text{in } \Omega_k \\ \boldsymbol{v}_k = \boldsymbol{v}_* \quad &\text{in } \partial\Omega_k \\ \boldsymbol{v}_k = \boldsymbol{v}_\infty \ &\text{for } |x| = R_k \end{aligned}\right\} \tag{2.6}$$

where R_k is a sequence such that $R_k \to \infty$ as $k \to \infty$. Then we prove an uniform estimate of the Dirichlet energy

$$\int_{\Omega_k} |\nabla \boldsymbol{v}_k|^2 \, dx \le c \qquad \text{for all } k; \tag{2.7}$$

for some constant c independent of k. It implies, in turn, that a subsequence $\boldsymbol{v}_{k_l}$ weakly converges to a limit $\boldsymbol{v}_L$, which is a solution to the problem (2.1), (2.2) satisfying (2.7). This solution v_L is called *Leray's solution.* And this approach is called the method of "invading domains".

Leray's arguments were made for two- as well as three-dimension with a major difference between these cases. As opposed to the three-dimensional case, Leray was not able to show that2 his solution satisfies the condition (2.3) at infinity in the two-dimensional case. In the three-dimensional case, Leray proved the following inequality:

$$\int_\Omega \frac{|\boldsymbol{v}(x) - \boldsymbol{v}_\infty|^2}{|x|^2} dx \le 4 \int_\Omega |\nabla \boldsymbol{v}(x)|^2 dx.$$

This inequality, in turn, implies (2.3) in a generalized sense. On the other hand, in the two-dimensional case, we only have the weaker inequality

$$\int_\Omega \frac{|\boldsymbol{v}(x) - \boldsymbol{v}_\infty|^2}{|x|^2 \log^2 |x|} dx \le 4 \int_\Omega |\nabla \boldsymbol{v}(x)|^2 dx.$$

This inequality does not even ensure the convergence at infinity. (There is a simple counterexample of a function that satisfies the finite Dirichlet energy condition but grows at infinity in two dimension.) A number of remarkable partial results have been made to resolve this issue of convergence at infinity in two-dimension, but still this is one of the big open problems in this area. For more history, see Section 2.1.

In addition, Leray's argument left out another long-standing open question. Note that the condition that 'total flux' on the boundary $\partial\Omega$ vanishes, that is,

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}\, d\sigma = 0, \tag{2.8}$$

is a necessary condition for bounded domains Ω due to the incompressibility condition $\mathrm{div}\boldsymbol{v} = \mathbf{0}$. This condition (2.8), however, is no longer a priori a compatibility condition in the case of exterior domains. In addition, note that condition (2.5), that Leray imposed, is even stronger than (2.8). Leray's method has been extensively studied in order to relax condition (2.5) and to remove condition (2.8) eventually. A. Russo in 2009 [43] relaxed condition (2.5) into the one of small flux when $\partial\Omega$ has a single connected component. (See Theorem 2.1.1.) There was a major recent breakthrough made by Korobkov, Pileckas, Russo [37] in 2020 saying that, in two dimension, there exists a solution to (2.1), (2.2) with $\boldsymbol{f} \equiv \mathbf{0}$ under assumption (2.8). In other words, they relaxed condition (2.5) of Leray into (2.8). For more details about their theorems, and sketch of one of their proofs, see Section 2.1 and Theorems 2.1.3, 2.1.4, 2.1.5. However, a further step, which is to get rid of condition (2.8), is still open. In other words, it is open to establish an existence theorem under no restrictions on total flux on the boundary $\partial\Omega$ by using Leray's method.

On the other hand, we can also apply a perturbation theory to investigate problem (2.1), (2.2), (2.3). For the case $\boldsymbol{v}_\infty = \mathbf{0}$, we can linearize equation (2.1) around

$\boldsymbol{v}_\infty = \mathbf{0}$, which leads to the following linear problem:

$$\begin{cases} \nu\Delta\boldsymbol{u} - \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega \end{cases} \tag{2.9}$$

with the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_* \tag{2.10}$$

$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0 \tag{2.11}$$

These equations are called the Stokes equations. (We will use the letter $\boldsymbol{u}$ when we deal with a linearized problem.) These equations have been extensively studied. However, this Stokes approximation involve a serious obstacle to the study of a perturbation theory of problem (2.1), (2.2), (2.3) in the case $\boldsymbol{v}_\infty = \mathbf{0}$; the Stokes problem above is not always solvable. In other words, the problem is solvable if and only if the boundary value $\boldsymbol{u}_*$ and the external force $\boldsymbol{f}$ satisfy a certain condition. In particular, if $\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{u}_*$ is a nonzero constant vector, then the problem (2.9), (2.10), (2.11) does not admit a solution. This phenomena is called the *Stokes paradox.* For more details about the Stokes paradox, see Theorem 2.3.3. And this is the case only in two dimension, which again gives another major difficulty in the study of the two-dimensional case compared to the three-dimensional one.

In addition, Hamel in 1916 found examples in two dimension, which provides another reason the two-dimensional case is more involved. Hamel's examples are given by

$$
\begin{aligned}
v_r &= -\frac{\lambda}{r}, \\
v_\theta &= \frac{\omega}{\lambda - 2}\frac{1}{r}\left(1 - r^{-\lambda+2}\right) \\
p &= -\lambda \int \left(\frac{1}{2}\frac{dv_r^2}{dr} - \frac{v_\theta^2}{r}\right)
\end{aligned}
\tag{2.12}
$$

This is a solution to

$$
\begin{cases}
\Delta \boldsymbol{v} - \lambda(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla p = \mathbf{0} \quad \text{in } \Omega, \\
\operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega \\
\boldsymbol{v}|_{\partial\Omega} = -\lambda \boldsymbol{e}_r \\
\lim\limits_{|x|\to\infty} \boldsymbol{v}(x) = \mathbf{0}
\end{cases}
\tag{2.13}
$$

for arbitrary constant ω and $\lambda \neq 2$ when Ω is the complement of a unit ball in $\mathbb{R}^2$. Note that, when λ is sufficiently close to 1, this family of examples provide solutions which decay slower than any negative power of r. Hence in the case $\boldsymbol{v}_\infty = \mathbf{0}$, it is unclear what asymptotic behavior solutions to the main problem (2.1), (2.2), (2.3) might *a priori* have. Moreover, this family of examples assumes the same boundary data for all ω, which provides examples of nonuniqueness in the case of $\boldsymbol{v}_\infty = \mathbf{0}$.

However, if we consider the case of the non-vanishing vector prescribed at infinity, that is, $\boldsymbol{v}_\infty \neq \mathbf{0}$, then the situation is more manageable.

To understand the nature of the case $\boldsymbol{v}_\infty \neq \mathbf{0}$, we first write the problem (2.1), (2.2), (2.3) in a dimensionless form. Indeed, in the case $\boldsymbol{v}_\infty \neq \mathbf{0}$, via scaling and rotation, we can normalize the vector $\boldsymbol{v}_\infty$ to the vector $\boldsymbol{e}_1 = \langle 1, 0\rangle$ which leads to the following dimensionless form:

$$\begin{cases} \Delta \boldsymbol{w} - \lambda(\boldsymbol{w} \cdot \nabla)\boldsymbol{w} - \nabla\pi = \lambda \boldsymbol{F} \quad \text{in } \Omega \\ \operatorname{div} \boldsymbol{w} = 0 \quad \text{in } \Omega \end{cases} \tag{2.14}$$

with the boundary conditions

$$\boldsymbol{w}|_{\partial\Omega} = \boldsymbol{w}_* \tag{2.15}$$

$$\lim_{|x|\to\infty} \boldsymbol{w}(x) = \boldsymbol{w}_\infty = \boldsymbol{e}_1 \tag{2.16}$$

where $\lambda > 0$. (In this chapter, we will use the letter $\boldsymbol{w}$ when we deal with the Navier-Stokes equations in the dimensionless form (2.14) with the normalized vector $\boldsymbol{w}_\infty = e_1$ whereas $\boldsymbol{v}_\infty$ still denotes a prescribed vector that is not re-scaled.) In this dimensionless form, Reynolds number λ being small amount to small $\boldsymbol{v}_\infty$ or big viscosity ν in the original form. Our main results of this chapter are to establish an existence theorem of a solution $(\boldsymbol{w}, \pi)$ to the problem (2.14), (2.15), (2.16) (see Theorems 2.0.3, 2.0.8).

In a series of papers [17],[49],[18] from 1965 to 1967, Finn and Smith considered the case of $\boldsymbol{v}_\infty \neq \boldsymbol{0}$ in two dimension and proved existence of a solution to (2.14), (2.15). (2.16) with $\boldsymbol{F} = \boldsymbol{0}$ if the norm of the quantity $(\boldsymbol{w}_* - \boldsymbol{w}_\infty)/|\log \lambda|$ is less than a certain constant depending on Ω, λ_0 (some upper bound of λ which is independent of data $\boldsymbol{w}_*$). See Theorem 4.1 of [18] [1]. This is the first existence theorem of the problem (2.14), (2.15), (2.16) at least with some smallness assumption on data because Leray's method is not complete in the sense that Leray's solution has not been proved to satisfy the boundary condition at infinity (2.3). And this result is meaningful as it rules out the famous Stokes paradox for the nonlinear problem at least for small data.

[1] This statement might appear to be different from Finn and Smith's, but they are equivalent because here we have assumed $\boldsymbol{w}_\infty = \langle 1, 0 \rangle$.

Indeed, they first linearize the stationary Navier-Stokes equations (2.14) (with $\boldsymbol{F} = \boldsymbol{0}$) around $\boldsymbol{w}_\infty = \boldsymbol{e}_1$. Setting $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{w}_\infty$, problem (2.14) (with $\boldsymbol{F} = \boldsymbol{0}$) turns into

$$\begin{cases} \Delta \boldsymbol{u} - \lambda \dfrac{\partial \boldsymbol{u}}{\partial x_1} - \nabla \pi = \lambda(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \quad \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega \end{cases} \tag{2.17}$$

with the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_* \tag{2.18}$$

$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = \boldsymbol{0} \tag{2.19}$$

where $x = (x_1, x_2)$ and $\boldsymbol{u}_* = \boldsymbol{w}_* - \boldsymbol{w}_\infty$. In [17], Finn and Smith studied on the linear equation obtained by formally removing the nonlinear term in (2.17):

$$\begin{cases} \Delta \boldsymbol{u} - \lambda \dfrac{\partial \boldsymbol{u}}{\partial x_1} - \nabla p = \boldsymbol{0} \qquad \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega \end{cases}$$

with the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_*,$$

$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0.$$

This linear equation is called the Oseen equations, originally proposed by Oseen. (see the introduction of Chapter V in Galdi [21] for detailed history of the Oseen equations.) The non-homogeneous boundary value problem of the Oseen equations are solvable even in two dimension in contrast to the Stokes problem.

Second, due to the term $\partial \boldsymbol{u}/\partial x_1$ in the equation (2.17), a solution $\boldsymbol{u}$ has anisotropic asymptotic behaviors at infinity. With help of this anisotropic asymptotic behavior, in

1965, Smith [49] made a painstaking estimate of the nonlinear contribution associated to the nonlinear term in (2.17). Combining these results of [17], [49] with a fixed point argument, they proved existence of a solution to (2.17), (2.18), (2.19),with $\boldsymbol{F} = \boldsymbol{0}$, which in turn yields existence of a solution to (2.14), (2.15), (2.16) . (Here the boundary condition at infinity is fulfilled in the sense that $\boldsymbol{w}$ converges to $\boldsymbol{w}_\infty$ uniformly.) In this regard, see also [15], [16], [39], [14].

On the other hand, Galdi also studied the same problem and prove an existence theorem as follows.

Theorem 2.0.1 (Theorem XII.5.1 Galdi [21]). *Let $\Omega \subset \mathbb{R}^2$ be an exterior domain of class C^2. Let $\boldsymbol{F} \in L^q(\Omega), \boldsymbol{w}_* \in W^{2-1/q,q}(\partial\Omega)$ for some $1 < q < 6/5$, and let $\boldsymbol{w}_\infty = \boldsymbol{e}_1$. Then there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0]$, if*

$$|\log \lambda|^{-1} \|\boldsymbol{w}_* - \boldsymbol{w}_\infty\|_{2-1/q,q,\partial\Omega} + \lambda^{2/q-1} \|\boldsymbol{F}\|_q < \frac{1}{32c^2}$$

for some constant $c = c(\Omega, q, \lambda_0)$, then there exists a solution $\boldsymbol{w}, p$ to (2.14), (2.15), (2.16) such that $\boldsymbol{w} - \boldsymbol{w}_\infty \in X^{2,q}(\Omega)$ and $p \in D^{1,q}(\Omega)$. Furthermore,

$$\|\boldsymbol{w} - \boldsymbol{w}_\infty\|_{X^{1,q}} \leq c^2 (\lambda^{2(1-1/q)} |\log \lambda|^{-1} \|\boldsymbol{w}_* - \boldsymbol{w}_\infty\|_{2-1/q,q,\partial\Omega} + \lambda \|F\|_q).$$

If there exists another solution $\tilde{\boldsymbol{w}}, \tilde{p}$ corresponding to the same data such that

$$c\lambda^{-2(1-1/q)} \|\tilde{\boldsymbol{w}} - \boldsymbol{w}_\infty\|_{X^{1,q}} < 13/64,$$

then $\boldsymbol{w} \equiv \tilde{\boldsymbol{w}}, p \equiv \tilde{p} + const.$

For notations and function spaces, see the end of this introduction and Section 2.2.

Even though Theorem 2.0.1 is taken from Theorem XII.5.1 in Galdi 2011 [21], the result was originally proved in Theorem 4.1 of Galdi 1993 [23].

Galdi used a fixed point argument but studied the Oseen equations in a way different from Finn and Smith's. Galdi's work involves Sobolev-type function spaces as opposed to Finn and Smith's. In this thesis, we mainly make use of these results of Galdi on the Oseen equations rather than Finn and Smith's.

And it is worth to notice that Finn and Smith's results as well as Galdi's do not impose any assumption on flux as like (2.8).

However, both Galdi's result and Finn and Smith's have some limitation. With λ fixed, data $\boldsymbol{w}_*$ must be chosen close to $\boldsymbol{w}_\infty = \langle 1, 0\rangle$ to ensure existence of a solution.

In this chapter, an improvement has been made to overcome this limitation. The main theorem of this chapter implies that with λ fixed (in a certain range), there exists a solution $(\boldsymbol{w}, \pi)$ to (2.14), (2.15), (2.16) even for some data $\boldsymbol{w}_*$ which is big in the sense that $\boldsymbol{w}_*$ is "far" from $\boldsymbol{w}_\infty$. See Theorem 2.0.3 below.

The key idea toward this end comes from A. Russo and Starita [44] in 2008.

Theorem 2.0.2 (Theorem 3.6 of Russo, Starita [44]). *Let Ω be an exterior Lipschitz domain in $\mathbb{R}^3$ and $\boldsymbol{v}_\infty = \mathbf{0}$. Let β be a harmonic function in Ω satisfying $\nabla\beta(x) = O(|x|^{-2})$. Assume $\boldsymbol{v}_* - \mu\nabla\beta \in L^\infty(\partial\Omega)$. Then there exists a countable subset $G \subset \mathbb{R}$ such that for any $\frac{\mu}{\nu} \notin G$, if*

$$\|\boldsymbol{v}_* - \mu\nabla\beta\|_{L^\infty(\partial\Omega)} \leq c \tag{2.20}$$

for some constant $c = c(\Omega, \mu, \nu, \nabla\beta)$, then there exists a solution $(\boldsymbol{v}, p)$ to (2.1), (2.2), (2.3) *with $\boldsymbol{f} = \mathbf{0}$ such that*

$$(\boldsymbol{v}, p) \in [L^\infty_\sigma(\Omega, r) \cap C^\infty(\Omega)] \times [L^\infty(\mathbb{R}^2 \setminus B_R, r^2 \log r) \cap C^\infty(\Omega)] \tag{2.21}$$

and $\boldsymbol{v}$ converges to $\boldsymbol{v}_*$ nontangentially.

The notation $L^\infty(\Omega, r), L^\infty(\mathbb{R}^2 \setminus B_R, r^2 \log r)$ refers to a weighted L^∞ space with weights $r, r^2 \log r$ respectively. [2]

They proved existence of a solution (vanishing at infinity) to the stationary Navier-Stokes equations in *three* dimensional exterior domain with the boundary condition (2.20), which allows some large data $\boldsymbol{w}_*$ because μ can be chosen large.

To explain their idea, if $\boldsymbol{v}$ is a solution to the stationary Navier-Stokes equations (2.1) and β is a harmonic function in Ω such that $\nabla\beta$ satisfies some decaying condition at infinity, then setting $\boldsymbol{u} = \boldsymbol{v} - \mu\nabla\beta$ for a scalar parameter μ leads to equivalent equations for $\boldsymbol{u}$ that have linear terms involving $\mu\nabla\beta$:

$$\begin{cases} \nu\Delta\boldsymbol{u} - \nabla p = \boldsymbol{f} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mu(\nabla\beta\cdot\nabla)\boldsymbol{u} + \mu(\boldsymbol{u}\cdot\nabla)\nabla\beta \quad \text{in } \Omega \\ \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega \\ \qquad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_* \\ \lim\limits_{|x|\to\infty} \boldsymbol{u}(x) = 0 \end{cases}$$

(They, in fact, did not involve an external force in their work.) Here $\nabla\beta$ is called a potential flow. They proved compactness of a volume potential associated with these linear terms, which is actually a linear operator of $\boldsymbol{u}$. Then they applied a spectral theorem of compact linear operators along with a fixed point argument to make a conclusion as desired. What they eventually proved implies existence of a solution $\boldsymbol{u}$ for data $\boldsymbol{v}_*$ close to $\mu\nabla\beta$, whose parameter μ can be chosen large outside a countable subset of $\mathbb{R}$.

The idea of perturbing around a potential flow originated from H. Fujita and H. Morimoto [19] in 1997 where they used the idea in a bounded domain. And after A. Russo and Starita's work [44], A. Russo and Tartaglione [45] in 2011 extended the result of [44] to higher dimension. But all of these papers only concern with the case of vanishing vector prescribed at infinity (and do not involve external force).

[2] the authors of [44] did not explicitly mention about the pressure or the radius R in their statement or in the proof, and it is unclear to me how the radius R is determined.

To incorporate this idea into our main problem (2.14), (2.15), (2.16) in the case $\boldsymbol{v}_\infty \neq \mathbf{0}$ and in two dimension, we fix a harmonic function β in Ω such that $\nabla\beta$ decays at infinity, and we derive another form of the stationary Navier-Stokes equations rather than (2.17) by setting $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{w}_\infty - \mu\nabla\beta$ where μ is a scalar parameter if $\boldsymbol{w}$ is a solution to (2.14) with $\boldsymbol{w}_\infty = e_1$ Then for $\boldsymbol{u}_* = \boldsymbol{w}_* - \boldsymbol{w}_\infty - \mu\nabla\beta$ on $\partial\Omega$ and

$$p = \pi + \frac{\lambda\mu^2}{2}|\nabla\beta|^2 + \lambda\mu\frac{\partial\beta}{\partial x_1},$$

the equations (2.14) turn into

$$\begin{cases} \Delta\boldsymbol{u} - \lambda\dfrac{\partial\boldsymbol{u}}{\partial x_1} - \nabla p = \lambda(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \lambda\mu(\nabla\beta\cdot\nabla)\boldsymbol{u} + \lambda\mu(\boldsymbol{u}\cdot\nabla)\nabla\beta + \lambda\boldsymbol{F} \quad \text{in } \Omega \\ \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega \end{cases} \tag{2.22}$$

with the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_* \tag{2.23}$$

$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0 \tag{2.24}$$

Here we have used the fact that $\nabla\beta$ is the gradient of a harmonic function. We will use this equivalent form (2.22) in the proof of our main result.

Here is one of our main theorems in this chapter.

Theorem 2.0.3. *Let Ω be an exterior domain in $\mathbb{R}^2$ of class C^2. Let $\boldsymbol{w}_\infty = \boldsymbol{e}_1, q \in (1, 6/5)$ and $\lambda \in (0, \lambda_0]$ for some constant $\lambda_0 = \lambda_0(\Omega)$. Fix a harmonic function β in Ω. Assume $\nabla\beta \in X^{2,q}(\Omega)$. Let $\mu \in \mathbb{R}\setminus G$ for some countable subset $G = G(\lambda, \Omega, q, \nabla\beta)$ of $\mathbb{R}$. If $\boldsymbol{w}_* \in W^{2-1/q,q}(\partial\Omega)$ and $\boldsymbol{F} \in L^q(\Omega)$ satisfies*

$$\|\boldsymbol{w}_* - \boldsymbol{w}_\infty - \mu\nabla\beta\|_{2-1/q,q,\partial\Omega} + \|\boldsymbol{F}\|_{q,\Omega} < \frac{1}{4c_1c_2} \tag{2.25}$$

for some constants $c_j = c_j(\Omega, q, \lambda, \mu, \nabla\beta) > 0, j = 1, 2,$ *then there exists a solution* $(\boldsymbol{w}, \pi)$ *to* (2.14), (2.15),(2.16) *such that*

$$(\boldsymbol{w} - \boldsymbol{w}_\infty - \mu\nabla\beta) \in X^{2,q}(\Omega), \quad \pi + \frac{\lambda\mu^2}{2}|\nabla\beta|^2 + \lambda\mu\frac{\partial\beta}{\partial x_1} \in Y^{1,q}(\Omega) \tag{2.26}$$

$$\|\boldsymbol{w} - \boldsymbol{w}_\infty - \mu\nabla\beta\|_{X^{2,q}(\Omega)} \leq 2c_1(\|\boldsymbol{w}_* - \boldsymbol{w}_\infty - \mu\nabla\beta\|_{2-1/q,q,\partial\Omega} + \|\boldsymbol{F}\|_{q,\Omega}). \tag{2.27}$$

If there exists another solution $(\tilde{\boldsymbol{w}}, \tilde{\pi})$ *corresponding to the same data* $\boldsymbol{w}_*, \boldsymbol{F}$ *such that*

$$(\tilde{\boldsymbol{w}} - \boldsymbol{w}_\infty - \mu\nabla\beta) \in X^{2,q}(\Omega), \quad \tilde{\pi} + \frac{\lambda\mu^2}{2}|\nabla\beta|^2 + \lambda\mu\frac{\partial\beta}{\partial x_1} \in Y^{1,q}(\Omega),$$
$$\|\tilde{\boldsymbol{w}} - \boldsymbol{w}_\infty - \mu\nabla\beta\|_{X^{2,q}(\Omega)} \leq \frac{1}{2c_2}, \tag{2.28}$$

then $\boldsymbol{w} \equiv \tilde{\boldsymbol{w}}, \pi \equiv \tilde{\pi} +$ *const.*

A proof of Theorem 2.0.3 is written in Section 2.8 (page 63).

Remark 2.0.4. In this Theorem 2.0.3, as mentioned earlier, fixing $\lambda \in (0, \lambda_0]$ (this smallness assumption does not depend on the data $\boldsymbol{w}_*, \boldsymbol{F}$), μ can be chosen large because G is a countable set. Hence large boundary value $\boldsymbol{w}_*$ is allowed, which is not the case for Finn and Smith's result and Galdi's.

Remark 2.0.5. An example of $\nabla\beta \in X^{2,q}(\Omega), 1 < q < 6/5$ is $\beta = \log|x|$. Note that this example $\nabla\beta$ have nonzero flux on the boundary $\partial\Omega$. Therefore, as μ may be large, $\mu\nabla\beta$ may have a large total flux.

Remark 2.0.6. The author of this thesis does not know whether the exceptional set $G = G(\lambda, \Omega, q, \nabla\beta)$ is empty or not. But in the special case of $\Omega = \{x \in \mathbb{R}^2 : R_1 < |x| < R_2\}$ and $\beta = \log|x|$, H. Morimoto proved that the exceptional set G that he got by applying the same method as the one for Theorem 2.0.3 is empty. See, for example, [41].

Remark 2.0.7. Theorems 2.0.3, 2.0.8 pertain to the dimensionless form (2.14). If we use the original form (2.1) with (2.2), (2.3), then let $d = \text{diam}(\Omega)$, and via scaling and rotation, the problem (2.1), (2.2), (2.3) turns into the problem (2.14), (2.15), (2.16) with $\lambda = (|\boldsymbol{v}_\infty| d)/\nu$ (and with diam(Ω) normalized to be 1). Hence small Reynolds number λ amounts to small velocity $|\boldsymbol{v}_\infty|$ at infinity or small obstacle d or large viscosity ν. Therefore, all the theorems of Finn and Smith, Galdi along with Theorem 2.0.8 mean that a solution exists if the velocity $|\boldsymbol{v}_\infty|$ at infinity or the size of the obstacle is small, or if viscosity ν is large.

Theorem 2.0.3 pertains to the case $\boldsymbol{v}_\infty \neq \mathbf{0}$ in two-dimension whereas Russo and Starita's result deal with the case $\boldsymbol{v}_\infty = \mathbf{0}$ in three-dimension.

The constants c_1, c_2 depend on λ. But Theorem 2.0.3 does not provide information about how c_1, c_2 behave as λ varies. However, we can prove a theorem with estimates with constants independent of $\lambda \in (0, \lambda_0]$ provided that $\mu\lambda^{2/q-5/3}$ is sufficiently small. (Note that the power $\frac{2}{q} - \frac{5}{3} > 0$ for $q \in (1, 6/5)$)

Theorem 2.0.8. *Let Ω be an exterior domain in $\mathbb{R}^2$ of class C^2. Let $\boldsymbol{w}_\infty = \boldsymbol{e}_1, q \in (1, 6/5)$ and $\lambda \in (0, \lambda_0]$ for some constant $\lambda_0 = \lambda_0(\Omega)$. Fix a harmonic function β in Ω and assume $\nabla\beta \in X^{2,q}(\Omega)$. Let $\mu \in \mathbb{R} \setminus G$ for some countable subset $G = G(\lambda, \Omega, q, \nabla\beta)$ of $\mathbb{R}$. Assume in addition that $c_3\mu\lambda^{\frac{2}{q}-\frac{5}{3}} < 1$ for some constant $c_3 = c_3(\Omega, q, \lambda_0, \nabla\beta)$. If $\boldsymbol{w}_* \in W^{2-1/q,q}(\partial\Omega)$ and $\boldsymbol{F} \in L^q(\Omega)$ satisfies*

$$\frac{c_4^2(|\log\lambda|^{-1}\|\boldsymbol{w}_* - \boldsymbol{w}_\infty - \mu\nabla\beta\|_{2-1/q,q,\partial\Omega} + \lambda^{\frac{2}{q}-1}\|\boldsymbol{F}\|_{q,\Omega})}{(1 - c_3\mu\lambda^{\frac{2}{q}-\frac{5}{3}})^2} < \frac{1}{4} \tag{2.29}$$

for some constant $c_4 = c_4(\Omega, q, \lambda_0)$, then there exists a solution $(\boldsymbol{w}, \pi)$ to (2.14), (2.15),(2.16) *such that*

$$(\boldsymbol{w} - \boldsymbol{w}_\infty - \mu\nabla\beta) \in X^{2,q}(\Omega), \quad \pi + \frac{\lambda\mu^2}{2}|\nabla\beta|^2 + \lambda\mu\frac{\partial\beta}{\partial x_1} \in Y^{1,q}(\Omega)$$

and

$$\|\boldsymbol{w}-\boldsymbol{w}_\infty-\mu\nabla\beta\|_{X^{2,q}(\Omega)} \leq \frac{c_4(\lambda^{2(1-1/q)}|\log\lambda|^{-1}\|\boldsymbol{w}_*-\boldsymbol{w}_\infty-\mu\nabla\beta\|_{2-1/q,q,\partial\Omega}+\lambda\|\boldsymbol{F}\|_{q,\Omega})}{1-c_3\mu\lambda^{\frac{2}{q}-\frac{5}{3}}}$$

If there exists another solution $(\tilde{\boldsymbol{w}},\tilde{\pi})$ *corresponding to the same data* $\boldsymbol{w}_*,\boldsymbol{F}$ *such that*

$$(\tilde{\boldsymbol{w}}-\boldsymbol{w}_\infty-\mu\nabla\beta)\in X^{2,q}(\Omega),\quad \tilde{\pi}+\frac{\lambda\mu^2}{2}|\nabla\beta|^2+\lambda\mu\frac{\partial\beta}{\partial x_1}\in Y^{1,q}(\Omega),$$
$$\|\tilde{\boldsymbol{w}}-\boldsymbol{w}_\infty-\mu\nabla\beta\|_{X^{2,q}(\Omega)}\leq\frac{1-c_3\mu\lambda^{\frac{2}{q}-\frac{5}{3}}}{8c_0}\lambda^{2(1-/q)}, \tag{2.30}$$

then $\boldsymbol{w}\equiv\tilde{\boldsymbol{w}},\pi\equiv\tilde{\pi}+const.$

A proof of Theorem 2.0.8 is written in Section 2.8 (page 63).

Notations

In this thesis, we denote a vector-valued or tensor-valued function in boldface: For example, we denote a scalar field by p, a vector field by $\boldsymbol{v}=(v_1,v_2)$ and a tensor field by $\boldsymbol{T}=\{T_{jk}\}$.

In this chapter, for a vector field $\boldsymbol{u}$ defined on Ω, $1\leq q\leq\infty$ we write

$$\|\boldsymbol{u}\|_q=\|\boldsymbol{u}\|_{L^q(\Omega)},\qquad |\boldsymbol{u}|_{m,q}=\|D^m\boldsymbol{u}\|_{L^q(\Omega)}=\sum_{|\alpha|=m}\|D^\alpha\boldsymbol{u}\|_{L^q(\Omega)},\ m=1,2,\dots$$

where the summation is taken over all multi-index α with $|\alpha|=m$. Here, for $\alpha=(\alpha_1,\alpha_2)$,

$$D^\alpha\boldsymbol{u}=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\boldsymbol{u}$$

denotes the α-th weak derivative of $\boldsymbol{u}$. For an arbitrary domain $\mathcal{A}$, we write

$$\|\boldsymbol{u}\|_{q,\mathcal{A}}=\|\boldsymbol{u}\|_{L^q(\mathcal{A})},\qquad |\boldsymbol{u}|_{m,q,\mathcal{A}}=\|D^m\boldsymbol{u}\|_{L^q(\mathcal{A})}.$$

We denote the norm of $W^{m-1/q,q}(\partial\Omega), m = 1, 2, \ldots$ by

$$\|\boldsymbol{u}\|_{m-1/q,q,\partial\Omega} = \|\boldsymbol{u}\|_{W^{m-1/q,q}(\partial\Omega)}.$$

For $1 \leq q \leq \infty$, $m \in \mathbb{N}$, we denote

$$D^{m,q} = D^{m,q}(\Omega) = \{\boldsymbol{u} \in L^1_{loc}(\Omega) : D^m \boldsymbol{u} \in L^q(\Omega)\}.$$

We denote by $D_0^{m,q}(\Omega)$ by the completion of the normed space $\{C_0^\infty(\Omega); |\cdot|_{m,q}\}$. And we denote by $D_0^{-m,q}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in the norm

$$|f|_{-m,q} = \sup\left\{\left|\int fu\right| : u \in D_0^{m,q}(\Omega), |u|_{m,q} = 1\right\}. \tag{2.31}$$

In addition, for a vector field $\boldsymbol{v}$ and pressure field p, we denote by $\boldsymbol{T}(\boldsymbol{v}, p) = \{T_{jk}(\boldsymbol{v}, p)\}$ the *Cauchy stress tensor* given by

$$T_{jk}(\boldsymbol{v}, p) = -p\delta_{jk} + \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j}\right).$$

For an exterior domain Ω with the origin of coordinates in Ω^c, we set

$$\Omega_r = \Omega \cap B_r(0) = \Omega \cap B_r,$$
$$\Omega^r = \Omega - \overline{\Omega}_r.$$

Outline of this chapter

This chapter is divided into three parts: Leray's method (Section 2.1), perturbation methods (all the other sections except the last one), Liouville problem in three dimension (Section 2.12). In Section 2.1, we provide more detailed history of Leray's method up to very recent breakthroughs by Korobkov, Pileckas, Russo.

For perturbation methods, Section 2.2 contains definitions, notations, properties of function spaces that Galdi used. In Section 2.3, we present on basic results of the Stokes equations including the Stokes paradox. In Section 2.4, we study an existence theorem of the Oseen equations and a compactness theorem, which will be used in our main results of this chapter. Section 2.5 provides definitions of the Stokes and the Oseen fundamental solutions and present on their asymptotic behavior. In Section 2.6, we study representation formula of solutions to the Stokes and the Oseen equations. Section 2.7 furnishes study of the nonlinear problem by perturbing around $\boldsymbol{w}_\infty = \boldsymbol{e}_1$. In Section 2.8, we provide more history of perturbation methods around a potential flow and our main results of this topic with their proofs. In the next Section 2.9, we present a recent result by using perturbation methods in case $\boldsymbol{v}_\infty = \mathbf{0}$. In Section 2.10 we briefly study asymptotic behavior of a solution to the nonlinear problem. In Section 2.11 we present a result about the behavior of a solution to the nonlinear problem in the limit of vanishing Reynolds number.

In the last section 2.12, we briefly present on history of a Liouville problem in three dimension of the stationary Navier-Stokes equations.

2.1 Leray's Method

This section is devoted to provide more history of Leray's method.

As mentioned in the introduction of this chapter, study of the non-homogeneous boundary value problem of the stationary Navier-Stokes equations in a two-dimensional exterior domain was initiated by J. Leray in 1933 [40]. He proved existence of a solution to the problem (2.1),(2.2) with finite Dirichlet energy (2.4) under assumption (2.5) on the flux of the boundary function on each connected component of the boundary. But he was not able to prove that his solution satisfies the boundary condition (2.3) at infinity only in the two-dimensional case.

Leray left out several important open problems other than the validity of (2.3):

removing a condition on the flux of the boundary function and non-triviality of his solution.

Regarding the validity of (2.3), in the three-dimensional case, Leray proved validity of the boundary condition (2.3) at infinity using the condition of finite Dirichlet energy condition. But in two dimension, the condition of finite Dirichlet energy alone no longer implies convergence at infinity because there are examples in two dimension with finite Dirichlet energy condition which grows at infinity. Proving the validity of the boundary condition in Leray's method in two dimension is one of the major challenging open problems, which is still open to date.

To overcome this difficulty, there have been numerous remarkable results. Gilbarg and Weinberger [50], [27] proved that for arbitrary D-solution $\boldsymbol{v}$, if $\boldsymbol{v}$ is bounded, then there is a constant vector $\tilde{\boldsymbol{v}}_\infty$ such that

$$\lim_{r\to\infty}\int_0^{2\pi}|\boldsymbol{v}(r,\theta)-\tilde{\boldsymbol{v}}_\infty|^2 d\theta = 0 \tag{2.32}$$

(along with more information about the asymptotic behavior of $\boldsymbol{v}$). Note that it does not give information about the relation between $\boldsymbol{v}_\infty, \tilde{\boldsymbol{v}}_\infty$ and it still remains open. (Note that Leray's construction of a solution involves the prescribed vector $\boldsymbol{v}_\infty$ at infinity.) Amick [1] proved that if $\boldsymbol{v}$ vanishes on the boundary, then $\boldsymbol{v}$ is bounded. However if $\boldsymbol{v}$ vanishes on the boundary, then the solution could even be the trivial one. Amick [1] proved non-triviality of Leray's solution with some symmetry assumption on the domain.

Regarding condition (2.5), that is,

$$\int_{\partial\Omega_i}\boldsymbol{v}_*\cdot\boldsymbol{n}\,d\sigma = 0 \quad \text{for all } i=1,\cdots,M,$$

where the complement of the domain Ω is written as a disjoint union of several connected components, that is, $\Omega^c = \cup_{i=1}^M \Omega_i$. A. Russo [43] relaxed this condition into a condition of small flux but in the case $M = 1$ ($\partial\Omega$ has a single connected component).

Theorem 2.1.1 (Theorem 4 of A. Russo [43])**.** *Let $\Omega = \mathbb{R}^2 \backslash \overline{\Omega}_0$ where $\partial\Omega_0$ is connected and Lipshitz. Let $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ and $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$. If*

$$\nu > \frac{\xi}{2\pi} \left| \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} d\sigma \right| \tag{2.33}$$

where

$$\xi = \sup \left\{ \left| \int_{\mathbb{R}^2} \log|x| \mathrm{div}(\boldsymbol{w} \cdot \nabla \boldsymbol{w}) \right|; \boldsymbol{w} \in D_\sigma^{1,2}(\mathbb{R}^2), \|\boldsymbol{w}\|_{D_\sigma^{1,2}(\mathbb{R}^2)} = 1 \right\}$$

then there exists a weak solution $\boldsymbol{v} \in W_\sigma^{1,2}(\Omega)$ to

$$\begin{cases} \nu\Delta\boldsymbol{v} - \boldsymbol{v} \cdot \nabla\boldsymbol{v} = \nabla p + \boldsymbol{f} & \text{in } \Omega \\ \mathrm{div}\boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}_*. \end{cases}$$

Here the subscript σ of $D_\sigma^{1,2}, W_\sigma^{1,2}$ implies that we consider divergence free vector fields, that is, $D_\sigma^{1,2} = \{\boldsymbol{w} \in D^{1,2} : \mathrm{div}\boldsymbol{w} = 0\}$, $W_\sigma^{1,2} = \{\boldsymbol{v} \in W^{1,2}(\Omega) : \mathrm{div}\boldsymbol{v} = 0\}$.

Very recently, Korobkov, Pileckas, Russo have made major breakthroughs for various open problems in this topic with remarkable methods. They proved that a Dirichlet solution converges uniformly to some vector $\tilde{\boldsymbol{v}}_\infty$ at infinity without any additional assumptions in 2019 [36] and that a Dirichlet solution to (2.1), (2.2) with $\boldsymbol{f} = \boldsymbol{0}$ exist under the vanishing total flux assumption in 2020 [37] ; they also proved uniform boundedness of a Dirichlet solution in 2020 [37] and non-triviality of Leray's solution in 2021 [38]. In addition, they also established an existence theorem of a solution in an exterior axially symmetric domain in three dimension in 2018 [35].

Their idea is to study stream functions and head pressures and geometry of their level sets by using Bernouli's law for a Sobolev solution to the stationary Euler equations, which was also recently proved. This method is very robust to address many problems in the topic of the stationary Navier-Stokes equations in a two-dimensional exterior domain.

We state here some of their recent theorems.

Theorem 2.1.2 (Theorem 1.2 of Korobkov, Pileckas, Russo [37])**.** *Let $\Omega \subset \mathbb{R}^2$ be an exterior domain of class C^2. Suppose that $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$ and*

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} d\sigma = 0. \tag{2.34}$$

Then there exists a D-solution $\boldsymbol{v}$ to (2.1), (2.2) *with $\boldsymbol{f} \equiv \mathbf{0}$.*

Theorem 2.1.3 (Theorem 1.1 of Korobkov, Pileckas, Russo [38])**.** *Let Ω be an exterior domain in $\mathbb{R}^2$ with smooth compact boundary and $\boldsymbol{v}_\infty \neq \mathbf{0}$. Take a sequence $\boldsymbol{v}_k$ of solutions to* (2.6) *with $\boldsymbol{v}_* = \mathbf{0}$, and take further arbitrary weakly convergence subsequence $\boldsymbol{v}_{k_l}$ that weakly converges to $\boldsymbol{v}$. Then the limiting solution $\boldsymbol{v}$ to* (2.1), (2.2) *with $\boldsymbol{f} = \mathbf{0}, \boldsymbol{v}_* = \mathbf{0}$ is nontrivial. In particular, Leray's solution is nontrivial.*

Theorem 2.1.4 (Theorem 1.2 of Korobkov, Pileckas, Russo [36])**.** *Let $\boldsymbol{v}$ be a D-solution to the stationary Naiver-Stokes equations* (2.1) *with $\boldsymbol{f} = \mathbf{0}$ in an exterior domain $\Omega \subset \mathbb{R}^2$. Then $\boldsymbol{v}$ converges uniformly at infinity to the constant vector $\tilde{\boldsymbol{v}}_\infty$ defined in* (2.32)*.*

Despite of these remarkable series of papers by Korobkov, Pileckas, Russo, there are still challenging open problems in the direction of Leray's method. It is still open to relax condition (2.34) in order to allow at least small total flux, and it is open to show $\boldsymbol{v}_\infty = \tilde{\boldsymbol{v}}_\infty$ (or they do not match).

In addition, as mentioned earlier, they solved a big open problem in a bounded domain by using a similar method in 2015.

Theorem 2.1.5 (Theorem 1.1 of Korobkov, Pileckas, Russo [34])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class C^2. If $\boldsymbol{f} \in W^{1,2}(\Omega)$ and $\boldsymbol{v}_* \in W^{3/2,2}(\partial\Omega)$ satisfy*

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} \, d\sigma = 0,$$

then there exists a weak solution $\boldsymbol{v}$ to the problem

$$\begin{aligned} \nu\Delta\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla p = \boldsymbol{f} \quad & \textit{in } \Omega \\ \mathrm{div}\boldsymbol{v} = 0 \quad & \textit{in } \Omega \\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}_*. & \end{aligned} \tag{2.35}$$

To illustrate their methods, we present here the main idea of their proof of Theorem 2.1.5. To this end, we first study Leray's argument *reductio ad absurdum.* Let $H(\Omega)$ be the subspace of all solenoidal vector fields from $D_0^{1,2}(\Omega)$ equipped with $\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}$. Let $\boldsymbol{V}$ be a solenoidal extension of $\boldsymbol{v}_*$ and $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{V}$. Then by Leray-Schauder Thereom, there exists a weak solution $\boldsymbol{v}$ to the problem (2.35) if and only if all solutions $\boldsymbol{w}$ to the probem

$$\begin{aligned} \nu\int_\Omega \nabla\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}dx - \lambda\int_{\partial\Omega} ((\boldsymbol{w}+\boldsymbol{V})\cdot\nabla)\,\boldsymbol{\eta}\cdot\boldsymbol{w}dx & \\ - \lambda\int_\Omega \boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\cdot\boldsymbol{V}dx = \lambda\int_\Omega \boldsymbol{V}\cdot\nabla\boldsymbol{\eta}\cdot\boldsymbol{V}dx, & \quad \text{for all } \boldsymbol{\eta} \in H(\Omega) \end{aligned}$$

are uniformly bounded in $H(\Omega)$ with respect to $\lambda \in [0,1]$. Assuming this is not true, we can prove the following lemma.

Lemma 2.1.6 (Lemma 3.1 of Korobkov, Pileckas, Russo [34])**.** *Let Ω be a bounded domain of class C^2. If there exists no weak solution to* (2.35)*, then there exists $\boldsymbol{u}, p$ that satisfies the following properties;*

- $\boldsymbol{u} \in W^{1,2}(\Omega), p \in W^{1,q}(\Omega), 1 < q < 2$ *is a solution to the Euler system*

$$\begin{aligned}(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla p = \mathbf{0} \quad & in\ \Omega \\ \mathrm{div}\boldsymbol{v} = 0 \quad & in\ \Omega \\ \boldsymbol{v}|_{\partial\Omega} = \mathbf{0}, &\end{aligned}$$

- *there exists a sequence* $\boldsymbol{u}_k \in W^{1,2}(\Omega), p_k \in W^{1,q}(\Omega)$ *and numbers* $\nu_k \to 0+, \lambda_k \to \lambda_0 > 0$ *such that* $\|\boldsymbol{u}_k\|_{W^{1,2}(\Omega)}, \|p_k\|_{W^{1,q}(\Omega)}$ *are uniformly bounded for every* $q \in [1, 2)$, *the pairs* $\boldsymbol{u}_k, p_k$ *satisfies*

$$\begin{aligned}\nu_k \Delta \boldsymbol{u}_k + (\boldsymbol{u}_k \cdot \nabla)\boldsymbol{u}_k + \nabla p_k = \boldsymbol{f}_k \quad & in\ \Omega \\ \mathrm{div}\boldsymbol{u}_k = 0 \quad & in\ \Omega \\ \boldsymbol{u}_k|_{\partial\Omega} = \boldsymbol{v}_{*k} &\end{aligned}$$

where $\boldsymbol{f}_k = \frac{\lambda_k \nu_k^2}{\nu^2}\boldsymbol{f}, \boldsymbol{v}_{*k} = \frac{\lambda_k \nu_k}{\nu}\boldsymbol{v}_*$, *and*

$$\|\nabla \boldsymbol{u}_k\|_{L^2(\Omega)} \to 1, \quad \boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \text{ in } W^{1,2}(\Omega), \quad p_k \rightharpoonup p \text{ in } W^{1,q}(\Omega) \text{ for all } q \in [1.2). \tag{2.36}$$

Moreover, $\boldsymbol{u}_k \in W^{3,2}_{loc}(\Omega), p_k \in W^{2,2}_{loc}(\Omega)$.

Now the goal is to reach a contradiction. Korobkov, Pileckas, Russo derived a contradiction as follows. For $i = 1, 2, \ldots$ and sufficiently large k, they construct a set $E_i \subset \Omega$ consisting of level lines of $\Phi_k := p_k + \frac{1}{2}|\boldsymbol{u}_k|^2$ such that $\Phi_k|_{E_i} \to 0$ as $i \to \infty$ and E_i separates the boundary component where $\Phi := p + \frac{1}{2}|\boldsymbol{u}|^2 = 0$ from the boundary components where $\Phi < 0$. By using Coarea formula, they established an estimate from below for $\int_{E_i} |\nabla \Phi_k|$. On the other hand, they established an estimate from above for $\int_{E_i} |\nabla \Phi_k|^2$. From these estimates, we can reach a contradiction as $i \to \infty$.

2.2 Sobolev-Type Function Spaces

We first introduce appropriate function spaces that are used in Galdi's approach. (See Galdi [24] p.20-23). For $1 < q < 3/2$, let $q^* = 2q/(2-q), s_1 = 3q/(3-q), s_2 = 3q/(3-2q)$. Note $q < s_1 < q^* < s_2$. For $1 < q < 3/2, \lambda > 0, \boldsymbol{u} = (u_1, u_2) \in L^1_{loc}(\Omega)$, define

$$\langle \boldsymbol{u} \rangle_{\lambda,q} = \lambda(\|u_2\|_{q^*} + |u_2|_{1,q}) + \lambda^{2/3}\|\boldsymbol{u}\|_{s_2} + \lambda^{1/3}|\boldsymbol{u}|_{1,s_1},$$

(Recall the Oseen equations have an anisotropical structure, so it is natural to work with anisotropic Sobolev-type norms like above.) Similarly, for an arbitrary domain $\mathcal{A}$ in $\mathbb{R}^2$, we define

$$\langle \boldsymbol{u} \rangle_{\lambda,q,\mathcal{A}} = \lambda(\|u_2\|_{q^*,\mathcal{A}} + |u_2|_{1,q,\mathcal{A}}) + \lambda^{2/3}\|\boldsymbol{u}\|_{s_2,\mathcal{A}} + \lambda^{1/3}|\boldsymbol{u}|_{1,s_1,\mathcal{A}},$$

We define

$$X^{1,q}(\Omega) = \{\boldsymbol{u} \in L^1_{loc}(\Omega) : \text{div}\boldsymbol{u} = 0 \text{ weakly, } \langle \boldsymbol{u} \rangle_{\lambda,q} < \infty\}$$
$$X^{2,q}(\Omega) = \{\boldsymbol{u} \in X^{1,q}(\Omega) : D^2\boldsymbol{u} \in L^q(\Omega)\}.$$

In these definitions of $X^{1,q}(\Omega)$ and $\langle \boldsymbol{u} \rangle_{\lambda,q}$, scaling by λ does not affect the definition of $X^{1,q}(\Omega)$. In other words, even if we define $\langle \boldsymbol{u} \rangle_{\lambda,q}$ with λ replaced by 1, it would define the same space $X^{1,q}(\Omega)$. But we use the scaled norm $\langle \cdot \rangle_{\lambda,q}$ because we will make estimates of the scaled norm $\langle \boldsymbol{u} \rangle_{\lambda,q}$ for a solution $\boldsymbol{u}$ to a fluid equation with a Reynolds number λ; and these estimates will involve constants independent of λ.

These X-spaces are set up for a velocity field $\boldsymbol{u}$; we also introduce a space for pressure p:

$$Y^{1,q}(\Omega) = \{p \in L^{q^*}(\Omega) : Dp \in L^q(\Omega)\}.$$

The spaces $X^{1,q}, X^{2,q}$ are reflexive, separable Banach spaces when equipped with the following norms respectively:

$$\|\boldsymbol{u}\|_{X^{1,q}} = \langle \boldsymbol{u} \rangle_{\lambda,q}$$
$$\|\boldsymbol{u}\|_{X^{2,q}} = \langle \boldsymbol{u} \rangle_{\lambda,q} + |\boldsymbol{u}|_{2,q}.$$

The space $Y^{1,q}$ is also a reflexive, separable Banach space endowed with the norm:

$$\|p\|_{Y^{1,q}} = \|p\|_{q^*} + |p|_{1,q}.$$

We observe that the continuous embedding holds: $X^{m,q}(\Omega) \hookrightarrow W^{m,q}(\Omega_R)$ for $m = 1, 2$ and large $R > 0$ where $\Omega_R = \Omega \cap B_R(0)$ and $W^{m,q}$ refers to the Sobolev spaces. If Ω is locally Lipschitz, then the trace map is continuous:

$$\boldsymbol{u} \in X^{m,q}(\Omega) \mapsto \boldsymbol{u}|_{\partial\Omega} \in W^{m-1/q,q}(\partial\Omega), \qquad m = 1, 2.$$

We observe that for every $\boldsymbol{u} \in X^{2,q}(\Omega)$, it holds that

$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0 \qquad \text{uniformly.}$$

For this convergence, see Remark 2.2.5 below.

The space that Galdi used in [24] is slightly different from the space $X^{1,q}(\Omega)$ that we define above. To understand the difference, we define a space $\tilde{X}^{1,q}(\Omega)$ by

$$\tilde{X}^{1,q}(\Omega) = \left\{ \boldsymbol{u} \in L^1_{loc}(\Omega) : \text{div}\boldsymbol{u} = 0, \langle \boldsymbol{u} \rangle_q + \left\| \frac{\partial \boldsymbol{u}}{\partial x_1} \right\|_q < \infty \right\}$$
$$\langle \boldsymbol{u} \rangle_q = \|u_2\|_{q^*} + |u_2|_{1,q} + \|\boldsymbol{u}\|_{s_2} + |\boldsymbol{u}|_{1,s_1}.$$

The space $\tilde{X}^{1,q}(\Omega)$ equipped with the norm $\langle \boldsymbol{u} \rangle_q + \|\partial \boldsymbol{u}/\partial x_1\|_q$ is a Banach space. This is the space defined by Galdi, for example, in [24]. And we will use some results of this paper [24].

Remark 2.2.1. We claim that $\tilde{X}^{1,q}(\Omega) = X^{1,q}(\Omega)$ and their norms are equivalent. It is obvious to see that $\tilde{X}^{1,q}(\Omega)$ is continuously embedded in $X^{1,q}(\Omega)$. On the other hand, if $\boldsymbol{u} \in X^{1,q}(\Omega)$, then $Du_2 \in L^q(\Omega)$. Then as $\boldsymbol{u}$ is divergence free, it holds that

$$\frac{\partial u_1}{\partial x_1} = -\frac{\partial u_2}{\partial x_2} \in L^q(\Omega).$$

Therefore, $\partial u_1/\partial x_1 \in L^q(\Omega)$ and

$$\left\| \frac{\partial u}{\partial x_1} \right\|_q \leq C \|Du_2\|_q \tag{2.37}$$

for some constant C. Hence, $X^{1,q}(\Omega)$ is also continuously embedded in $\tilde{X}^{1,q}(\Omega)$. Therefore, we can use results of Galdi in [24] with $\tilde{X}^{1,q}(\Omega)$ replaced by $X^{1,q}(\Omega)$.

The rest of this section is devoted to investigate the asymptotic behavior of a function in a homogeneous Sobolev space in an exterior domain. All of the material in this section is taken from Galdi's book [21].

First of all, the following preliminary result gives us information about the asymptotic behavior of a function $u \in D^{1,q}(\Omega), 1 \leq q < N$.

Lemma 2.2.2 (Lemma II.6.3 of Galdi [21]). *Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be an exterior domain and let $u \in D^{1,q}(\Omega), 1 \leq q < N$. Then, there exists a unique $u_0 \in \mathbb{R}$ such that, for all $R > \operatorname{diam}(\Omega^c)$*

$$\int_{S^{N-1}} |u(R,\omega) - u_0|^q d\omega \leq \gamma_0 R^{q-n} \int_{\Omega \cap B_R} |\nabla u|^q \tag{2.38}$$

where $\gamma_0 = [(q-1)/(n-q)]^{q-1}$ if $q > 1$ and $\gamma_0 = 1$ if $q = 1$.

The key idea of the proof in Galdi [21] is that we first show that $u(r,\omega)$ converges to $u^* \in L^q(S^{N-1})$ as $r \to \infty$ and then we establish the inequality (2.38) with

$$u_0 = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u^*(\omega) d\omega.$$

The main tools used are Hölder inequality and Wirtinger inequality (see equation (II.5.17) of Galdi [21]).

We next introduce a theorem about the Sobolev inequality that a function from $D^{1,q}(\Omega), 1 \leq 1 < N$, possibly modified by the addition of a uniquely determined constant, enjoys.

Theorem 2.2.3 ([21]). *Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be a locally Lipschitz exterior domain. Let $u \in D^{1,q}(\Omega), 1 \leq q < N$. Then for $w = u - u_0$ with u_0 defined in Lemma 2.2.2, $w \in L^{Nq/(N-q)}(\Omega)$ and*

$$\|w\|_{Nq/(N-q)} \leq \gamma_1 |w|_{1,q}$$

for some constant γ_1 independent of u.

This theorem is just part of Theorem II.6.1 of Galdi [21]. See the theorem in Galdi [21] for the full statement.

The main idea of the proof in Galdi [21] is that we first estimate a product of w and a cut-off function "near" infinity by the sum of $|w|_{1,q}$ and L^q norm of w in some bounded domain, which we control by $|w|_{1,q}$ with help of Lemma 2.2.2, and we estimate w "near" $\partial\Omega$ in a similar manner.

Theorem 2.2.4 ([21]). *Let $\Omega \subset \mathbb{R}^N, N \geq 2,$ be an exterior domain and let*

$$u \in L^s(\Omega) \cap D^{1,q}(\Omega), \text{ for some } s \in [1, \infty) \text{and some } q \in (N, \infty).$$

Then $u(x)$ converges to 0 uniformly as $|x| \to \infty$.

This theorem is just part of Theorem II.9.1 of Galdi [21]. See the theorem in Galdi [21] for the full statement.

The main idea of the proof in Galdi [21] is that, for a fixed x, we estimate $|u(x)|$ by the sum of L^1 norm of u near x and L^q norm of ∇u near x, both of which converge to zero as $|x| \to \infty$ by using the assumptions of Theorem 2.2.4.

Remark 2.2.5. With help of these Theorems 2.2.3, 2.2.4, we can prove that for $\boldsymbol{u} \in X^{2,q}(\Omega)$,

$$\lim_{|x|\to\infty} |\boldsymbol{u}(x)| = 0 \quad \text{uniformly}.$$

Indeed, recall that for $\boldsymbol{u} = (u_1, u_2) \in X^{2,q}(\Omega)$, it holds that

$$\boldsymbol{u} \in L^{s_2} \cap D^{1,s_1} \cap D^{2,q} \quad \text{in } \Omega$$

for

$$s_1 = \frac{3q}{3-q}, \quad s_2 = \frac{3q}{3-2q}, \quad 1 < q < \frac{3}{2}.$$

Then as $D\boldsymbol{u} \in L^{s_1} \cap D^{1,q}$ in Ω, it follows, from Theorem 2.2.3, that $\boldsymbol{u} \in D^{1,q^*}(\Omega)$. Then as $\boldsymbol{u} \in L^{s_2} \cap D^{1,q^*}$ in Ω and $q^* > 2$, the claim follows by Theorem 2.2.4. This remark comes from Remark XII.5.1 of Galdi [21] in 2011 and Remark 1.6 of Galdi [24] in 2004. [3]

2.3 The Stokes Equations

In this section, we are concerned with the linear equations in an exterior domain: The Stokes equations. Most of the material in this section comes from Galdi [24]. See also Chang and Finn 1961 [10], R. Russo 2010 [46], Sections V.5, 7 of Galdi [21].

[3] For Galdi's two remarks and this Remark 2.2.5, all of them pertain to slightly different function spaces, but the essential idea is the same.

First of all, considering the problem (2.1), (2.2), (2.3) with $\boldsymbol{v}_\infty = \mathbf{0}$, linearizing the nonlinear equations (2.1) around $\boldsymbol{v}_\infty = \mathbf{0}$ leads to the Stokes equations

$$\begin{cases} \Delta \boldsymbol{u} - \nabla p = \boldsymbol{f} \quad \text{in } \Omega \\ \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega. \end{cases} \tag{2.39}$$

with the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_*, \tag{2.40}$$

$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0. \tag{2.41}$$

(For simplicity, we assume $\nu = 1$.)

To study the problem (2.39), (2.40), (2.41), we can first investigate the problem with $\boldsymbol{u}_* = \mathbf{0}$ and without the boundary condition (2.41) at infinity .

Lemma 2.3.1 (Lemma 1.3 of Galdi [24])**.** *Let Ω be an exterior domain of class C^2. Let $\mathcal{S}_q$ be the linear subspace of $D_0^{1,q}(\Omega) \times L^q(\Omega), 1 < q < \infty$ constituted by the distributional solutions $(\boldsymbol{u}, \pi) \in D_0^{1,q}(\Omega) \times L^q(\Omega)$ to the following problem:*

$$\begin{cases} \Delta \boldsymbol{u} = \nabla p \quad \textit{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \quad \textit{in } \Omega \\ \boldsymbol{u}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

If $1 < q \le 2$, then $\mathcal{S}_q = \{(\mathbf{0}, 0)\}$. If $2 < q < \infty$, then $\dim \mathcal{S}_q = 2$. In this latter case, there exists a basis $\{\boldsymbol{h}^{(i)}, p^{(i)}\}_{i=1,2}$ in $\mathcal{S}_q$ satisfying the following properties

1. For all $1 < q < \infty$ and $i = 1, 2$, we have

$$(\boldsymbol{h}^{(i)}, p^{(i)}) \in [W^{2,q}_{loc}(\overline{\Omega}) \times L^q_{loc}(\overline{\Omega})] \cap [C^\infty(\Omega) \times C^\infty(\Omega)];$$

2. *There exists* $\boldsymbol{h}_\infty^{(i)} \in \mathbb{R}^2$ *and* $p_\infty^{(i)} \in \mathbb{R}, i = 1, 2$ *such that as* $|x| \to \infty$*, the following representation holds*

$$\boldsymbol{h}^{(i)}(x) = \boldsymbol{h}_\infty^{(i)} - \boldsymbol{U}(x) \cdot \boldsymbol{e}_i + O(|x|^{-1}) \quad p^{(i)}(x) = p_\infty^{(i)} + \boldsymbol{q}(x) \cdot \boldsymbol{e}_i + O(|x|^{-2}).$$

3. *For* $i = 1, 2$*, we have*

$$\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{h}^{(i)}, p^{(i)}) \cdot \boldsymbol{n} = \boldsymbol{e}_i.$$

This Lemma 2.3.1 was originally proved in Galdi and Simader 1990 [25]. (See Theorem 4.1 of [25].)

The main problem (2.39) , (2.40), (2.41) in two-dimension is not solvable for all data $\boldsymbol{u}^*, \boldsymbol{f}$.

Here $\boldsymbol{U}, \boldsymbol{q}$ are Stokes fundamental solutions which will be defined in (2.50).

Lemma 2.3.2 (Lemma 1.4 of Galdi [24])**.** *Let* Ω *be an exterior domain of class* C^2*. Then for every* $\boldsymbol{f} \in D_0^{-1,q}(\Omega), 1 < q < 2$ *satisfying* $\langle \boldsymbol{f}, \boldsymbol{h}^{(i)} \rangle = 0, i = 1, 2$*, the Stokes equations* (2.39) *has one and only one solution* $(\boldsymbol{u}, p) \in D_0^{1,q}(\Omega) \times L^q(\Omega)$*, in the sense of distributions.*

This Lemma 2.3.2 was originally proved in Galdi and Simader 1990 [25].

Theorem 2.3.3 (Theorem 1.1 of Galdi [24])**.** *Let* Ω *be an exterior domain of class* C^2*. Let* $\boldsymbol{u}_* \in W^{1-1/q,q}(\partial\Omega), 1 < q < \infty$*, and let* $(\boldsymbol{u}_0, p_0) \in W_{loc}^{1,q}(\overline{\Omega}) \times L_{loc}^q(\overline{\Omega})$ *satisfy the Stokes equations* (2.39) *in the sense of distribution,* (2.40) *in the trace sense, and* (2.41) *in the following averaged sense*

$$\lim_{r\to\infty} \int_0^{2\pi} |\boldsymbol{u}_0(r, \theta) - \boldsymbol{u}_\infty| d\theta = 0.$$

Then $\boldsymbol{u}_*$ and $\boldsymbol{u}_\infty$ must satisfy the following condition

$$\boldsymbol{u}_{\infty,i} = \int_{\partial\Omega} \boldsymbol{u}_* \cdot \boldsymbol{T}(\boldsymbol{h}^{(i)}, p^{(i)}) \cdot \boldsymbol{n}, \quad i = 1, 2. \tag{2.42}$$

Conversely, let $\boldsymbol{u}_* \in W^{1-1/q,q}(\partial\Omega), 1 < q < \infty$ and $\boldsymbol{u}_\infty \in \mathbb{R}^2$ satisfy (2.42). *Then there is a unique solution $\boldsymbol{u}_0, p_0 \in C^\infty(\Omega)$ to* (2.39) *that assumes the boundary data* (2.40) *in the sense of trace and that satisfies* (2.41) *uniformly pointwise. Moreover, as $|x| \to \infty$,*

$$\boldsymbol{u}_0(x) = \boldsymbol{u}_\infty + \boldsymbol{\zeta}(|x|)$$
$$D^\alpha \boldsymbol{\zeta}(x) = O(|x|^{-1-|\alpha|}), \qquad \textit{all } |\alpha| \geq 0.$$

In particular, if $\boldsymbol{u}_*$ is a nonzero constant vector, then the problem with $\boldsymbol{f} = \boldsymbol{0}$ does not have a solution. This phenomena is called the *Stokes paradox*, which attracts so much attention from various authors for long time. And this paradox makes it harder to study the case $\boldsymbol{v}_\infty = 0$ of the nonlinear problem (2.1), (2.2), (2.3) in two dimension.

In addition, Stokes found an explicit solution to problem (2.9), (2.10), (2.11) in some special case, which does not exhibit a wake region past the body. (See the introduction of Chapter V of Galdi [21].)

2.4 The Oseen Equations

Now we are concerned with the other linear equations: the Oseen equations. Consider the dimensionless form (2.14) (with $\boldsymbol{w}_\infty = \langle 1, 0 \rangle$), and linearize the equation (2.14) around $\boldsymbol{w}_\infty = \langle 1, 0 \rangle$ to obtain the following problem:

$$\begin{cases} \Delta \boldsymbol{u} - \lambda \dfrac{\partial \boldsymbol{u}}{\partial x_1} - \nabla p = \lambda \boldsymbol{F} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \end{cases} \tag{2.43}$$

with the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_*, \tag{2.44}$$

$$\lim_{|x|\to\infty} \boldsymbol{u}(x) = 0. \tag{2.45}$$

These linear equations (2.43) are called the Oseen equations. When it comes to flow past a body, Oseen equations are more accurate to describe actual flow compared to Stokes equations in several aspects: For the Oseen equations, the problem (2.43), (2.44), (2.45) admits a solution even in the two-dimensional case (without restrictions on data in contrast to the analogous problem of the Stokes equations); solutions to the Oseen equations even exhibit *wake phenomena*. (For wake phenomena, see Section 2.5.)

In order to study the Oseen equations in an exterior domain, we can first define a generalized solution to the problem (2.43), (2.44), (2.45) as follows.

Definition 2.4.1 (Definition VII.1.1 of Galdi [21])**.** A vector field $\boldsymbol{u} : \Omega \to \mathbb{R}^N$ is called a *q-generalized solution* to (2.43), (2.44), (2.45) if

1. $\boldsymbol{u} \in D^{1,2}(\Omega)$;
2. $\boldsymbol{u}$ is weakly divergence free in Ω;
3. $\boldsymbol{u}$ assumes $\boldsymbol{u}_*$ on $\partial\Omega$ in the trace sense, or if $\boldsymbol{u}_* = \mathbf{0}$, then $\boldsymbol{u} \in D_0^{1,2}(\Omega)$;
4. $\displaystyle\lim_{|x|\to\infty} \int_{S^{N-1}} |\boldsymbol{u}(x)| = 0$;
5. $$\int \nabla\boldsymbol{u} : \nabla\boldsymbol{\varphi} + \lambda \int \frac{\partial \boldsymbol{u}}{\partial x_1} \cdot \boldsymbol{\varphi} = -\lambda[\boldsymbol{F}, \boldsymbol{\varphi}]$$

   for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega; \mathbb{R}^N)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω.

Here $[\cdot,\cdot]$ refers to duality pairing between $D_0^{-1.2}(\Omega), D_0^{1,2}(\Omega)$.

Proposed by Finn and Smith in [17], in order to solve the Oseen equations (2.43) with (2.44), (2.45), we can first solve the modified problem:

$$\begin{aligned} \Delta \boldsymbol{u} - \lambda \frac{\partial \boldsymbol{u}}{\partial x_1} - \varepsilon \boldsymbol{u} - \nabla p = \lambda \boldsymbol{F} \quad & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \qquad & \text{in } \Omega \\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_* \qquad & \\ \lim_{|x|\to\infty} \boldsymbol{u}(x) = \mathbf{0}. \qquad & \end{aligned} \tag{2.46}$$

for $\varepsilon \in (0,1]$ along with suitable estimates uniform in $\varepsilon \in (0,1]$. Then we show that these solutions to the modified problem converge to a solution to the original problem (2.43), (2.44), (2.45).

Galdi also used the same idea of utilizing the modified equations (2.46) but, other than that, his proofs are totally different.

Galdi proved the following theorem.

Theorem 2.4.2 (Theorem VII.5.1 of Galdi [21])**.** *Let Ω be a two-dimensional, locally Lipschitz exterior domain. Then given*

$$\begin{aligned} &\boldsymbol{F} \in D_0^{-1,2}(\Omega) \cap L^q(\Omega), \quad 1 < q < 3/2, \\ &\boldsymbol{u}_* \in W^{1/2,2}(\partial\Omega), \end{aligned}$$

then there exists a unique weak solution $\boldsymbol{u}$ to (2.43), (2.44), (2.45)*. Moreover, for all $R >$ diam(Ω^c), this solution verifies*

$$\begin{aligned} \boldsymbol{u} &\in D^{2,q}(\Omega^R) \cap D^{1,3q/(3-q)}(\Omega^R) \cap L^{3q/(3-2q)}(\Omega) \\ u_2 &\in L^{2q/(2-q)}(\Omega) \cap D^{1,q}(\Omega) \\ \frac{\partial u_1}{\partial x_1} &\in L^q(\Omega) \\ p &\in D^{1,q}(\Omega^R), \end{aligned}$$

where p is the pressure field associated to $\boldsymbol{v}$ by Lemma VII.1.1 of Galdi [21]*. Finally, the following estimate holds:*

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{2,\Omega_R} + |\boldsymbol{u}|_{1,2} + \lambda\left(\|u_2\|_{2q/(2-q)} + |u_2|_{1,q} + \left\|\frac{\partial u_1}{\partial x_1}\right\|_q\right) \\
&+ \min\{1, \lambda^{2/3}\}\|\boldsymbol{u}\|_{3q/(3-2q)} + \lambda^{1/3}|\boldsymbol{u}|_{1,3q/(3-q),\Omega^R} + |\boldsymbol{u}|_{2,q,\Omega^R} + |p|_{1,q,\Omega^R} \\
&\leq c\left\{\lambda\left(\|\boldsymbol{F}\|_q + (1+\lambda)|\boldsymbol{F}|_{-1,2}\right) + (1+\lambda)^2\|\boldsymbol{u}_*\|_{1/2,2,\partial\Omega}\right\}
\end{aligned}
$$

where $c = c(q, \Omega, R)$.

In Theorem 2.4.2, $D_0^{-1,2}(\Omega)$ refers to the completion of $C_0^\infty(\Omega)$ in the norm

$$
|f|_{-1,2} = \sup\left\{\left|\int fu\right| : u \in D_0^{1,2}(\Omega), |u|_{1,2} = 1\right\}. \tag{2.47}
$$

Unlike Theorem 2.4.2 above, we can also prove existence, uniqueness and corresponding estimates of a solution $\boldsymbol{u}, p$ to (2.43), (2.44), (2.45) in $D^{2,q}(\Omega)$. Galdi proved a theorem as follows.

Theorem 2.4.3 (Theorem VII.7.1 of Galdi [21])**.** *Let Ω be an exterior domain in $\mathbb{R}^2$ of class C^2. Given $\boldsymbol{F} \in L^q(\Omega), \boldsymbol{u}_* \in W^{2-1/q,q}(\partial\Omega), 1 < q < 3/2$, there exists a unique solution $(\boldsymbol{u}, p)$ to the Oseen problem* (2.43)*,* (2.44)*,* (2.45) *such that*

$$
\begin{aligned}
&\boldsymbol{u} \in L^{s_2}(\Omega) \cap D^{1,s_1}(\Omega) \cap D^{2,q}(\Omega), \\
&u_2 \in L^{2q/(2-q)}(\Omega) \cap D^{1,q}(\Omega), \\
&p \in D^{1,q}(\Omega)
\end{aligned}
$$

where $s_1 = 3q/(3-q), s_2 = 3q/(3-2q)$*. Moreover,* $\boldsymbol{u}, p$ *verify*

$$\lambda\left(\|u_2\|_{2q/(2-q)} + |u_2|_{1,q}\right) + \min\{1, \lambda^{2/3}\}\|\boldsymbol{u}\|_{s_2} + \lambda\left\|\frac{\partial \boldsymbol{u}}{\partial x_1}\right\|_q + \min\{1, \lambda^{1/3}\}|\boldsymbol{u}|_{1,s_1} + |\boldsymbol{u}|_{2,q} + |p|_{1,q} \leq c(\lambda\|\boldsymbol{F}\|_q + \|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega}).$$

The constant c *depends on* q, Ω, λ*.*

This is part of Theorem VII.7.1 of Galdi [21]. The part omitted here is for higher dimension and higher derivatives.

Even though Theorem 2.4.3 is taken from Theorem VII.7.1 of Galdi 2011 [21], the result was originally proved in Galdi 1992 [22] (Theorem 2.2 of [22]).

As pointed out in the statement, the constant c involved in the estimate of Theorem 2.4.3 depends on λ. However, Galdi established a theorem with estimates that involve a constant independent of λ.

We state this theorem as follows, which we employ to prove our main theorems (contained in the introduction of the this chapter with its proofs in Section 2.8.).

Theorem 2.4.4 (Theorem 1.6 of Galdi [24])**.** *Let* Ω *be an exterior domain of class* C^2*. Given*

$$\boldsymbol{F} \in L^q(\Omega), \quad \boldsymbol{u}_* \in W^{2-1/q,q}(\partial\Omega), \quad 1 < q < 6/5,$$

there exists a unique solution $(\boldsymbol{u}, p) \in X^{2,q}(\Omega) \times Y^{1,q}(\Omega)$ *to the Oseen problem* (2.43)*,* (2.44)*,* (2.45)*. Moreover, there exists* $\lambda_0 = \lambda_0(\Omega) \in (0, 1]$ *such that for all* $\lambda \in (0, \lambda_0]$*, this solution* $(\boldsymbol{u}, p)$ *satisfies the estimate*

$$\|\boldsymbol{u}\|_{X^{2,q}} + |p|_{1,q} \leq c_0(\lambda^{2(1-1/q)}|\log\lambda|^{-1}\|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} + \lambda\|\boldsymbol{F}\|_q) \tag{2.48}$$

where the positive constant c_0 *depends only on* q, Ω *and* λ_0 *but independent of* λ*.*

The basic idea of the estimate of this Theorem 2.4.4 can be found in Remark 2.6.6 in Section 2.6

Even though Theorem 2.4.4 is taken from Theorem 1.6 of Galdi in 2004 [24], a similar statement was originally proved in Galdi 1993 [23] (Lemma 3.5 of [23]), which can be also found in Galdi 2011 [21] (Lemma XII.5.2 of [21]).

Note that in the estimate (2.48), the norm $\|\boldsymbol{u}\|_{X^{2,q}}$ on the left hand side depends on λ. For more details of this lemma, see also Theorem VII.7.1 of [21]

Remark 2.4.5. In the statement of this theorem 2.4.4, a pair $(\boldsymbol{u}, p)$ is a solution to the Oseen problem (2.43),(2.44),(2.45) in sense that:

- $(\boldsymbol{u}, p)$ satisfies the Oseen equations (2.43) almost everywhere in Ω
- $(\boldsymbol{u}, p)$ assumes the boundary value $\boldsymbol{u}_*$ in the trace sense
- $\boldsymbol{u}(x)$ converges to $\mathbf{0}$ uniformly as $|x| \to \infty$.

For the rest of this section, we formally define an operator $K_\beta(\boldsymbol{u}) = (\nabla\beta \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\nabla\beta$ and study this operator.

Lemma 2.4.6 (Lemma 1.6 of [24])**.** *Let $\mathcal{A}$ be an arbitrary domain in $\mathbb{R}^2$. For $\boldsymbol{u}, \boldsymbol{v} \in X^{1,q}(\mathcal{A}), 1 < q \leq 6/5$, the following inequality holds that for all $\lambda > 0$*

$$\|\boldsymbol{u} \cdot \nabla \boldsymbol{v}\|_{q,\mathcal{A}} \leq 4\lambda^{-1-2(1-1/q)} \langle \boldsymbol{u} \rangle_{\lambda,q,\mathcal{A}} \langle \boldsymbol{v} \rangle_{\lambda,q,\mathcal{A}}.$$

Recall for an arbitrary domain $\mathcal{A}$ in $\mathbb{R}^2$, we write

$$\langle \boldsymbol{u} \rangle_{\lambda,q,\mathcal{A}} = \lambda(\|u_2\|_{q^*,\mathcal{A}} + |u_2|_{1,q,\mathcal{A}}) + \lambda^{2/3}\|\boldsymbol{u}\|_{s_2,\mathcal{A}} + \lambda^{1/3}|\boldsymbol{u}|_{1,s_1,\mathcal{A}},$$

Sketch of the proof of Lemma 2.4.6 in Galdi [24]. By using the divergence free condition of $\boldsymbol{u}$ and Hölder inequality, we obtain

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_{q,\mathcal{A}} \leq \|u_1\|_{s_2,\mathcal{A}}|v_2|_{1,3/2,\mathcal{A}} + \|u_2\|_{3,\mathcal{A}}|\boldsymbol{v}|_{1,s_1,\mathcal{A}}.$$

Now we apply elementary L^q-interpolation inequalities on $|v_2|_{1,3/2,\mathcal{A}}, \|u_2\|_{3,\mathcal{A}}$ (note $q < 3/2 < q^* < 3 < s_2$), the claim follows. $\square$

By applying this lemma to the operator K_β, it holds that

$$K_\beta : X^{2,q}(\Omega) \to L^q(\Omega)$$

is well-defined and that it satisfies

$$\|K_\beta(u)\|_q \leq 8\lambda^{-1-2(1-1/q)}\langle\boldsymbol{u}\rangle_{\lambda,q}\langle\nabla\beta\rangle_{\lambda,q}. \tag{2.49}$$

Furthermore, we can also prove compactness of this operator K_β

Lemma 2.4.7 (Lemma 1.7 of [24])**.** *The operator $K_\beta : X^{2,q}(\Omega) \to L^q(\Omega)$ is compact for $q \in (1, 6/5]$.*

Sketch of proof of Lemma 2.4.7. Let $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}} \subset X^{2,q}(\Omega)$ with $\|\boldsymbol{u}_k\|_{X^{2,q}(\Omega)} = 1$. Then by reflexivity of $X^{2,q}(\Omega)$ and the Rellich theorem, there exists a subsequence, still denoted by $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}}$, and a function $\boldsymbol{u} \in X^{2,q}(\Omega)$ such that

$$\lim_{k\to\infty}\langle\boldsymbol{u}_k - \boldsymbol{u}\rangle_{\lambda,q,\Omega_R} = 0, \quad \forall R > \mathrm{diam}(\Omega^c).$$

We also find

$$\|K_\beta(\boldsymbol{u}_k - \boldsymbol{u})\|_q \leq c(\langle\nabla\beta\rangle_{q,\Omega_R}\langle\boldsymbol{u}_k - \boldsymbol{u}\rangle_{q,\Omega_R} + \langle\nabla\beta\rangle_{q,\Omega^R}\langle\boldsymbol{u}_k - \boldsymbol{u}\rangle_{q,\Omega^R})$$

Then the lemma follows. $\square$

2.5 The Stokes and the Oseen Fundamental Solutions

In this section, we study fundamental solutions to the Stokes and the Oseen equations especially about their asymptotic behaviors. Most of material of this section originates from Sections IV.2 and VII. 3 of Galdi [21]. Note that we only investigate the two-dimensional case.

The Stokes fundamental solution is given by

$$U_{ij}(x-y) = \left(\delta_{ij}\Delta - \frac{\partial^2}{\partial y_i \partial y_j}\right)\Phi(|x-y|) \tag{2.50}$$

$$q_j(x-y) = -\frac{\partial}{\partial y_j}\Delta\Phi(|x-y|) \tag{2.51}$$

where $\Phi(|x-y|) = \frac{1}{8\pi}|x-y|^2\log(|x-y|)$. Hence as either $|x| \to 0$ or $|x| \to \infty$, it holds that

$$U(x) = O(\log|x|), \quad D^\alpha U(x) = O(|x|^{-|\alpha|}), \quad |\alpha| \geq 1$$

and

$$D^\alpha q(x) = O(|x|^{-1-|\alpha|}), \quad |\alpha| \geq 0.$$

On the other hand, for the Oseen equations in the form of

$$\Delta\boldsymbol{u} - 2\lambda\frac{\partial\boldsymbol{u}}{\partial x_1} - \nabla p = 0, \quad \mathrm{div}\boldsymbol{u} = 0,$$

the Oseen fundamental solution is given by

$$E_{ij}(x-y) = \left(\delta_{ij}\Delta - \frac{\partial^2}{\partial y_i \partial y_j}\right)\Phi(x-y) \tag{2.52}$$

$$e_j(x-y) = -\frac{\partial}{\partial y_j}\left(\Delta - 2\lambda\frac{\partial}{\partial y_1}\right)\Phi(x-y) \tag{2.53}$$

where

$$\Phi(x-y) = \frac{1}{4\pi\lambda}\int_0^{x_1-y_1}\left\{\log\sqrt{\tau^2+(x_2-y_2)^2} + K_0\left(\lambda\sqrt{\tau^2+(x_2-y_2)^2}\right)e^{-\lambda\tau}\right\}d\tau$$
$$-\frac{1}{4\pi}\int_0^{y_2-x_2}(y_2-x_2-\tau)K_0(\lambda|\tau|)d\tau. \tag{2.54}$$

and K_0 denotes the modified Bessel function of the second kind.

Now from the property of K_0, as $\lambda|x-y| \to 0$, it holds that

$$E_{ij}(x-y) = U_{ij}(x-y) - \frac{1}{4\pi}\delta_{ij}\log\frac{1}{2\lambda} + o(1). \tag{2.55}$$

On the other hand, we can also study asymptotic behavior of E as $\lambda r \to \infty$. Denote by φ the angle made by a ray that starts from x and is directed toward y with the direction of the positive x_1-axis. And set $r = |x-y|$, $s = \lambda r(1-\cos\varphi)$. The Oseen fundamental solution E has the following asymptotic behavior:

$$E_{11}(x-y) = -\frac{\cos\varphi}{4\pi\lambda r} + \frac{e^{-s}}{4\sqrt{2\lambda\pi r}}\left(1+\cos\varphi - \frac{1-3\cos\varphi}{8\lambda r} + \mathcal{R}(\lambda r)\right)$$
$$E_{12}(x-y) = E_{21}(x-y) = \frac{\sin\varphi}{4\pi\lambda r} - \frac{e^{-s}\sin\varphi}{4\sqrt{2\lambda\pi r}}\left(1+\frac{3}{8\lambda r} + \mathcal{R}(\lambda r)\right)$$
$$E_{22}(x-y) = \frac{\cos\varphi}{4\pi\lambda r} + \frac{e^{-s}}{4\sqrt{2\pi}(\lambda r)^{3/2}}\left(s - \frac{1+3\cos\varphi}{8} + \mathcal{R}(\lambda r)\right)$$

where the remainder $\mathcal{R}(t)$ satisfies

$$\frac{d^k\mathcal{R}}{dt^k} = O(t^{-2-k}), \quad \text{as } t \to \infty, \quad k \geq 0.$$

The Oseen fundamental solution has a "nonsymmetric" structure. And the Oseen fundamental solution vanishes at infinity whereas the Stokes fundamental solution grows logarithmically at infinity, which is closely related to the Stokes paradox.

The Oseen fundamental solution even exhibit a wake region. When y is interior to the parabola $|y|(1-\cos\varphi)=1$,

$$|E_{11}(y)| \leq \frac{C}{|y|^{1/2}} \quad \text{as } |y| \to \infty.$$

On the other hand, when $(1-\cos\varphi) \geq |y|^{-1+2\sigma}$ for some $\sigma \in [0,1/2]$,

$$|E_{11}(y)| \leq \frac{C}{|y|^{1/2+\sigma}} \quad \text{as } |y| \to \infty.$$

This parabolic region is called a wake region.

For the remaining components of E, it holds that

$$|E_{j2}(y)| \leq \frac{C}{|y|}, \quad j=1,2 \quad \text{as } |y| \to \infty.$$

For much more information about asymptotic behavior of the Oseen fundamental solution, see Smith [49], page 347-348 and Kračmar, Novotný, Pokorý in 2001 [39]

2.6 Representation of Solutions to the Stokes and the Oseen Equations

In this section, we investigate representation formula of solutions to the Stokes and the Oseen equations. Most of the material is taken from Sections V.3 and VII.6 of Galdi [21]. But see also Chang and Finn [10], Finn and Smith [17], Galdi [24].

We define the *Cauchy stress tensor* $T_{ij}(\boldsymbol{u}, p)$ for a pair of a vector field $\boldsymbol{u} = \boldsymbol{u}(x)$

and a pressure field $p = p(x)$ by

$$T_{ij}(\boldsymbol{u}, p) = -p\delta_{ij} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For the Stokes equations, we define the *Stokes-Fujita truncated fundamental solution* $U_{ij}^{(R)}, q_j^{(R)}$ by replacing Φ by $\psi_R\Phi$ in the definition (2.50) of the Stokes fundamental solution U_{ij}, q_j, that is,

$$\begin{aligned} U_{ij}^{(R)}(x-y) &= \left(\delta_{ij}\Delta - \frac{\partial^2}{\partial y_i \partial y_j}\right)(\psi_R(|x-y|)\Phi(|x-y|)) \\ q_j^{(R)}(x-y) &= -\frac{\partial}{\partial y_j}\Delta\left(\psi_R(|x-y|)\Phi(|x-y|)\right) \end{aligned}$$

where $\psi_R(|x-y|) = \psi(|x-y|/R), R > 0$ and $\psi = \psi(t)$ is a smooth function in $\mathbb{R}$ that equals one for $|t| \leq 1/2$ and zero for $|t| \geq 1$. This idea was first introduced by Fujita in 1961 [20].

By using $U_{ij}^{(R)}, q_j^{(R)}$ we can prove a theorem about representation formula of a solution to the Stokes equations as follows.

Theorem 2.6.1 (Theorem V.3.2 of Galdi [21])**.** *Let* Ω *be an exterior domain of class* C^2 *(in* $\mathbb{R}^N, N \geq 2$*) and let* $\boldsymbol{u} \in W^{2,q}_{loc}(\overline{\Omega}), q \in (1, \infty)$ *satisfy the Stokes equations* (2.39) *with* $\boldsymbol{f} \in L^q(\Omega)$ *in* Ω *in the sense of distribution. Assume that the support of* $\boldsymbol{f}$ *is bounded. Then if at least one of the following conditions is satisfied as* $|x| \to \infty$*:*

- $|\boldsymbol{u}(x)| = o(|x|)$,
- $\displaystyle\int_{|x|\leq r} \frac{|\boldsymbol{u}(x)|^t}{(1+|x|)^{N+t}} dx = o(\log r)$, *some* $t \in (1, \infty)$,

there exists vector and scalar constants $\boldsymbol{u}_\infty, p_\infty$ *such that for almost all* $x \in \Omega$

$$
\begin{aligned}
u_j(x) =& u_{\infty j} + \sum_{i=1}^{N} \int_\Omega U_{ij}(x-y) f_i(y) dy \\
& - \sum_{i,l=1}^{N} \int_{\partial\Omega} \left[U_{ij}(x-y) T_{il}(\boldsymbol{u}, p)(y) - u_i(y) T_{il}(\boldsymbol{U}_j, q_j)(x-y) \right] n_l(y) d\sigma_y \\
p(x) =& p_\infty - \sum_{i=1}^{N} \int_\Omega q_i(x-y) f_i(y) dy \\
& + \sum_{i,l=1}^{N} \int_{\partial\Omega} \left[q_i(x-y) T_{il}(\boldsymbol{u}, p)(y) - 2u_i(y) \frac{\partial q_l(x-y)}{\partial x_i} \right] n_l(y) d\sigma_y
\end{aligned}
$$

where $\boldsymbol{U}_j$ *is the vector field whose i-component is* U_{ij}. *Denote the term of the integral on the boundary* $\partial\Omega$ *by* $u_j^{(1)}(x)$ *and* $p^{(1)}(x)$ *respectively. Moreover, as* $|x| \to \infty$, $\boldsymbol{u}^{(1)}(x), p^{(1)}(x)$ *are infinitely differentiable and the following asymptotic representation hold:*

$$
\begin{aligned}
u_j^{(1)}(x) &= \sum_{i=1}^{N} \mathcal{T}_i U_{ij}(x) + \sigma_j(x) \\
p^{(1)}(x) &= \sum_{i=1}^{N} \mathcal{T}_i q_i(x) + \eta(x)
\end{aligned}
$$

where

$$
\mathcal{T}_i = \int_{\partial\Omega} f_i - \sum_{l=1}^{N} \int_{\partial\Omega} T_{il}(\boldsymbol{u}, p) n_l,
$$

and, for all $|\alpha| > 0$

$$
\begin{aligned}
D^\alpha \boldsymbol{\sigma}(x) &= O(|x|^{1-N-|\alpha|}), \\
D^\alpha \eta(x) &= O(|x|^{-N-|\alpha|}).
\end{aligned}
$$

This theorem is Theorem V.3.2 of Galdi [21], which is an extension of classical results of Chang and Finn in 1961 [10].

The main idea of the proof is that we apply the Green's formula with the Stokes-Fujita truncated fundamental solution in Ω and we can control a resulting volume integral involving $\boldsymbol{v}$ by using the growth conditions.

A simple application of this representation formula is to prove a uniqueness result, which furnishes another form of the *Stokes paradox*.

Theorem 2.6.2 (Theorem V.3.5. of Galdi [21]). *Let* $(\boldsymbol{u}, p)$ *be a regular solution to*

$$\begin{cases} \Delta \boldsymbol{u} - \nabla p = \mathbf{0} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u}|_{\partial\Omega} = \mathbf{0} \end{cases} \tag{2.56}$$

in a exterior domain Ω *of* $\mathbb{R}^N$ *of class* C^1 . *Then if as* $|x| \to \infty$,

$$\boldsymbol{u}(x) = \begin{cases} o(\log |x|) & \text{if } N = 2 \\ o(1) & \text{if } N > 2. \end{cases} \tag{2.57}$$

then it follows that $\boldsymbol{u} \equiv \mathbf{0}$.

This provides another form of the Stokes paradox in two dimension; condition (2.57) in two dimension means that there is no solution $\boldsymbol{u}$ to the Stokes equations such that $\boldsymbol{u}|_{\partial\Omega} = \mathbf{0}$ and $\boldsymbol{u}$ converges to a nonzero constant vector.

The main idea of the proof of this theorem in Galdi [21] is to just take dot-multiply the Stokes equations by $\boldsymbol{u}$ and integrate by parts over $\Omega_R = \Omega \cap B_R$ and use the asymptotic behavior from Theorem 2.6.1 to control the resulting surface integrals as $R \to \infty$.

This statement of Theorem 2.6.2 is taken from Galdi [21], and it is an extension of classical results by Chang and Finn in 1961 [10].

Now regarding the Oseen equations, we define the *Oseen-Fujita truncated fundamental solution* $E_{ij}^{(R)}, e_j^{(R)}$ in a similar way as we did for the Stokes fundamental solution. We replace Φ by $\psi_R\Phi$ in the definition (2.52) of the Oseen fundamental solution, that is,

$$\begin{aligned} E_{ij}^{(R)}(x-y) &= \left(\delta_{ij}\Delta + \frac{\partial^2}{\partial y_i \partial y_j}\right)(\psi_R(x-y)\Phi(x-y)) \\ e_j^{(R)}(x-y) &= -\frac{\partial}{\partial y_j}\left(\Delta + 2\lambda\frac{\partial}{\partial y_1}\right)(\psi_R(x-y)\Phi(x-y)). \end{aligned}$$

Then again as we did for the Stokes equations, we apply the Green's formula with this truncated fundamental solution and control a resulting volume integral involving $\boldsymbol{v}$ by using some growth condition on $\boldsymbol{v}$ to obtain a theorem about a representation formula and asymptotic behavior of a solution to the Stokes equations in an exterior domain as follows.

Theorem 2.6.3 (Theorem VII.6.2 of Galdi [21])**.** *Let Ω be a exterior domain in $\mathbb{R}^N$ of class C^2 and let $\boldsymbol{u} \in W^{2,q}(\overline{\Omega}), q \in (1,\infty)$ satisfy the Oseen equations* (2.43) *with $\boldsymbol{F} \in L^q(\Omega)$. Assume that the support of $\boldsymbol{f}$ is bounded. Then if at least one of the following conditions is satisfied as $|x| \to \infty$:*

- $\displaystyle\int_{S^{N-1}} |\boldsymbol{u}(x)| = o(|x|)$,
- $\displaystyle\int_{|x|\le r} \frac{|\boldsymbol{u}(x)|^t}{(1+|x|)^{N+t}} dx = o(\log r)$, *some* $t \in (1,\infty)$,

there exist vector and scalar constants $\boldsymbol{v}_0, p_0$ such that for almost all $x \in \Omega$ we have

$$\begin{aligned} u_j(x) =& u_{0j} + \lambda \sum_{i=1}^N \int_\Omega E_{ij}(x-y) f_i(y) dy + \sum_{i,l=1}^N \int_{\partial\Omega} \big[u_i(y) T_{il}(\boldsymbol{E}_j, e_j)(x-y) \\ & - E_{ij}(x-y) T_{ij}(\boldsymbol{u}, p)(y) - \lambda u_i(y) E_{ij}(x-y)\delta_{1l} \big] n_l d\sigma_y \\ p(x) =& p_0 - \lambda \sum_{i=1}^N \int_\Omega e_i(x-y) f_i(y) dy + \sum_{i,l=1}^N \int_{\partial\Omega} \Big\{ e_i(x-y) T_{il}(\boldsymbol{u}, p)(y) \\ & - 2u_i(y) \frac{\partial}{\partial x_l} e_i(x-y) - \lambda [e_1(x-y) u_l(y) - u_i(y) e_i(x-y)\delta_{1l}] \Big\} n_l d\sigma_y \end{aligned}$$

where $\boldsymbol{E}_j$ is the vector field whose i-component is E_{ij}. Denote the terms of the integrals on the boundary $\partial\Omega$ by $u_j^{(1)}(x), p^{(1)}(x)$. Moreover, as $|x| \to \infty$, $\boldsymbol{u}^{(1)}(x), p^{(1)}(x)$ are infinitely differentiable and there the following asymptotic representations hold:

$$\begin{aligned} u_j^{(1)}(x) &= \sum_{i=1}^N E_{ij}(x) \mathcal{M}_i + \sigma_j(x) \\ p^{(1)}(x) &= -\sum_{i=1}^N e_i(x) \mathcal{M}_i^* + \eta(x), \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}_i &= -\sum_{l=1}^N \int_{\partial\Omega} [T_{il}(\boldsymbol{u}, p) + \lambda \delta_{1l} u_i] n_l + \lambda \int_\Omega f_i \\ \mathcal{M}_i^* &= -\sum_{l=1}^N \int_{\partial\Omega} \{T_{il}(\boldsymbol{u}, p) + \lambda [\delta_{1l} u_i - \delta_{1i} u_l]\} n_l + \lambda \int_\Omega f_i \end{aligned}$$

and, for all $|\alpha| \geq 0$,

$$\begin{aligned} D^\alpha \boldsymbol{\sigma}(x) &= O(|x|^{-N+|\alpha|/2}) \\ D^\alpha \eta(x) &= O(|x|^{-N-|\alpha|}). \end{aligned}$$

This Theorem 2.6.3 is taken from Galdi [21] and it is an extension of a classical result of Chang and Finn in 1961 [10].

As an application of Theorem 2.6.3, we can derive a fundamental estimate for the integral $\mathcal{I}(\boldsymbol{u})$ given by

$$\mathcal{I}(\boldsymbol{u}) = \int_{\partial\Omega} T(\boldsymbol{u}, p) \cdot \boldsymbol{n}$$

where $T(\boldsymbol{u}, p)$ is the Cauchy stress tensor.

Theorem 2.6.4 (Theorem VII.8.1 of Galdi [21])**.** *Let Ω be a two-dimensional exterior domain of class C^2. Assume for some $q \in (1, 2]$,*

$$\boldsymbol{u}_* \in W^{2-1/q,q}(\partial\Omega)$$

and denote by $\boldsymbol{u}, p$ the solution to

$$\begin{cases} \Delta \boldsymbol{u} - \lambda \dfrac{\partial \boldsymbol{u}}{\partial x_1} = \nabla p & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_* & \\ \lim\limits_{|x| \to \infty} \boldsymbol{u}(x) = \boldsymbol{0}. & \end{cases} \tag{2.58}$$

Then there exists constants $\tilde{\lambda}_0$ and $c = c(\Omega, q, \tilde{\lambda}_0)$ such that

$$\left| \int_{\partial\Omega} T(\boldsymbol{u}, p) \cdot \boldsymbol{n} \right| \le c |\log \lambda|^{-1} \|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} \tag{2.59}$$

for all $\lambda \in (0, \tilde{\lambda}_0]$

Again this Theorem 2.6.4 is taken from Galdi [21] but the basic idea of the proof in Galdi [21] is due to Finn and Smith in 1967 [17], and this estimate plays an essential role in investigating the nonlinear problem.

Remark 2.6.5. The main idea of the proof of Theorem VII.8.1 of Galdi [21] is to use the representation formula for a solution to the Oseen equations (Theorem 2.6.3) and approximate the Oseen fundamental solution by the Stokes fundamental solution from (2.55) giving rise to the constant $\tilde{\lambda}_0$. From the construction of $\tilde{\lambda}_0$, it follows that $\tilde{\lambda}_0$ only depends on Ω (independent of $\boldsymbol{v}_*$).

Remark 2.6.6. Regarding Theorem 2.4.4, the main idea of the estimate (2.48) is to use the representation formula of a solution to the Oseen equations in an exterior domain (Theorem 2.6.3) and use summability properties of the Oseen fundamental solution, that can be obtained from the asymptotic behavior of it, and finally use the estimate (2.59) above. And in this process, we need to take $\lambda_0 = \min\{1, \tilde{\lambda}_0\}$. Hence λ_0 only depends on Ω.

2.7 The Nonlinear Problem: Perturbation around a constant vector

We can solve the nonlinear problem, (2.1), (2.2), (2.3) by linearizing the Navier-Stokes equations around $\boldsymbol{v}_\infty$. This section is devoted to the case $\boldsymbol{v}_\infty \neq \mathbf{0}$. In this case, as mentioned in the introduction, we can rotate and normalize, without loss of generality, $\boldsymbol{v}_\infty$ to be $\boldsymbol{w}_\infty = \langle 1, 0 \rangle$.

In addition to Theorem 2.0.1, Galdi also proved an existence theorem as follows.

Theorem 2.7.1 (Theorem 2.1 of Galdi [24]). *Let Ω be an exterior domain in $\mathbb{R}^2$ of class C^2. Let $\boldsymbol{u}_* \in W^{2-1/q.q}(\partial\Omega), 1 < q < 6/5$. Then there exists a constant λ_0 such that if for some $\lambda \in (0, \lambda_0]$,*

$$|\log \lambda|^{-1} \|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} < \frac{1}{64c^2}$$

for some constant $c > 0$, then there exists a solution $(\boldsymbol{u}, p) \in X^{2,q}(\Omega) \times Y^{1,q}(\Omega)$ to *the problem*

$$\begin{cases} \Delta \boldsymbol{u} - \lambda \dfrac{\partial \boldsymbol{u}}{\partial x_1} = \lambda \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p, \\ \mathrm{div} \boldsymbol{u} = 0 \\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_* \\ \lim\limits_{|x|\to\infty} \boldsymbol{u}(x) = \boldsymbol{0}. \end{cases}$$

This solution can be written in the form of a series

$$\boldsymbol{u}(x) = \sum_{n=0}^{\infty} \lambda^n \boldsymbol{u}_n(x,\lambda), \quad p(x) = \sum_{n=0}^{\infty} \lambda^n p_n(x,\lambda)$$

where $(\boldsymbol{u}_0, p_0)$ *is the solution to the Oseen problem*

$$\begin{cases} \Delta \boldsymbol{u}_0 - \lambda \dfrac{\partial \boldsymbol{u}_0}{\partial x_1} = \nabla p, \\ \mathrm{div} \boldsymbol{u}_0 = 0 \\ \boldsymbol{u}_0|_{\partial\Omega} = \boldsymbol{u}_* \\ \lim\limits_{|x|\to\infty} \boldsymbol{u}_0(x) = \boldsymbol{0}, \end{cases}$$

and for $n \geq 0$

$$\begin{cases} \Delta \boldsymbol{u}_{n+1} - \lambda \dfrac{\partial \boldsymbol{u}_{n+1}}{\partial x_1} = \nabla p_{n+1} + \sum\limits_{k=0}^{n} \boldsymbol{u}_k \cdot \nabla \boldsymbol{u}_{n-k}, \\ \mathrm{div} \boldsymbol{u}_{n+1} = 0 \\ \boldsymbol{u}_{n+1}|_{\partial\Omega} = \boldsymbol{u}_* \\ \lim\limits_{|x|\to\infty} \boldsymbol{u}_{n+1}(x) = \boldsymbol{0}, \end{cases}$$

and these two series converge in $X^{2,q}(\Omega), Y^{1,q}(\Omega)$ *respectively. Furthermore, this so-*

lution satisfies

$$\langle \boldsymbol{u} \rangle_{\lambda,q} + |\boldsymbol{u}|_{2,q} + |p|_{1,q} \le 2c\lambda^{2(1-1/q)} |\log \lambda|^{-1} \|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega}.$$

If $(\tilde{\boldsymbol{u}}, p_1) \in X^{2,q}(\Omega) \times Y^{1,q}(\Omega)$ *is another solution corresponding to the same data and such that* $\lambda^{-2(1-1/q)} \langle \tilde{\boldsymbol{u}} \rangle_{\lambda,q} < 1/(8c)$*, then* $\boldsymbol{u} = \tilde{\boldsymbol{u}}, p = p_1$.

In fact, Galdi used a function space in Theorem 2.1 of [24], that looks different. But actually they are not different. See Remark 2.2.1.

As Galdi did above, Finn and Smith also found a solution in the form of a series and used a contraction mapping theorem in 1967 [18], but Galdi and Finn, Smith used a totally different function space and studied the Oseen problems by means of a different method.

R. Russo and Simader [47] in 2006 also studied the case $\boldsymbol{v}_\infty \neq \boldsymbol{0}$ and applied a perturbation method using the Oseen equations in three-dimension.

Theorem 2.7.2 (Theorem 4.1 of Russo, Simader [47])**.** *Let* Ω *be an exterior domain of* $\mathbb{R}^3$ *and let* $\boldsymbol{v}_* \in L^\infty(\partial\Omega)$*. If* $\|\boldsymbol{v}_* - \boldsymbol{v}_\infty\|_{L^\infty(\partial\Omega)}$ *is sufficiently small, then there is a* C^∞ *solution* $(\boldsymbol{v}, p)$ *to* (2.1), (2.2), (2.3) *with* $\nu = 1, \boldsymbol{F} = \boldsymbol{0}$*. Moreover,* $\boldsymbol{v}$ *converges to* $\boldsymbol{v}_*$ *nontangentially and* $\boldsymbol{v} = \boldsymbol{v}_\infty + O(r^{-1})$*. If* $\boldsymbol{v}_* \in C^{0,\mu}(\partial\Omega)$*, with* $\mu \in [0, \alpha)$ *and* α *depending on the Lipschitz character of* $\partial\Omega$*, then* $\boldsymbol{v} \in C^{0,\mu}(\overline{\Omega})$ *and the value* $\boldsymbol{v}_*$ *is taken in the classical sense.*

2.8 The Nonlinear Problem: Perturbation around a Potential Flow

Here, we study the nonlinear problem, non-homogeneous boundary value problem of the Navier-Stokes equations in an exterior domain, by using a perturbation method around a potential flow.

Due to the Stokes paradox in two dimension, we need to investigate the two cases $\boldsymbol{v}_\infty = \mathbf{0}, \boldsymbol{v}_\infty \neq \mathbf{0}$ separately.

For the case $\boldsymbol{v}_\infty = \mathbf{0}$, A. Russo and Starita in 2008 [44] linearized the Navier-Stokes equations around a potential flow $\nabla\beta$ where β is a harmonic function and solved the nonlinear problem (Theorem 2.0.2, which is already stated in the introduction). This theorem pertains to the three-dimensional problem.

Later, A. Russo and Tartaglione in 2011 [45] extended the result [44] to higher dimensions.

Theorem 2.8.1 (Theorem 1 of Russo, Tartaglione [45])**.** *Let Ω be an exterior domain of $\mathbb{R}^N, N \geq 4$ of class C^2 and let $\beta \in D^{2,q}(\Omega), q > N/2$ be a harmonic function vanishing at infinity. There is a discrete at-most countable subset G of $\mathbb{R} \setminus (-\alpha, \alpha)$ with $\alpha = (N-2)\sqrt{N}/((N-1)\|\nabla\beta\|_{L^N(\Omega)})$ such that if $\mu \notin G$, then a positive constant $c_0 = c_0(N, \beta, \mu, \Omega)$ exists such that if $\|\boldsymbol{v}_* - \mu\nabla\beta\|_{1-2/N,N/2,\partial\Omega} \leq c_0$, then there exists a solution $(\boldsymbol{v}, p)$ to* (2.1)*,* (2.2)*,* (2.3) *with $\boldsymbol{v}_\infty = \mathbf{0}, \boldsymbol{f} = \mathbf{0}, \nu = 1$ such that*

$$(\boldsymbol{v}, p) \in [D^{1,N/2}_{\sigma,0}(\overline{\Omega}) \cap C^\infty(\Omega)] \times [L^{N/2}(\Omega) \cap C^\infty(\Omega)].$$

We can even extend these results by Russo, Starita and Tartaglione to two dimension but for the case $\boldsymbol{v}_\infty \neq \mathbf{0}$. In this case, we can use the Oseen equations, which are solvable in contrast to the Stokes equations in two dimension.

Our first main result of this topic is already stated in the introduction, Theorem 2.0.3. Hence we leave several remarks about the theorem here.

Definition 2.8.2. In the statement of Theorem 2.0.3, $(\boldsymbol{w}, \pi) \in X^{2,q}(\Omega) \times Y^{1,q}(\Omega)$ is called a solution to (2.14), (2.15), (2.16) in the following sense:

1. $(\boldsymbol{w}, \pi)$ satisfies the stationary Navier-Stokes equations (2.14) almost everywhere in Ω,

2. $\boldsymbol{w}$ assumes the boundary data $\boldsymbol{w}_*$ in the trace sense.

3. $\boldsymbol{w}(x)$ converges to $\boldsymbol{w}_\infty$ uniformly as $|x| \to \infty$.

2.8.1 A Proof of Theorems 2.0.3, 2.0.8

In order to prove Theorems 2.0.3, 2.0.8, we can first prove a lemma below which is a simple application of the contraction mapping theorem. This lemma or some similar statements must be well known.

For a Banach space X and an operator $T : X \to X$, an element $\boldsymbol{u} \in X$ satisfying $T[\boldsymbol{u}] = \boldsymbol{u}$ is called a fixed point of T.

Lemma 2.8.3. *Let X be a Banach space equipped with a norm $\|\cdot\|_X$ and B_r the open ball of radius $r > 0$ centered at the origin in the space X . And let $\mathcal{B} : X \times X \to X$ be a bilinear continuous operator. Fix an element $\boldsymbol{u}^0 \in X$. Define an operator $T : X \to X$ by $T[\boldsymbol{u}] = \boldsymbol{u}^0 + \mathcal{B}[\boldsymbol{u}, \boldsymbol{u}]$ for all $\boldsymbol{u} \in X$. Assume $\|\boldsymbol{u}^0\|_X \le a$ and $\|\mathcal{B}\|_{OP} \le b$ for some constants $a, b > 0$ where*

$$\|\mathcal{B}\|_{OP} = \sup_{\boldsymbol{u},\boldsymbol{v} \neq \boldsymbol{0},} \frac{\|\mathcal{B}[\boldsymbol{u}, \boldsymbol{v}]\|_X}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}.$$

If $ab < 1/4$, then there exists a fixed point $\boldsymbol{u} \in B_{2a}$ of T. Moreover, if there exists another fixed point $\boldsymbol{v} \in B_{1/(2b)}$ of T, then $\boldsymbol{u} = \boldsymbol{v}$.

Note in Lemma 2.8.3 above, if $ab < 1/4$, then the ball B_{2a}, where existence of a fixed point is established, is strictly smaller than the ball $B_{1/(2b)}$, where uniqueness of a fixed point is guaranteed.

Proof of Lemma 2.8.3 . For $\boldsymbol{u} \in B_{2a}$, note that

$$\|T[\boldsymbol{u}]\|_X \le \|\boldsymbol{u}^0\|_X + \|\mathcal{B}[u, u]\|_X \le a + b\|\boldsymbol{u}\|_X^2 \le a + 4a^2 b.$$

Furthermore, $a + 4a^2 b < 2a$ due to the assumption $ab < 1/4$. Therefore, the operator T maps B_{2a} to B_{2a}. Moreover, the operator $T : B_{2a} \to B_{2a}$ is a contraction mapping.

Indeed, it holds that

$$T[\boldsymbol{u}] - T[\boldsymbol{v}] = \mathcal{B}[u,u] - \mathcal{B}[v,v] = \mathcal{B}[u,u-v] - \mathcal{B}[v-u,v].$$

Hence

$$\|T[\boldsymbol{u}] - T[\boldsymbol{v}]\|_X \le b(\|\boldsymbol{u}\|_X + \|\boldsymbol{v}\|_X)\|\boldsymbol{u} - \boldsymbol{v}\|_X.$$

If $\boldsymbol{u}, \boldsymbol{v} \in B_{2a}$, then

$$\|T[\boldsymbol{u}] - T[\boldsymbol{v}]\|_X \le 4ab\|\boldsymbol{u} - \boldsymbol{v}\|_X$$

Therefore, the operator $T : B_{2a} \to B_{2a}$ is a contraction mapping if $ab < 1/4$. By virtue of the contraction mapping theorem, it implies that there exists a fixed point $\boldsymbol{u} \in B_{2a}$ of T.

Next, if there exists another fixed point $\boldsymbol{v} \in B_{1/(2b)}$ of T, then

$$\|\boldsymbol{u} - \boldsymbol{v}\|_X = \|T[\boldsymbol{u}] - T[\boldsymbol{v}]\|_X \le b(\|\boldsymbol{u}\|_X + \|\boldsymbol{v}\|_X)\|\boldsymbol{u} - \boldsymbol{v}\|_X \le b\left(2a + \frac{1}{2b}\right)\|\boldsymbol{u} - \boldsymbol{v}\|_X.$$

As we have assumed $ab < 1/4$, it holds that $b(2a + 1/(2b)) < 1$ and thus $\boldsymbol{u} = \boldsymbol{v}$. $\square$

Now we are ready to prove the main Theorems 2.0.3, 2.0.8. Let $1 < q < 6/5$. We consider this mapping $(\boldsymbol{u}^*, \lambda \boldsymbol{F}) \mapsto \boldsymbol{u}$ defined in Theorem 2.4.4. Denote this mapping by $L[\boldsymbol{u}^*, \lambda \boldsymbol{F}] = \boldsymbol{u}$. Note that the operator $L : W^{2-1/q,q}(\partial\Omega) \times L^q(\Omega) \to X^{2,q}(\Omega)$ is bilinear.

Now to prove the main Theorems 2.0.3, 2.0.8, fix $\boldsymbol{u}_* \in W^{2-1/q,q}(\partial\Omega), \boldsymbol{F} \in L^q(\Omega)$.

We consider the following particular problem: find a solution $(\boldsymbol{u}^0, p^0)$ to

$$\begin{cases} \Delta \boldsymbol{u}^0 - \lambda \dfrac{\partial \boldsymbol{u}^0}{\partial x_1} - \nabla p^0 = \lambda \boldsymbol{F} \qquad \text{in } \Omega, \\ \qquad \text{div}\boldsymbol{u}^0 = 0 \qquad \text{in } \Omega, \\ \qquad \boldsymbol{u}^0|_{\partial\Omega} = \boldsymbol{u}^*, \\ \lim_{|x|\to\infty} \boldsymbol{u}^0(x) = 0. \end{cases}$$

By Theorem 2.4.4, there exists a solution $\boldsymbol{u}^0$ to this problem and we can write $\boldsymbol{u}^0 = L[\boldsymbol{u}^*, \lambda\boldsymbol{F}]$, and also it holds that for $\lambda \in (0, \lambda_0]$

$$\|L[\boldsymbol{u}^*, \lambda\boldsymbol{F}]\|_{X^{2,q}(\Omega)} \leq c_0(\lambda^{2(1-1/q)}|\log\lambda|^{-1}\|\boldsymbol{u}^*\|_{W^{2-1/q,q}(\partial\Omega)} + \lambda\|\boldsymbol{F}\|_q). \tag{2.60}$$

The next step is to investigate the following problem: given $\boldsymbol{u} \in X^{2,q}(\Omega)$, solve for $(\boldsymbol{u}^1, p^1)$ the problem

$$\begin{cases} \Delta \boldsymbol{u}^1 - \lambda \dfrac{\partial \boldsymbol{u}^1}{\partial x_1} - \nabla p^1 = \lambda\mu(\nabla\beta\cdot\nabla)\boldsymbol{u} + \lambda\mu(\boldsymbol{u}\cdot\nabla)\nabla\beta \, (= \lambda\mu K_\beta(\boldsymbol{u})) \qquad \text{in } \Omega, \\ \qquad \text{div}\boldsymbol{u}^1 = 0 \qquad \text{in } \Omega, \\ \qquad \boldsymbol{u}^1|_{\partial\Omega} = 0, \\ \lim_{|x|\to\infty} \boldsymbol{u}^1(x) = 0. \end{cases} \tag{2.61}$$

First of all, define an operator $\mathcal{L}_1$ on $X^{2,q}(\Omega)$ by

$$\mathcal{L}_1[\boldsymbol{u}] = L[\boldsymbol{0}, K_\beta(\boldsymbol{u})], \qquad \boldsymbol{u} \in X^{2,q}(\Omega)$$

Then by Theorem 2.4.4 and (2.49), this operator $\mathcal{L}_1 : X^{2,q}(\Omega) \to X^{2,q}(\Omega)$ is well-defined and satisfies the estimate that for all $\boldsymbol{u} \in X^{2,q}(\Omega)$, $\lambda \in (0, \lambda_0]$

$$\|\mathcal{L}_1[\boldsymbol{u}]\|_{X^{2,q}(\Omega)} \leq c_0 \|K_\beta(\boldsymbol{u})\|_q \leq 8c_0 \lambda^{-1-2(1-1/q)} \langle \nabla\beta \rangle_{\lambda,q} \langle \boldsymbol{u} \rangle_{\lambda.q}. \tag{2.62}$$

As $\langle \nabla\beta \rangle_{\lambda,q} \leq \lambda^{1/3} \langle \nabla\beta \rangle_q$ for all $\lambda \leq 1$, it holds that

$$\|\mathcal{L}_1[\boldsymbol{u}]\|_{X^{2,q}(\Omega)} \leq 8c_0 \lambda^{-\frac{2}{3}-2\left(1-\frac{1}{q}\right)} \langle \nabla\beta \rangle_q \langle \boldsymbol{u} \rangle_{\lambda,q}.$$

Therefore, the operator norm $\|\mathcal{L}_1\|_{OP}$ satisfies that for all $\lambda \in (0, \lambda_0]$

$$\|\mathcal{L}_1\|_{OP} \leq 8c_0 \lambda^{-\frac{2}{3}-2\left(1-\frac{1}{q}\right)} \langle \nabla\beta \rangle_q.$$

Note that $\boldsymbol{u}^1 = \lambda\mu\mathcal{L}_1[\boldsymbol{u}]$ and for all $\lambda \in (0, \lambda_0]$

$$\lambda\mu \|\mathcal{L}_1\|_{OP} \leq 8c_0 \mu \lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)} \langle \nabla\beta \rangle_q. \tag{2.63}$$

In addition, we have

Lemma 2.8.4. $\mathcal{L}_1 : X^{2,q}(\Omega) \to X^{2,q}(\Omega)$ *is compact for* $q \in (1, 6/5]$.

Proof. By Lemma 2.4.7, the operator $K_\beta : X^{2,q}(\Omega) \to L^q(\Omega)$ is compact for $q \in (1, 6/5]$. Note $L[\mathbf{0}, \cdot] : L^q(\Omega) \to X^{2,q}(\Omega)$ is a continuous operator. As $\mathcal{L}_1[\cdot] = L[\mathbf{0}, K_\beta(\cdot)]$ and K_β is compact, the operator $\mathcal{L}_1 : X^{2,q}(\Omega) \to X^{2,q}(\Omega)$ is compact. $\square$

Next we consider the problem: given $\boldsymbol{u} \in X^{2,q}(\Omega)$, solve for $(\boldsymbol{u}^2, p^2)$ the problem

$$\begin{cases} \Delta \boldsymbol{u}^2 - \lambda \dfrac{\partial \boldsymbol{u}^2}{\partial x_1} - \nabla p^2 = \lambda (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}^2 = 0 & \text{in } \Omega, \\ \boldsymbol{u}^2|_{\partial\Omega} = 0, & \\ \lim\limits_{|x| \to \infty} \boldsymbol{u}^2(x) = 0. & \end{cases} \tag{2.64}$$

First, we define an operator $\mathcal{N} : X^{2,q}(\Omega) \times X^{2,q}(\Omega) \to X^{2,q}(\Omega)$ by $\mathcal{N}[\boldsymbol{u}, \boldsymbol{v}] = \lambda L[\boldsymbol{0}, (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}]$ for all $\boldsymbol{u}, \boldsymbol{v} \in X^{2,q}(\Omega)$. By Theorem 2.4.4 along with Lemma 2.4.6, the operator $\mathcal{N}$ is well-defined, bilinear and continuous. Morever, the operator $\mathcal{N}$ satisfies that for all $\lambda \in (0, \lambda_0]$, $\boldsymbol{u}, \boldsymbol{v} \in X^{2,q}(\Omega)$,

$$\|\mathcal{N}[\boldsymbol{u}, \boldsymbol{v}]\|_{X^{2,q}(\Omega)} \leq c_0 \lambda \|\boldsymbol{u} \cdot \nabla \boldsymbol{v}\|_{L^q(\Omega)} \leq 4c_0 \lambda^{-2(1-1/q)} \langle \boldsymbol{u} \rangle_{\lambda,q} \langle \boldsymbol{v} \rangle_{\lambda,q}. \tag{2.65}$$

Going back to the problem (2.64), it holds that $\boldsymbol{u}^2 = \mathcal{N}[\boldsymbol{u}, \boldsymbol{u}]$.

In view of Lemma 2.8.4, there exists a countable subset $\tilde{G}(\Omega, \lambda, q, \nabla\beta)$ of $\mathbb{R}$ such that for any $\lambda\mu \in \mathbb{R} \setminus \tilde{G}$, the operator $(I - \lambda\mu\mathcal{L}_1) : X^{2,q}(\Omega) \to X^{2,q}(\Omega)$ is invertible where I is an identity operator on $X^{2,q}(\Omega)$. Define

$$G = \frac{1}{\lambda}\tilde{G} = \left\{ \frac{g}{\lambda} \in \mathbb{R} : g \in \tilde{G} \right\}. \tag{2.66}$$

Then for any $\mu \in \mathbb{R} \setminus G$, the operator $(I - \lambda\mu\mathcal{L}_1)$ is invertible. And we denote the operator norm of the inverse $(I - \lambda\mu\mathcal{L}_1)^{-1}$ by

$$\mathcal{R} = \mathcal{R}(\Omega, \lambda, \mu, q, \nabla\beta) = \|(I - \lambda\mu\mathcal{L}_1)^{-1}\|_{OP}. \tag{2.67}$$

Moreover, if we assume $8c_0\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)}\langle\nabla\beta\rangle_q < 1$, then by (2.63),

$$\mathcal{R} \leq \frac{1}{1 - 8c_0\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)}\langle\nabla\beta\rangle_q}. \tag{2.68}$$

Now we consider the main problem (2.22), (2.23), (2.24). Recall we have fixed $\boldsymbol{u}^* \in W^{2-1/q,q}(\partial\Omega), \boldsymbol{F} \in L^q(\Omega), 1 < q < 6/5$. Let $\mu \in \mathbb{R} \setminus G$. Finding a solution $\boldsymbol{u} \in X^{2,q}(\Omega)$ to (2.22),(2.23), (2.24) is equivalent to find $\boldsymbol{u} \in X^{2,q}(\Omega)$ such that

$$L[\boldsymbol{u}^*, \lambda\boldsymbol{F}] + \lambda\mu\mathcal{L}_1[\boldsymbol{u}] + \mathcal{N}[\boldsymbol{u}, \boldsymbol{u}] = \boldsymbol{u}.$$

Furthermore, this is equivalent to find $\boldsymbol{u} \in X^{2,q}(\Omega)$ such that

$$(I - \lambda\mu\mathcal{L}_1)^{-1}\left[L[\boldsymbol{u}_*, \lambda\boldsymbol{F}] + \mathcal{N}[\boldsymbol{u}, \boldsymbol{u}]\right] = \boldsymbol{u}. \tag{2.69}$$

Here the inverse $(I-\lambda\mu\mathcal{L}_1)^{-1}$ is well-defined because $\mu \in \mathbb{R}\backslash G$. We define an operator T by

$$T[\boldsymbol{u}] = (I - \lambda\mu\mathcal{L}_1)^{-1}\left[L[\boldsymbol{u}^*, \lambda\boldsymbol{F}] + \mathcal{N}[\boldsymbol{u}, \boldsymbol{u}]\right]. \tag{2.70}$$

Then finding a solution $\boldsymbol{u} \in X^{2,q}(\Omega)$ to (2.69) is equivalent to finding a fixed point $\boldsymbol{u}$ of the operator T. To this end, we apply Lemma 2.8.3.

In order to apply Lemma 2.8.3, we first observe that

$$\|(I - \lambda\mu\mathcal{L}_1)^{-1}L[\boldsymbol{u}_*, \lambda\boldsymbol{F}]\|_{X^{2,q}(\Omega)} \leq c_0\mathcal{R}(\lambda^{2(1-1/q)}|\log\lambda|^{-1}\|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} + \lambda\|\boldsymbol{F}\|_{L^q(\Omega)}) \tag{2.71}$$

$$\leq c_1(\|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} + \|\boldsymbol{F}\|_{L^q(\Omega)}) \tag{2.72}$$

where we define the constant $c_1 = c_1(\Omega, q, \lambda, \mu, \nabla\beta)$ by

$$c_1 = c_0\mathcal{R}\max\{\lambda^{2(1-1/q)}|\log\lambda|^{-1}, \lambda\}. \tag{2.73}$$

Moreover, if we assume $8c_0\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)}\langle\nabla\beta\rangle_q < 1$, then by (2.68), for all $\boldsymbol{u} \in X^{2,q}()$

$$\begin{aligned}
&\|(I - \lambda\mu\mathcal{L}_1)^{-1}L[\boldsymbol{u}^*, \lambda\boldsymbol{F}]\|_{X^{2,q}(\Omega)} \\
&\quad\leq \frac{c_0}{1 - 8c_0\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)}\langle\nabla\beta\rangle_q}(\lambda^{2(1-1/q)}|\log\lambda|^{-1}\|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} + \lambda\|\boldsymbol{F}\|_{L^q(\Omega)}) \\
&\qquad= \frac{c_0}{1 - c_3\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)}}(\lambda^{2(1-1/q)}|\log\lambda|^{-1}\|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} + \lambda\|\boldsymbol{F}\|_{L^q(\Omega)})
\end{aligned} \tag{2.74}$$

where we define

$$c_3 = 8c_0\langle\nabla\beta\rangle_q. \tag{2.75}$$

Next we observe that by (2.65), for all $\boldsymbol{u} \in X^{2,q}(\Omega)$

$$\|(I - \lambda\mu\mathcal{L}_1)^{-1}\mathcal{N}[\boldsymbol{u}, \boldsymbol{u}]\|_{X^{2,q}(\Omega)} \leq 4c_0\lambda^{-2(1-1/q)}\mathcal{R}\langle\boldsymbol{u}\rangle^2_{\lambda,q}. \tag{2.76}$$

Hence

$$\|(I - \lambda\mu\mathcal{L}_1)^{-1}\mathcal{N}\|_{OP} \leq 4c_0\lambda^{-2(1-1/q)}\mathcal{R} =: c_2. \tag{2.77}$$

Moreover, if we assume $8c_0\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)}\langle\nabla\beta\rangle_q < 1$ (which is equivalent to $c_3\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)} < 1$), then by (2.68),

$$\|(I - \lambda\mu\mathcal{L}_1)^{-1}\mathcal{N}\|_{OP} \leq \frac{4c_0\lambda^{-2(1-1/q)}}{1 - c_3\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)}}. \tag{2.78}$$

Therefore, in general (without the assumption $c_3\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)} < 1$), by applying Lemma 2.8.3 with

$$a = c_1(\|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} + \|\boldsymbol{F}\|_{q,\Omega}), \quad b = c_2,$$

it follows that if

$$\|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} + \|\boldsymbol{F}\|_{q,\Omega} \leq \frac{1}{4c_1c_2},$$

then there exists a fixed point $\boldsymbol{u} \in B_{2a}$ of the operator T defined in (2.70). Moreover, if there exists another fixed point $\boldsymbol{v} \in B_{1/(2b)}$ of the operator T, then $\boldsymbol{u} = \boldsymbol{v}$. Pressure associated to each of the vector fields $L[\boldsymbol{u}_*, \lambda\boldsymbol{F}], \lambda\mu\mathcal{L}_1[\boldsymbol{u}], \mathcal{N}[\boldsymbol{u}, \boldsymbol{u}]$ exists in $Y^{1,q}(\Omega)$ according to Theorem 2.4.4,and the sum of all these pressures is a pressure field associated to the fixed point $\boldsymbol{u}$. The pair $(\boldsymbol{u}, p)$ solves the main problem (2.14), (2.15), (2.16). This finishes the proof of Theorem 2.0.3.

Next, if we assume $c_3\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)} < 1$, then we now apply Lemma 2.8.3 with

$$a = \frac{c_0}{1 - c_3\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)}}(\lambda^{2(1-1/q)}|\log\lambda|^{-1}\|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} + \lambda\|\boldsymbol{F}\|_{L^q(\Omega)}) \tag{2.79}$$

$$b = \frac{4c_0\lambda^{-2(1-1/q)}}{1 - c_3\mu\lambda^{\frac{1}{3}-2\left(1-\frac{1}{q}\right)}}. \tag{2.80}$$

Then we obtain that if

$$\frac{4c_0^2}{(1 - c_3\mu\lambda^{1/3-2(1-1/q)})^2}(|\log\lambda|^{-1}\|\boldsymbol{u}_*\|_{2-1/q,q,\partial\Omega} + \lambda^{1-2(1-/q)}\|\boldsymbol{F}\|_{q,\Omega}) < \frac{1}{4},$$

then there exists a fixed point $\boldsymbol{u} \in B_{2a}$ of the operator T. Moreover, if there exists another fixed point $\boldsymbol{v} \in B_{1/(2b)}$ of the operator T, then $\boldsymbol{u} = \boldsymbol{v}$. This finishes the proof of Theorem 2.0.8.

2.9 The Nonlinear Problem: Perturbation for flows vanishing at infinity

In this section, we present an interesting result of Hillairet and Wittwer in 2013 [32]. Consider the nonlinear problem

$$\begin{cases} \Delta\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla p = 0 \quad \text{in } \Omega \\ \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega \\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}_* \\ \lim\limits_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{v}_\infty \end{cases} \tag{2.81}$$

In case $\boldsymbol{v}_\infty \neq \mathbf{0}$, we can essentially linearize the Navier-Stokes equations around $\boldsymbol{v}_\infty \neq \mathbf{0}$, which leads to the Oseen equations, which we can solve. However, in case $\boldsymbol{v}_\infty = \mathbf{0}$, linearization of the Navier-Stokes equations around $\boldsymbol{v}_\infty = \mathbf{0}$ leads to the Stokes problem

$$
\begin{aligned}
\Delta \boldsymbol{v} - \nabla p = \mathbf{0} &\quad \text{in } \Omega \\
\operatorname{div} \boldsymbol{v} = 0 &\quad \text{in } \Omega \\
\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}_* & \\
\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0. &
\end{aligned}
$$

In two dimension, due to the Stokes paradox, this Stokes problem is solvable if and only if the boundary function $\boldsymbol{v}_*$ satisfies condition (2.42) (with $\boldsymbol{u}_\infty = \mathbf{0}$), that is,

$$
\mathbf{0} = \int_{\partial\Omega} \boldsymbol{v}_* \cdot T(\boldsymbol{h}^{(i)}, p^{(i)}) \cdot \boldsymbol{n}, \quad i = 1, 2
$$

according to Theorem 2.3.3. This is the reason the Stokes paradox poses a difficulty in studying the nonlinear problem in case $\boldsymbol{v}_\infty = \mathbf{0}$ by perturbation methods.

Moreover, Hamel's example (2.12) provides solutions that decay slower than any negative power of $|x|$. Hence we don't know a priori asymptotic behavior of a solution to the nonlinear problem (2.81) in case $\boldsymbol{v}_\infty = \mathbf{0}$. But these examples have nonzero flux.

However, instead of perturbing around $\boldsymbol{v}_\infty = \mathbf{0}$, Hillairet and Wittwer in 2013 [32] perturbed around $\mu x^\perp/|x|^2$ where $x^\perp = \langle -x_2, x_1 \rangle$ and μ is a scalar. And they studied the problem in a special domain, $\mathbb{R}^2 \setminus B_1(0)$ where $B_1(0)$ is the ball of radius 1 centered at the origin.

Theorem 2.9.1 (Theorem 2 of Hillairet, Wittwer [32])**.** *Let* $\mu_0 > \mu_{crit} = \sqrt{48}$ *and* *let* $\boldsymbol{v}_* \in C^\infty(\partial\Omega), \Omega = \mathbb{R}^2 \setminus B_1(0)$ *satisfy*

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} \, d\sigma = 0.$$

If $\boldsymbol{v}_*$ *is sufficiently close to* $\mu_0 \boldsymbol{e}_\theta$ *, then the problem* (2.81) *with* $\boldsymbol{v}_\infty = \boldsymbol{0}$ *and* $\Omega = \mathbb{R}^2 \setminus B_1(0)$ *has at least one solution* $(\boldsymbol{v}, p) \in C^\infty(\mathbb{R}^2 \setminus B_1(0)) \times C^\infty(\mathbb{R}^2 \setminus B_1(0))$. *Moreover, there exist* μ *close to* μ_0 *such that*

$$\lim_{r\to\infty} r \left\| \boldsymbol{v}(r,\theta) - \frac{\mu}{r}\boldsymbol{e}_\theta \right\|_{L^\infty(-\pi,\pi)} = 0. \tag{2.82}$$

In addition, for the topic of asymptotic behavior of a solution to the nonlinear problem with zero velocity at infinity, we refer to the work of Guillod and Wittwer in 2015 [29].

2.10 The Nonlinear Problem: Asymptotic Behavior

In this section, we present results about the asymptotic behavior of a solution to the Navier-Stokes equations in a two-dimensional exterior domain in the case $\boldsymbol{v}_\infty \neq \boldsymbol{0}$. If $\boldsymbol{v}_\infty = \boldsymbol{0}$, Hamel's examples provide explicit solutions that decay slower than any negative power of $|x|$. On the other hand, in case of $\boldsymbol{v}_\infty \neq \boldsymbol{0}$, a solution to the Navier-Stokes equations behaves like the Oseen fundamental solution at large distances.

Theorem 2.10.1 (Theorem XII.8.1 of Galdi [21])**.** *Let* $\boldsymbol{F}^q(\Omega) \in L^q(\Omega)$ *with bounded support,* $\boldsymbol{w}_* \in W^{2-1/q_0,q_0}(\partial\Omega)$ *for some* $q_0 > 2$*, all* $q \in (1, q_0]$*. And let* $(\boldsymbol{w}, p)$ *be a* *solution to*

$$\begin{cases} \Delta \boldsymbol{w} - \lambda(\boldsymbol{w} \cdot \nabla)\boldsymbol{w} - \nabla p = \lambda \boldsymbol{F} \\ \operatorname{div} \boldsymbol{w} = 0 \\ \boldsymbol{w}|_{\partial\Omega} = \boldsymbol{w}_* \\ \lim_{|x| \to \infty} \boldsymbol{w}(x) = \boldsymbol{w}_\infty = e_1 \end{cases}$$

such that $\boldsymbol{w} \in D^{1,2}(\Omega)$. *Then for all sufficiently large* $|x|$, *we have*

$$\boldsymbol{w} = \boldsymbol{w}_\infty + \boldsymbol{m} \cdot \boldsymbol{E}(x) + \boldsymbol{\mathcal{V}}(x)$$

where

$$m_i = \lambda \int_\Omega \boldsymbol{f}_i - \sum_{l=1}^{2} \int_{\partial\Omega} [T_{il}(\boldsymbol{u}, p) + \lambda(\delta_{1l} u_i - u_i u_l)] n_l, \quad i = 1, 2$$

and $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{w}_\infty$ *and* $\boldsymbol{\mathcal{V}}(x)$ *verifies the estimate*

$$\boldsymbol{\mathcal{V}}(x) = O(|x|^{-1+\varepsilon})$$

for arbitrary small $\varepsilon > 0$.

2.11 Limit of Vanishing Reynolds Number

We present here an interesting result about the behavior of a solution to the Navier-Stokes equations in two-dimensional exterior domains in the limit of vanishing Reynolds number.

The basic idea comes from the works of Finn and Smith in 1967 [18], [17]. But here we state a result proved by Galdi. The following statement is taken from Galdi [21], but it was originally proved by Galdi in 1993 [23].

Theorem 2.11.1 (Theorem XII.9.1 of Galdi [21])**.** *Let* Ω *be an exterior domain in* $\mathbb{R}^2$ *of class* C^2*. Let* $\boldsymbol{w}_* \in W^{2-1/q,q}(\partial\Omega), 1 < q < 6/5$ *and let* $\boldsymbol{w}_\infty = \boldsymbol{e}_1$*. Let* $(\boldsymbol{w}, p)$ *be the solution constructed in Theorem 2.0.1 with* $\boldsymbol{F} = \boldsymbol{0}$*. Then, denoting by* $\boldsymbol{u}, \pi$ *the (uniquely determined) solution to the Stokes system*

$$\begin{cases} \Delta \boldsymbol{u} = \nabla \pi & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_* & \\ |\boldsymbol{u}|_{1,2,\Omega} < \infty. & \end{cases}$$

Then, as $\lambda \to 0$*,* $\boldsymbol{w}, p$ *tends to* $\boldsymbol{u}, \pi$ *in the following sense:*

$$\begin{aligned} \nabla \boldsymbol{w} &\rightharpoonup \nabla \boldsymbol{u} && \text{in } L^2(\Omega) \\ \boldsymbol{w} &\rightharpoonup \boldsymbol{u} && \text{in } W^{2,q}(\partial\Omega) \\ p &\rightharpoonup \pi && \text{in } W^{1,q}(\Omega_R) \end{aligned}$$

for all $R > \operatorname{diam}(\Omega^c)$*. Moreover, there is a vector* $\boldsymbol{u}_\infty \in \mathbb{R}^2$ *such that* $\lim_{|x|\to\infty} \boldsymbol{u}(x) = \boldsymbol{u}_\infty$ *and we have*

$$\boldsymbol{u}_\infty - e_1 = \frac{1}{4\pi} \lim_{\lambda \to} |\log \lambda| \int_{\partial\Omega} T(\boldsymbol{w}, p) \cdot \boldsymbol{n}.$$

Finally, the limit process preserves the condition at infinity, that is, $\boldsymbol{u}_\infty = \boldsymbol{e}_1$ *if and only if*

$$\int_{\partial\Omega} (\boldsymbol{w}_* - \boldsymbol{e}_1) \cdot T(\boldsymbol{h}^{(i)}, \pi^{(i)}) \cdot \boldsymbol{n} = 0 \quad i = 1, 2 \tag{2.83}$$

where $\{\boldsymbol{h}^{(i)}, p^{(i)}\}_{i=1,2}$ *is the basis in* $\mathcal{S}_q$ *constructed in Lemma 2.3.1. In the particular case where* Ω *is exterior to a unit circle, condition* (2.83) *reduces to*

$$\int_{\partial\Omega} (\boldsymbol{w}_{*i} + \delta_{1i}) = 0 \quad i = 1, 2.$$

2.12 A Liouville Problem in Three Dimension

This last section does not pertain to the problem of the stationary Navier-Stokes equations in two-dimensional exterior domains. Instead, we introduce short history of a Liouville problem of the Navier-Stokes equations in $\mathbb{R}^3$, which is still open to date.

For a smooth solution $(\boldsymbol{v}, p)$ to the Navier-Stokes problem

$$\begin{cases} \Delta \boldsymbol{v} = \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p & \text{in } \mathbb{R}^N \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \mathbb{R}^N \\ \lim_{|x| \to \infty} \boldsymbol{v}(x) = \boldsymbol{0} \end{cases} \tag{2.84}$$

such that

$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{v}(x)|^2 dx < \infty. \tag{2.85}$$

The trivial solution (identically vanishing) is apparently a solution to this problem. However, a natural question arises: is the trivial solution the only smooth solution to the problem? This has an affirmative answer for any dimension N other than three. The three-dimensional case still remains a big open problem.

Theorem 2.12.1 (Theorem X.9.5 of Galdi [21])**.** *Let $\boldsymbol{v}$ be a smooth solution to* (2.84) *in $\mathbb{R}^3$ such that*

$$\boldsymbol{v} \in L^{9/2}(\mathbb{R}^3). \tag{2.86}$$

Then $\boldsymbol{v} \equiv \boldsymbol{0}$

The basic idea of the proof by Galdi [21] is: 1) to multiply the Navier-Stokes equations by $\psi_R \boldsymbol{v}$ where ψ_R is a standard cut-off function ψ_R; 2) integrate by parts over $\mathbb{R}^3$; 3) prove summability properties of $\boldsymbol{v}, p$ by utilizing the representation formula of a solution to the Stokes equations (Theorem 2.6.1) and the uniqueness result, Theorem 2.6.2, 4) control all the other integrals other than the one involving $\psi_R|\nabla \boldsymbol{v}|^2$ by using the summability properties.

In higher dimension $n \geq 4$, we can replace the condition $\boldsymbol{v} \in L^{9/2}(\mathbb{R}^3)$ by the condition (2.85) (using Sobolev inequality). And we can solve the problem even in two dimension; using pointwise behavior of a function in $D^{1,2}(\Omega)$ and using the equation satisfied by the vorticity of $\boldsymbol{v}$, we can prove the vorticity is constant, which leads to the triviality of $\boldsymbol{v}$ using condition (2.85). See Theorem XII.3.1 of Galdi [21] and Gilbarg and Weinberger [27].

For the three-dimensional problem, there have been numerous outstanding partial results. To name a few, Chae in 2014 [6] proved the triviality of a solution $\boldsymbol{v}$ to (2.84) under the assumption that $\Delta \boldsymbol{v} \in L^{6/5}$. And recently, Chae in 2021 [7] proved the following theorem.

Theorem 2.12.2 (Theorem 1.1 of Chae [7]). *Let* $(\boldsymbol{v}, p)$ *be a smooth solution to* (2.84), (2.85) *in* $\mathbb{R}^3$ *and let* $Q = \frac{1}{2}|\boldsymbol{v}|^2 + p$ *be its head pressure. If either*

$$\sup_{x \in \mathbb{R}^3} \frac{|\boldsymbol{v}(x)|^2}{|Q(x)|} < \infty \quad \text{or} \quad \sup_{x \in \mathbb{R}^3} \frac{|p(x)|}{|Q(x)|} < \infty, \tag{2.87}$$

then $\boldsymbol{v} \equiv \mathbf{0}$ *and* $p = \text{constant}$ *in* $\mathbb{R}^3$.

In addition, Korobkov, Pileckas, Russo in 2015 [34] proved a Liouville type theorem in $\mathbb{R}^3$ for an axially symmetric D-solution with no swirl.

Theorem 2.12.3 (Theorem 1.1 of Korobkov, Pileckas, Russo [34]). *Let* $(\boldsymbol{v}, p)$ *be an axially symmetric* D*-solution with no swirl to* (2.84), *then* $\boldsymbol{v} \equiv \mathbf{0}$

Chapter 3

Some Integral Identities for the Euler and the Navier-Stokes Equations

Let Ω be a domain in $\mathbb{R}^N, N \geq 2$. In this chapter, we will mainly consider special domains, such as $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}_+ \times \mathbb{R}, (0,1) \times \mathbb{R}$ when $N = 2$.

In this chapter, we mainly consider the (incompressible) Euler ($\nu = 0$) or Navier-Stokes ($\nu > 0$) equations:

$$\begin{aligned} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla p = \nu \Delta \boldsymbol{v} \quad & \text{in } \Omega \times (0,T) \\ \operatorname{div}\boldsymbol{v} = 0 \qquad & \text{in } \Omega \times (0,T) \end{aligned} \tag{3.1}$$

where $\boldsymbol{v}(x,t) = (v_1(x,t), \cdots, v_N(x,t)) : \Omega \times (0,T) \to \mathbb{R}^N$ is a velocity field, $p : \Omega \times (0,T) \to \mathbb{R}$ is a pressure field. The parameter $\nu \geq 0$ is called viscosity. In some cases, we consider steady solutions to the Euler or the Navier-Stokes equations.

Dobrokhotov and Shafarevich [12] in 1994 proved some integral identities for the Navier Stokes and the Euler equations for a classical (twice differentiable) solution $(\boldsymbol{v}, p)$ in space dimension three. If $\boldsymbol{v}$ and its derivatives $\partial \boldsymbol{v}/\partial t, \partial \boldsymbol{v}/\partial x_j, j = 1,2,3$

decay faster than $|x|^{-4}$ at some time t, then

$$\int_{\mathbb{R}^3} (v_j(x,t)v_k(x,t) + \delta_{jk}p(x,t))\,dx = 0 \qquad \text{for all } j,k \in \{1,2,3\}. \tag{3.2}$$

The basic idea is to multiply j-th component of the equations by x_k and k-th component by x_j for fixed j,k and to integrate the sum of the resulting equations in a ball of radius R in $\mathbb{R}^3$ and pass to the limit as $R \to \infty$. (See Section 3.2) They established these integral identities in order to study the instantaneous spreading of the equations in the entire space. And they proved these integral identities only in space dimension three, but it can be easily extended to other space dimensions. Even though the integral identities are applicable to various problems (see below for various papers who used these integral identities), not only the integral identities (3.2) but also this paper [12] do not seem to be well-known to the best of my knowledge.

In this chapter, we will mainly focus on the study of the integral identities (3.2) of the Euler and Navier-Stokes equations (3.1) in terms of the form of the integral identities and assumptions on $(\boldsymbol{v},p)$ that are needed, and then we will also study Liouville type theorems of the equations (3.1) as an immediate corollary of the integral identities.

In this introduction, we will first focus on history of the integral identities (3.2) and then on one of Liouville type theorems of the stationary Euler equations (related to shear flows in two dimension and to Beltrami solutions in three dimension).

Some Integral Identities for the Euler and the Navier-Stokes equations

Dongho Chae in 2011 [3] proved the integral identities (3.2) for a weak solution: if $(\boldsymbol{v},p) \in L^1(0,T;L^2_{loc,\sigma}(\mathbb{R}^N)) \times L^1(0,T;\mathcal{S}'(\mathbb{R}^N))$ is a weak solution to the Euler or the Navier-Stokes equations (3.1) in $\mathbb{R}^N$ and if the weak solution satisfies $(\boldsymbol{v},p) \in$

$L^1(0,T;L^2(\mathbb{R}^N)) \times L^1(0,T;L^1(\mathbb{R}^N))$, then the integral identities

$$\int_{\mathbb{R}^N} (v_j(x,t)v_k(x,t) + p(x,t)\delta_{jk})\,dx = 0 \tag{3.3}$$

hold for almost every $t \in (0,T)$. (See Theorem 3.2.3. This statement comes from the proof of Theorem 1.1 of Chae [3], which is a Liouville type theorem. But proving these integral identities is the main part of the proof of Theorem 1.1 of Chae [3]. And here the subscript σ of $L^2_{loc,\sigma}$ means v is weakly divergence free.)

These integral identities (3.3) are used in many papers for various objectives. Chae, in the same paper [3], used these integral identities in order to prove a Liouville theorem for the Euler and the Navier-Stokes equations (3.1) in $\mathbb{R}^N$ as an immediate corollary of the integral identities. Jiu and Xin [33] also used these integral identities to establish a strong convergence criterion of approximate solution for the Euler equations in $\mathbb{R}^3$. Brandolese and Meyer [2] also studied these integral identities for the instantaneous spreading of the evolutionary Navier-Stokes equations in the entire space. Constantin and Chae [8] used the integral identities to prove a Liouville theorem for a Beltrami solution to the stationary Euler equations in $\mathbb{R}^3$ (see Theorem 3.0.5). Sharafutdinov [48] improved these integral identities for higher degree integral momentum. Hence, these integral identities seem to be applicable to various problems for the Euler or the Navier-Stokes equations. However, these integral identities are not well known and have not been studied extensively, to the best of my knowledge.

These types of integral identities 3.3 are also studied in a totally different context. Consider the system of equations

$$-\Delta \boldsymbol{v}(x) + \nabla_{\boldsymbol{v}} H(\boldsymbol{v}(x)) = 0, \quad x \in \mathbb{R}^N \tag{3.4}$$

for $\boldsymbol{v} \in H^1(\mathbb{R}^N, \mathbb{R}^M)$ and a potential function $H \in C^2$ with $H(0) = 0$. When $M = 1$, the equations turn into the Allen-Cahn equation. This system enjoys a similar integral identity,

$$\int_{\mathbb{R}^N} \left\{ \frac{1}{2} \left(\sum_{l=1}^{N-1} |\partial_{x_l} v(x)|^2 - |\partial_{x_N} v(x)|^2 \right) + H\left(v(x)\right) \right\} dx = 0, \tag{3.5}$$

(which in turn can be used to derive Pohozaev identity in the entire space). This is proved by Changfeng Gui in 2008 [28]. See (1.17) of [28]. To establish this integral identity (3.5), the author first proved integral identities on a hyperplane,

$$\int_{\mathbb{R}^{N-1}} \left\{ \frac{1}{2} \left(\sum_{l=1}^{N-1} |\partial_{x_l} v(x)|^2 - |\partial_{x_N} v(x)|^2 \right) + H\left(v(x)\right) \right\} dx'_N = \text{constant} \tag{3.6}$$

for all $x_N \in \mathbb{R}$. (Gui called it a Hamiltonian identity.) Here $x'_N = (x_1, \cdots, x_{N-1})$ and $dx'_N = dx_1 \cdots dx_{N-1}$. See Theorem 1.2 of [28]. Note that the integral of (3.6) is carried out on a hyperplane $\mathbb{R}^{N-1}$ while the one of (3.5) is in the entire space $\mathbb{R}^N$. The author used theses integral identities to prove Young's law for the contact angles in triple junction formation and to analyze structure of level curves of saddle solutions to the Allen-Cahn equations. (See Remark 3.7.5.)

Integral identities on a hyperplane were also found for the Naiver-Stokes and the Euler equations.

Theorem 3.0.1 (Theorem 1.1 of Chae [4] in 2012)**.** *Let* $(\boldsymbol{v}, p)$ *be a weak solution of the incompressible Euler or the Navier-Stokes equations* (3.30)*,* (3.31)*. Assume*

$$|v(\cdot, t)|^2 + |p(\cdot, t)| \in L^1(\mathbb{R}^N) \tag{3.7}$$

for some $t \in [0, \infty)$*. Then*

$$\int_{\mathbb{R}^{N-1}} \left(|v_k(x,t)|^2 + p(x,t)\right) dx'_k = 0 \tag{3.8}$$

for almost every $x_k \in \mathbb{R}$ *and for all* $k = 1, \ldots, N$.

Here $x'_k = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_N)$ and $dx'_k = dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_N$. Note that the integral identities (3.2) involve two indices j, k whereas these integral identities (3.8) only one index. We will call a type of integral identities involving two indices like (3.2) a *matrix form* whereas the other type with one index *a vector form.*

To compare the integral identities (3.8) on a hyperplane to the ones (3.3) in the entire space, the ones on a hyperplane hold in the vector form whereas the ones in the entire space hold in the matrix form. Theorem 3.0.1 does not give information about whether integral identities on a hyperplane also hold in the matrix form. However, one of our main results is that integral identities hold in the matrix form when it comes to the stationary Euler equations. (See Theorem 3.0.2.)

The basic idea of Chae's proofs in the papers [3], [4], [5] of integral identities in the entire space and on a hyperplane is to take divergence of the Navier-Stokes and the Euler equations to obtain

$$\sum_{l,m=1}^{N} \partial_{x_l} \partial_{x_m} (v_l v_m + p\delta_{lm}) = 0. \tag{3.9}$$

Then this equation (3.9) is in the form

$$\sum_{l,m=1}^{N} \partial_{x_l} \partial_{x_m} F_{lm}(x) = 0 \tag{3.10}$$

where $F(x) = (F_{lm}(x)) : \mathbb{R}^N \to M_{N\times N}(\mathbb{R})$ denotes a tensor field of order 2. This equation (3.10) is called a *double divergence free equation.* And Chae used a various special type of cut-off functions in the weak formulation of equation (3.10) to derive the integral identities (3.3), (3.8).

To prove the matrix form of the integral identities (3.3) in $\mathbb{R}^N$, he used cut-off functions given by $x_j^2\sigma_R(x)/2, x_j x_k \sigma_R(x)$ for fixed j, k where $\sigma_R(x) = \sigma(x/R)$ and $\sigma(x)$ is a standard cut-off function in $\mathbb{R}^N$, that equals one if $|x| < 1$ and vanishes if $|x| > 2$. And to prove the vector form of the integral identities (3.8) on a hyperplane, he used cut-off functions in the form of $e^{i\xi_j x_j}\sigma_R(x)$ for fixed j and used the method of Fourier transform. (For more details, see Theorems 3.2.1, 3.2.3, 3.3.1 and sketch of their proofs presented after the statements of the theorems.)

In this chapter, we derive the matrix or the vector form of integral identities (3.8) on a hyperplane with assumptions slightly different from Chae's in [4]. Here is one of our main results of this thesis.

Theorem 3.0.2. *Let* $(\boldsymbol{v}, p) \in L^1(0,T; L^2_{loc}(\mathbb{R}^N)) \times L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ *be a weak solution to the evolutionary Euler or Navier-Stokes equations* (3.30), (3.31). *Assume that for some* $t \in (0,T)$ *and for all* $k \in \{1, \ldots, N\}$

$$|\boldsymbol{v}(x,t)|^2 + |p(x,t)| \in L^1_{loc,x_k} L^1_{x_k'}, \tag{3.11}$$

$$\liminf_{|x_k|\to\infty} \left\| |\boldsymbol{v}(x,t)|^2 + |p(x,t)| \right\|_{L^1(\mathbb{R}^{N-1}; x_k')} = 0. \tag{3.12}$$

Then for the evolutionary Euler or Navier-Stokes equations, for all $k \in \{1, \ldots, N\}$

$$\int_{\mathbb{R}^{N-1}} \left(v_k^2(x,t) + p(x,t) \right) dx_k' = 0 \quad \textit{for a.e. } x_k \in \mathbb{R}, \tag{3.13}$$

whereas for the stationary Euler equations, for all $j, k \in \{1, \ldots, N\}$,

$$\int_{\mathbb{R}^{N-1}} (v_j(x) v_k(x) + p(x)\delta_{jk}) dx_k' = 0 \quad \textit{for a.e. } x_k \in \mathbb{R}. \tag{3.14}$$

For notations, see Section 3.1. For a proof, see page 124.

Remark 3.0.3. We compare our assumptions (3.11), (3.12) to Chae's assumption (3.7), that is, $|\boldsymbol{v}(\cdot,t)|^2 + |p(\cdot,t)| \in L^1(\mathbb{R}^N)$ for some t. If we only consider $\boldsymbol{v}, p$ satisfying that for some t,

$$|\boldsymbol{v}(x,t)|^2 + |p(x,t)| \leq \frac{C}{1+|x|^\alpha} \quad \text{in } \mathbb{R}^N, \quad \alpha > 0,$$

then assumption (3.7) is satisfied if $\alpha > N$ whereas assumptions (3.11), (3.12) are satisfied if $\alpha > N-1$. This is a main benefit of Theorem 3.0.2, (which makes it possible to prove Theorem 3.0.7).

Remark 3.0.4. In fact, Theorem 3.0.2 do not provide assumptions weaker than the one of Theorem 3.0.1 in general; In other words, the assumption (3.7) of Theorem 3.0.1 does not imply the assumptions (3.11), (3.12) of Theorem 3.0.2. However, we can simply replace assumption (3.12) by

$$\Big\| |\boldsymbol{v}(x,t)|^2 + |p(x,t)| \Big\|_{L^1(\mathbb{R}^{N-1};x_k')} \in L^1_{weak}(\mathbb{R};x_k). \tag{3.15}$$

Even with this replacement, we can reach the same conclusion, and assumption (3.7) of Theorem 3.0.1 implies (3.11) and (3.15). Hence assumptions (3.11), (3.15) of Theorem 3.0.2 are weaker than (3.7). However, a theorem with this replacement does not provide good conditions in terms of decaying of $|\boldsymbol{v}|^2 + |p|$.

The main idea of our proof for the evolutionary equations is to use cut-off functions different from the ones used by Chae in [4]. The main idea of our proof for the stationary Euler equations is to use the stationary Euler equations directly without taking the divergence of the equations. In order words, we use the divergence structure of the stationary Euler equations directly rather than using the double divergence structure of divergence of the stationary Euler equations. And we also used a different cut-off function.

These main ideas can also be used to understand connection between the similar integral identities for two different equations, the stationary Euler equations and equation (3.4). This is because we can find divergence structure out of equation (3.4) as well as the stationary Euler equations. See Remark 3.7.5.

We also prove the integral identities (3.8) on a section in a domain with boundary for the stationary Euler equations and establish, as an immediate application of the integral identities, Liouville type theorems, which provide a different approach to the result of Hamel and Nadirashvili [30], [31], Chae and Constantin [8]. We will see history of Liouville type theorems of the stationary Euler equations in detail below as it needs separate attention

As a matter of fact, the author of the present thesis in fact established the matrix form of the integral identities (3.8) on a hyperplane for the stationary Euler equations without even knowing Gui [28], Chae [4]. While the author of the present thesis was presenting these results to his adviser, Yanyan Li, he suggested to read the paper by Gui [28]. And the paper by Chae [4] was found by the author of the present thesis after having established these results.

Liouville type theorems of the stationary Euler equations

We now present history of Liouville type theorems of the stationary Euler equations.

First of all, in Chae [5], a Liouville type theorem is established in a straightforward way by using the integral identities (3.8): for a continuous weak solution $(\boldsymbol{v}, p) \in L^1(0,T; L^2_{loc,\sigma}(\mathbb{R}^N)) \times L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ to the Euler and the Navier-Stokes equations (3.1) in $\mathbb{R}^N$ satisfying $(\boldsymbol{v}(\cdot,t), p(\cdot,t)) \in L^2(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$ for some time $t \in (0,T)$, if $p(\cdot,t) \geq 0$ on almost every hyperplane of $\mathbb{R}^N$, then $\boldsymbol{v}(\cdot,t) = 0$ in $\mathbb{R}^N$. As this is just an immediate application of the integral identities (3.8) on a hyperplane, we can simply replace the condition of positivity of pressure on hyperplanes by positivity of integrals of pressure on hyperplanes: if pressure p satisfies that for some $t \in (0,T)$ and for all $k \in \{1, \ldots, N\}$

$$\int_{\mathbb{R}^{N-1}} p(x,t)\, dx'_k \geq 0 \qquad \text{for a.e. } x_k \in \mathbb{R} \tag{3.16}$$

then $\boldsymbol{v}(\cdot, t) \equiv 0$.

In addition, for Liouville type theorems of the stationary Euler equations, several researchers especially studied Liouville type properties of a Beltrami solution in $\mathbb{R}^3$. A pair of a vector field $\boldsymbol{v}$ and pressure p is called a *Beltrami solution* to the stationary Euler equations in $\mathbb{R}^3$ if $p + \frac{1}{2}|\boldsymbol{v}|^2 = c$ in $\mathbb{R}^3$ for some constant c and there exists a function $\lambda = \lambda(x)$ such that $\operatorname{curl}\boldsymbol{v} = \lambda \boldsymbol{v}$. (The pair satisfying these two conditions is in fact a solution to the stationary Euler equations.) See Definition 3.5.1.

Enciso and Peralta-Salas [13] in 2012 constructed a non-trivial Beltrami flow which decays of $O(1/|x|)$ as $|x| \to \infty$. On the other hand, Nadirashvili [42] in 2014 investigated a Liouville theorem for Beltrami solutions to the stationary Euler equations in $\mathbb{R}^3$: For a Beltrami solution $\boldsymbol{v} \in C^1(\mathbb{R}^3)$, if we assume that either $\boldsymbol{v} \in L^q(\mathbb{R}^3), 2 \leq q \leq 3$ or $\boldsymbol{v}(x) = o(1/|x|)$, then $\boldsymbol{v} \equiv 0$. See Theorem 3.5.2.

Chae and Constantin [8] in 2015 provided different, simple proofs of the following theorem.

Theorem 3.0.5 (Chae, Constantin [8])**.** *For a Beltrami solution $\boldsymbol{v}$ to the stationary Euler equations,*

- (Theorem 1.2 of [8]) *if $\boldsymbol{v} \in L^2(\mathbb{R}^3)$, then $\boldsymbol{v} \equiv 0$,*
- (Theorem 1.3 of [8]) *if $\boldsymbol{v} \in L^\infty_{loc}(\mathbb{R}^3)$ satisfies either $\boldsymbol{v} \in L^q(\mathbb{R}^3)$ for some $q \in [2, 3)$ or there exists $\varepsilon > 0$ such that $|\boldsymbol{v}(x)| = O(1/|x|^{1+\varepsilon})$ as $|x| \to \infty$, then $\boldsymbol{v} \equiv 0$.*

The first part of Theorem 3.0.5 is just part of the original Theorem 1.2 of [8]. In the omitted part, they provided an alternative assumption that includes λ (the function λ from the definition of Beltrami solutions).

Remark 3.0.6. For the first part of Theorem 3.0.5, Chae and Constantin proved it by using the integral identities (3.3) in the entire space. On the other hand, they argued differently without using the integral identities for the second part. For the proofs, see page 110. However, we can prove one of our main theorems below, whose statement is similar to the second part of Theorem 3.0.5 by using the integral identities (3.14) on a hyperplane of Theorem 3.0.2.

Theorem 3.0.7. *For a weak Beltrami solution $\boldsymbol{v}$ to the stationary Euler equations, assume that for all $k \in \{1, 2, 3\}$*

$$|\boldsymbol{v}(x)|^2 \in L^1_{loc,x_k} L^1_{x'_k} \tag{3.17}$$

$$\liminf_{|x_k| \to \infty} \left\| |\boldsymbol{v}(x)|^2 \right\|_{L^1(\mathbb{R}^2; x'_k)} = 0. \tag{3.18}$$

Or assume that $\boldsymbol{v} \in L^\infty_{loc}(\mathbb{R}^3)$ and that there exists $\varepsilon > 0$ such that $|\boldsymbol{v}(x)| = O(1/|x|^{1+\varepsilon})$ as $|x| \to \infty$. Then $\boldsymbol{v} \equiv \mathbf{0}$.

A proof of this Theorem 3.0.7 can be found on page 112.

Remark 3.0.8. Even if we replace condition (3.18) by

$$\left\| |\boldsymbol{v}(x)|^2 \right\|_{L^1(\mathbb{R}^2; x'_k)} \in L^1_{weak}(\mathbb{R}; x_k), \tag{3.19}$$

the conclusion, $\boldsymbol{v} \equiv \mathbf{0}$, is still true. And to compare these conditions (3.17), (3.19) to the conditions of Theorem 3.0.5, the condition $\boldsymbol{v} \in L^2(\mathbb{R}^3)$ of Theorem 3.0.5 implies conditions (3.17), (3.19).

In contrast, there are simple examples of nontrivial smooth, compactly-supported solutions to the stationary Euler equations in $\mathbb{R}^{2N}, N \in \mathbb{N}$: Choose $\varphi \in C_c^\infty[0, \infty)$ such that $\varphi(r) = 0$ for $r \leq r_0$ with some $r_0 > 0$. Then a pair of vector and pressure given by

$$v_{2j-1}(x) = -\varphi(|x|)x_{2j},\ v_{2j}(x) = \varphi(|x|)x_{2j-1},\ j = 1, \ldots, N, \quad p(x) = -\int_{|x|}^{\infty} s\varphi^2(s)ds \tag{3.20}$$

is a smooth, compactly-supported solution to the stationary Euler equations in $\mathbb{R}^{2N}$. This example was taken from Sharafutdinov [48] but, as it was pointed out in the paper, this example is highly likely well known.

Construction of a nontrivial, smooth, compactly-supported solution to the stationary Euler equations in odd dimension is still open with the exception of $\mathbb{R}^3$. Such an example in $\mathbb{R}^3$ has been recently constructed in Gavrilov [26] and Constantin, La and Vicol [11]. See Section 3.6.

On the other hand, Hamel and Nadirashvili in 2017, 2019 [30], [31] provide a new approach to Liouville type theorems of the stationary Euler equations in various special domains in $\mathbb{R}^2$. Their theorems pertain to a shear flow, which is by definition a vector field $\boldsymbol{v}$ in the form

$$\boldsymbol{v}(x) = V(x \cdot \boldsymbol{e}^{\perp})\boldsymbol{e}, \qquad \boldsymbol{e}^{\perp} = (-e_2, e_1) \tag{3.21}$$

for some vector $\boldsymbol{e} = (e_1, e_2) \in \mathbb{S}^1$ and a function $V : \mathbb{R} \to \mathbb{R}$. Any such vector field is a solution to the stationary Euler equations with $p =$ constant. Special examples include the trivial solution $\boldsymbol{v} \equiv 0$ and $\boldsymbol{v}(x) = (v_1(x_2), 0)$. Instead of asking what assumptions lead to the trivial solution $\boldsymbol{v} \equiv 0$, the authors ask what assumptions lead to a shear flow. In other words, the authors consider a solution $(\boldsymbol{v}, p)$ to the stationary Euler equations with some assumptions on $\boldsymbol{v}$, and instead of concluding that $\boldsymbol{v} \equiv 0$, their conclusion reads $\boldsymbol{v}$ is a shear flow.

Speaking of domains of Hamel and Nadirashvili's results, they consider special ones: $(0,1) \times \mathbb{R}, \mathbb{R}_+ \times \mathbb{R}$ and $\mathbb{R}^2$. Here we write their theorem in the case of $\Omega = (0,1) \times \mathbb{R}$. See Theorem 3.4.1 for the other domains.

Theorem 3.0.9 (Theorem 1.1 of [30])**.** *If* $\boldsymbol{v} \in C^2(\overline{\Omega})$ *is a solution to the stationary*

Euler equations in $\Omega = (0,1) \times \mathbb{R}$ *with* $v_1 = 0$ *on* $\partial\Omega$ *(tangential on* $\partial\Omega$*) such that*

$$\inf_{\Omega} |\boldsymbol{v}| > 0, \tag{3.22}$$

then $\boldsymbol{v}$ *is a shear flow, that is,* $\boldsymbol{v}(x) = (0, v_2(x_1))$ *for all* $x \in \Omega$*.*

As a matter of fact, Hamel and Nadirashvili investigated the problem in $\mathbb{R} \times (0,1)$, not $(0,1) \times \mathbb{R}$. But this is just a matter of rotation. And here we have just stated their theorem in $(0,1) \times \mathbb{R}$ for the sake of comparison to our results.

As for the other domains, $\Omega = \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}^2$, their statements are slightly different, but they needed to assume (3.22) for all the cases.

Their theorems have a limitation as it was pointed out by them: the condition $\inf_{\mathbb{R}^2} |\boldsymbol{v}| > 0$ is not equivalent to being a shear flow because one can construct various simple examples of a shear flow that does not satisfy this condition. For instance, $\boldsymbol{v} = (v_1(x_2), 0)$ is a shear flow but v_1 may vanish at many points, which does not satisfy the condition $\inf_{\mathbb{R}^2} |\boldsymbol{v}| > 0$.

However, we can also derive integral identities on a section of a domain with boundary, and, as an immediate corollary of that, we can establish assumptions on a solution $(\boldsymbol{v}, p)$ to the stationary Euler equations with some boundary condition, that are equivalent to being a shear flow. We state here one of our main theorems of this topic in the case $\Omega = (0,1) \times \mathbb{R}$. (The rest of our main theorems of this topic can be found in Theorems 3.4.2,3.4.3, 3.4.4, 3.4.9.)

Theorem 3.0.10. *Let* $(\boldsymbol{v}, p) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ *be a solution to the stationary Euler equations in* $\Omega = (0,1) \times \mathbb{R}$ *satisfying*

$$v_2|_{\partial\Omega} = 0 \tag{3.23}$$

and

$$\liminf_{|x_2|\to\infty} \int_0^1 (v_2^2(x) + |p(x)|)dx_1 = 0.$$

If the pressure p satisfies

$$\int_0^1 p(x)dx_1 \geq 0 \text{ for all } x_2 \in \mathbb{R}, \tag{3.24}$$

then $\boldsymbol{v}$ is a shear flow, that is, $\boldsymbol{v}(x) = (v_1(x_2), 0)$ for all $x \in \Omega$.

This is a simple application of Lemma 3.7.7, whose proof can be found on page 132.

The assumption (3.24) is equivalent to being a shear flow. If $\boldsymbol{v}$ is a shear flow, then from the stationary Euler equations, it holds that $p = 0$ up to a constant, which obviously satisfies (3.24). See also Remark 3.4.5.

One downside of our result is that we are making a non-standard boundary condition. But example (3.20) even satisfy both the non-standard boundary condition (3.23) and the standard boundary condition $\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$, that is, $v_1|_{\partial\Omega} = 0$. See also Remarks 3.4.6, 3.4.7

In addition, example (3.20) satisfies all the assumptions of Theorem 3.0.10 except (3.24) and is not a shear flows. Therefore, this is also an example that shows we cannot simply remove (3.24). (See Example 3.8.4 and Remark 3.4.8.)

This chapter is organized in the following way. Section 3.1 is just devoted for notations and definitions used in this chapter. Section 3.2 provides old theorems on integral identities in the entire space for the Euler and the Navier-Stokes equations and sketch of their proofs. In Section 3.3, we study integral identities on a hyperplane and provide our main results of this topic, Theorem 3.3.2 in addition to Theorem 3.0.2, which is already introduced in this introduction. Then in Section 3.4 we provide Liouville type theorems of the stationary Euler equations related to shear flows in two dimension with our main results, Theorem 3.4.2, 3.4.3, 3.4.4, 3.4.9 in addition to Theorem 3.0.10. Next, in Section 3.5, we investigate Liouville type theorems for Beltrami solutions to the stationary Euler equations in $\mathbb{R}^3$ with a proof of our main result of this topic, Theorem 3.0.7, which is already introduced in this introduction. Lastly, Section 3.7 contains proofs of majority of our main theorems, Theorems 3.0.2, 3.3.2, 3.0.10, 3.4.2, 3.4.3, 3.4.9, along with various approaches to establish integral identities on a hyperplane for equations with divergence structure.

3.1 Notations and Definitions

Throughout this chapter, we do not use the Einstein summation convention to avoid confusions.

In this paper, we denote a vector-valued or tensor-valued function in boldface: For example, we denote a scalar field by p, a vector field by $\boldsymbol{v} = (v_1, \ldots, v_N)$ and a tensor field by $\boldsymbol{F} = \{F_{jk}\}$.

For $k \in \{1, \ldots, N\}$, we denote the k-th variable of $x \in \mathbb{R}^N$ by x_k. For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we write $x_k' = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-1}$. For a vector $\boldsymbol{v}$, each component of $\boldsymbol{v}$ is denoted by v_j, $j = 1, \ldots, N$. A similar notation also applies to a tensor. Subscripts are reserved only for this purpose. For indices, the letters j, k are used to indicate fixed indices whereas the letters l, m are used to indicate dummy indices (when we write a summation over l and/or m).

For $k \in \{1, \ldots, N\}$ and a function $f(x) \in L^1_{loc}(\mathbb{R}^N)$, we write

$$f(x) \in L^1_{loc,x_k} L^1_{x_k'}$$

when

- for a.e. $x_k \in \mathbb{R}$, the map $x_k' \to f(x)$ is in $L^1(\mathbb{R}^{N-1})$, and

- the map $x_k \mapsto \int_{\mathbb{R}^{N-1}} |f(x)| dx'_k$ is in $L^1_{loc}(\mathbb{R})$.

A vector field $\boldsymbol{v} = (v_1, \ldots, v_N) \in L^1_{loc,x_k} L^1_{x'_k}$ if and only if $v_j \in L^1_{loc,x_k} L^1_{x'_k}$ for all $j = 1, \ldots, N$.

Similarly, for $k \in \{1, \ldots, N\}$ and a function $f(x,t) \in L^1_{loc}(0,T; L^1_{loc}(\mathbb{R}^N))$, we write

$$f(x,t) \in L^1_{loc,t} L^1_{loc,x_k} L^1_{x'_k}$$

when

- for a.e. $x_k \in \mathbb{R}$, a.e. $t \in \mathbb{R}$, the map $x'_k \mapsto f(x,t)$ is in $L^1(\mathbb{R}^{N-1})$,
- for a.e. $t \in \mathbb{R}$, the map $x_k \mapsto \|f(x,t)\|_{L^1(\mathbb{R}^{N-1}; x'_k)}$ is in $L^1_{loc}(\mathbb{R}; x_k)$,
- for all bounded intervals $I \subset \mathbb{R}$, the map $t \mapsto \int_I \int_{\mathbb{R}^{N-1}} |f(x,t)| dx'_k dx_k$ is in $L^1_{loc}(0,T)$.

We will call a type of integral identities involving two indices like (3.25) a *matrix form* whereas the other type with one index *a vector form*.

A smooth function with compact support is called a *test function*. The letter $\psi(x)$ is reserved for a test function in $\mathbb{R}^N$. And we write $\phi(t)$ to indicate a test function with respect to time variable t.

For a fixed index $k \in \{1, \ldots, N\}$, $\xi(x_k)$ is reserved for a test function of $x_k \in \mathbb{R}$. And we define specific test functions as follows. Fix $k \in \{1, \ldots, N\}$. Let $\sigma(x'_k)$ be a cut-off function in $C_c^\infty(\mathbb{R}^{N-1})$ such that

$$\sigma(x'_k) = \begin{cases} 1 & \text{if } |x'_k| < 1 \\ 0 & \text{if } |x'_k| > 2 \end{cases}$$

and $0 \leq \sigma(x'_k) \leq 1$ for $1 < |x'_k| < 2$. And for $R > 0$, we define $\sigma_R(x'_k) = \sigma(x'_k/R)$. Note that $\xi(x_k)\sigma(x'_k) \in C_c^\infty(\mathbb{R}^N)$. We will use this test function $\xi(x_k)\sigma(x'_k)$ frequently in Section 3.7.

3.2 Some Integral Identities in the Entire Space

This section is devoted to study the integral identities (3.14) in the entire space $\mathbb{R}^N$ for the Euler and the Navier-Stokes equations. We present first more details about the history of this topic.

In 1994, Dobrokhotov and Shafarevich [12] proved the integral identities in the entire space $\mathbb{R}^3$ of a classical solution $\boldsymbol{v}$ to the Euler and the Navier-Stokes equations: If $\boldsymbol{v}$ and its derivatives $\partial\boldsymbol{v}/\partial t, \partial\boldsymbol{v}/\partial x_j, j = 1, 2, 3$ decay faster than $|x|^{-4}$ at some time t, then

$$\int_{\mathbb{R}^3} (v_j(x,t)v_k(x,t) + \delta_{jk}p(x,t))\, dx = 0 \qquad \text{for all } j,k = 1,2,3. \tag{3.25}$$

Their main idea is to multiply j-th equations by x_k and k-th equations by x_j and integrate by parts the sum of the resulting equations in a ball B_R of radius R, which leads to

$$\partial_t \int_{B_R} (v_j x_k + v_k x_j) dx - 2\int_{B_R} (v_j v_k + p\delta_{jk}) dx = \text{some boundary integrals.}$$

As $v_j x_k + v_k x_j = \operatorname{div}(\boldsymbol{v} x_j x_k)$, the first volume integral can be written as a boundary integral by applying the divergence theorem. And applying the decay conditions, all the boundary terms vanish, which proves the integral identities above. .

In Brandolese and Meyer's paper in 2002 [2] and in Chae and Constantin in 2015 [8] independently, they provided a proof of the integral identities in the entire space by means of the Fourier transform.

Theorem 3.2.1 (Theorem 1.1 of Chae,Constantin [8]). *If $(\boldsymbol{v}, p)$ satisfies*

$$p = -\sum_{j,k=1}^{N} R_j R_k (v_j v_k) \tag{3.26}$$

where $R_j, j = 1, \dots, N$ denote the Riesz transforms, and if $|\boldsymbol{v}|^2 + |p| \in L^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} v_j v_k dx = -\delta_{jk} \int_{\mathbb{R}^N} p dx. \tag{3.27}$$

Here the Riesz transforms $R_j, j = 1, \dots, N$ are given by

$$R_j(f)(x) = C_N \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{(x_j - y_j) f(y)}{|x-y|^{N+1}} dy, \quad C_N = \frac{\Gamma((N+1)/2)}{\pi^{(N+1)/2}}. \tag{3.28}$$

Note that by taking the divergence of the Euler or the Navier-Stokes equations (3.1), we can obtain

$$\Delta p = - \sum_{j,k=1}^{N} \partial_{x_j} \partial_{x_k} (v_j v_k). \tag{3.29}$$

Therefore, the pressure in the Euler and the Navier-Stokes equations (3.1) is given in terms of the velocity up to addition of a harmonic function by (3.26).

Sketch of the proof of Theorem 3.2.1 in [8]. By the Fourier transform, we have

$$\hat{p}(\xi) = - \sum_{l,m=1}^{N} \frac{\xi_l \xi_m}{|\xi|^2} \widehat{v_l v_m}(\xi).$$

As $|\boldsymbol{v}|^2 + |p| \in L^1(\mathbb{R}^N)$, the left hand side and thus the right hand side is continuous at $\xi = 0$, which implies

$$\hat{p}(0) \delta_{jk} = -\widehat{v_j v_k}(0).$$

This, in turn, implies the integral identities (3.27). □

In 2011, Chae proved the integral identities (3.27) in the entire space by choosing a special type of test functions in a weak formulation of (3.29).

We first define a weak solution to the Euler or the Navier-Stokes equations.

Definition 3.2.2 (Weak Solution)**.** A pair $(\boldsymbol{v}, p) \in L^1(0,T; L^2_{loc}(\mathbb{R}^N)) \times L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ is called a weak solution to the Euler and the Navier-Stokes equations (3.1) in $\mathbb{R}^N$ if and only if for all $k \in \{1, \ldots, N\}$ and all $\psi(x) \in C_c^\infty(\mathbb{R}^N), \phi(t) \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} v_k(x,t)\psi(x)\partial_t\phi(t)dxdt \\
&+ \sum_{l=1}^N \int_0^T \int_{\mathbb{R}^N} v_l(x,t)v_k(x,t)\partial_{x_l}\psi(x)\phi(t)\,dxdt + \int_0^T \int_{\mathbb{R}^N} p(x,t)\,\partial_{x_k}\psi(x)\phi(t)\,dxdt \\
&\qquad\qquad = -\nu \sum_{l=1}^N \int_0^T \int_{\mathbb{R}^N} v_k \partial_{x_l}^2 \psi(x)\phi(t)dxdt \quad (3.30)
\end{aligned}$$

and for a.e. $t \in (0,T)$,

$$\sum_{j=1}^N \int_{\mathbb{R}^N} v_j(x,t)\partial_{x_j}\psi(x)dx = 0 \qquad \text{for all } \psi \in C_0^\infty(\mathbb{R}^N). \tag{3.31}$$

This definition comes from Definition 1.1 of Chae [3], but there are small differences between them. In Chae [3], he defined a weak solution for pressure in $L^1(0,T;\mathcal{S}'(\mathbb{R}^N))$, not in $L^1(0,T;L^1_{loc}(\mathbb{R}^N))$. Due to that difference, he wrote the dual pairing $\langle p(t), \partial_k\psi\rangle$ in place of the product $p(x,t)\partial_k\psi(x)$ in the definition above.

Theorem 3.2.3 (Chae [3])**.** *If* $(\boldsymbol{v}, p) \in L^1(0,T; L^2_{loc}(\mathbb{R}^N)) \times L^1(0,T;\mathcal{S}'(\mathbb{R}^N))$ *is a weak solution to the Euler or the Navier-Stokes equations* (3.1) *in* $\mathbb{R}^N$ *and* $(\boldsymbol{v}, p)$ *satisfies*

$$|\boldsymbol{v}| \in L^1(0,T;L^2(\mathbb{R}^N)), \quad |p| \in L^1(0,T;L^1(\mathbb{R}^N)), \tag{3.32}$$

then the integral identities (3.27) *in the entire space hold for almost every* $t \in (0,T)$.

This Theorem 3.2.3 is taken from Theorem 1.1 of Chae [3]. However, as a matter of fact, this statement of Theorem 3.2.3 is slightly different from Theorem 1.1 of Chae [3]. In Theorem 1.1 of [3], Chae divided cases by the sign of the integral of p in $\mathbb{R}^N$. In one case, he made a Liouville type result whereas, in the other case, he showed the integral identities (3.27). In addition, Chae also gave an alternative condition of p with $L^1(\mathbb{R}^N)$ replaced by the Hardy space $\mathcal{H}^q(\mathbb{R}^N)$. Moreover, Chae included the condition $|\boldsymbol{v}| \in L^1(0,T;L^2(\mathbb{R}^N))$ in his definition of a weak solution rather than stating it as an assumption explicitly.

Sketch of the Proof of Theorem 3.2.3 in [3]. We first use the test function given by

$$\frac{x_j^2}{2}\sigma_R(x)$$

for fixed j in a weak formulation of (3.29) where $\sigma_R(x) = \sigma(x/R)$ and $\sigma(x)$ is a standard cut-off function, which equals to one for $|x| < 1$ but vanishes for $|x| > 2$.

Any integrals involving derivatives of $\sigma_R(x)$ of any order vanish as $R \to \infty$. On the other hand, the integrals that do not contain derivatives of $\sigma_R(x)$ remain after taking $R \to \infty$. This yields us the integral identities (3.27) in the entire space in case $j = k$.

Now using

$$x_j x_k \sigma_R(x)$$

as a test function in the weak formulation, similarly we obtain the other cases of the integral identities (3.27) in the entire space for $j \neq k$. $\square$

And Chae also applied the same idea to the magnetohydrodynamic (MHD) equations in $\mathbb{R}^N, N \geq 2$.

$$\begin{aligned} &\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} - \nabla\left(p + \frac{1}{2}|\boldsymbol{b}|^2\right) + \nu\Delta\boldsymbol{v} \\ &\frac{\partial b}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{v} + \mu\Delta\boldsymbol{b} \\ &\operatorname{div}\boldsymbol{v} = \operatorname{div}\boldsymbol{b} = 0 \end{aligned} \tag{3.33}$$

where $\boldsymbol{v} = (v_1, \ldots, v_N), \boldsymbol{v} = \boldsymbol{v}(x,t)$ is the velocity of the flow, $p = p(x,t)$ is the scalar pressure, $\boldsymbol{b} = (b_1, \ldots, b_N), \boldsymbol{b} = \boldsymbol{b}(x,t)$ is the magnetic field.

Definition 3.2.4. A triple $(\boldsymbol{v}, \boldsymbol{b}, p) \in [L^1(0,T; L^2_{loc}(\mathbb{R}^N))]^2 \times L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ is said to be a weak solution to the MHD equation (3.33) ($\mu, \nu \geq 0$) on $\mathbb{R}^N \times (0,T)$ if for all $k \in \{1, \ldots, N\}$

$$
\begin{aligned}
&-\int_0^T \int_{\mathbb{R}^N} v_k(x,t)\psi(x)\phi'(t)\,dxdt - \sum_{l=1}^N \int_0^T \int_{\mathbb{R}^N} v_k(x,t)v_l(x,t)\partial_{x_l}\psi(x)\phi(t)\,dxdt \\
&\quad = -\sum_{l=1}^N \int_0^T \int_{\mathbb{R}^N} b_k(x,t)b_l(x,t)\partial_{x_l}\psi(x)\phi(t)\,dxdt + \int_0^T \int_{\mathbb{R}^N} p(x,t)\partial_{x_k}\psi(x)\phi(t)\,dxdt \\
&\quad + \frac{1}{2}\int_0^T \int_{\mathbb{R}^N} |\boldsymbol{b}(x,t)|^2 \partial_{x_l}\psi(x)\phi(t)\,dxdt + \nu \int_0^T \int_{\mathbb{R}^N} v_k(x,t)\Delta\psi(x)\phi(t)\,dxdt
\end{aligned}
\tag{3.34}
$$

and

$$
\begin{aligned}
&-\int_0^T \int_{\mathbb{R}^N} b_k(x,t)\psi(x)\phi'(t)\,dxdt - \sum_{l=1}^N \int_0^T \int_{\mathbb{R}^N} b_k(x,t)v_l(x,t)\partial_{x_l}\psi(x)\phi(t)\,dxdt \\
&\quad = -\sum_{l=1}^N \int_0^T \int_{\mathbb{R}^N} v_k(x,t)b_l(x,t)\partial_{x_l}\psi(x)\phi(t)\,dxdt + \mu \int_0^T \int_{\mathbb{R}^N} b_k(x,t)\Delta\psi(x)\phi(t)\,dxdt
\end{aligned}
\tag{3.35}
$$

for all $\xi \in C_0^\infty(0,T)$ and $\phi \in C_0^\infty(\mathbb{R}^N)$, and if both fields $\boldsymbol{v}, \boldsymbol{b}$ satisfy the divergence free condition in the weak sense as in (3.31).

Theorem 3.2.5 (Chae [3])**.** *Suppose $(\boldsymbol{v}, \boldsymbol{b}, p)$ is a weak solution to the magnetohydrodynamic equations (3.33) with $\mu, \nu \geq 0$ on $\mathbb{R}^N \times (0,T)$ satisfying*

$$
|\boldsymbol{v}|, |\boldsymbol{b}| \in L^1(0,T; L^2(\mathbb{R}^N)), \quad |p| \in L^1(0,T : L^1(\mathbb{R}^N)). \tag{3.36}
$$

Then

$$
\int_{\mathbb{R}^N} \left(v_j^2 + p + \frac{1}{2}|\boldsymbol{b}|^2 - b_j^2 \right) dx = 0 \quad \text{for all } j = 1, \ldots, N. \tag{3.37}
$$

This Theorem 3.2.5 is taken from the proof of Theorem 3.1 of Chae [3]. The Theorem 3.1 of Chae [3] is stated as a Liouville type theorem which can be obtained by applying the integral identities (3.37) in the entire space. The main idea of the proof is similar to the one of Theorem 3.2.3.

3.3 Some Integral Identities on a Hyperplane

This section is devoted to the study of the integral identities on a hyperplane either in the matrix form

$$\int_{\mathbb{R}^{N-1}} (v_j v_k + p\delta_{jk}) dx'_k = 0 \quad \text{for all } x_k \in \mathbb{R}, \quad j,k = 1,\ldots,N \tag{3.38}$$

or in the vector form

$$\int_{\mathbb{R}^{N-1}} (v_k^2 + p) dx'_k = 0 \quad \text{for a.e. } x_k \in \mathbb{R}, \quad k = 1,\ldots,N \tag{3.39}$$

for the Euler and the Navier-Stokes equations.

Note that by taking the divergence of the Euler and the Navier-Stokes equations, we obtain

$$\Delta p = -\sum_{l,m=1}^{N} \partial_{x_l}\partial_{x_m}(v_l v_m), \tag{3.40}$$

which can be re-written as

$$0 = \sum_{l,m=1}^{N} \partial_{x_l}\partial_{x_m}(v_l v_m + p\delta_{lm}).$$

This is in the form of *double divergence free equations*

$$0 = \sum_{l,m=1}^{N} \partial_{x_l}\partial_{x_m} F_{lm} = 0 \tag{3.41}$$

for a tensor field, F_{lm}.

As the main idea of Chae's proofs in [3] used the double divergence structure of (3.40), Chae in 2012 [4] and in 2013 [5] studied the double divergence free equations and proved the vector form of the integral identities on a hyperplane as follows.

Theorem 3.3.1 (Theorem 2.1 of Chae [5])**.** *If a tensor $F_{jk}(\cdot,t) \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ satisfies* (3.41) *in the sense of distribution, then*

$$\int_{\mathbb{R}^{N-1}} F_{kk}(x,t)dx'_k = 0 \quad \textit{for all } k = 1, \ldots, N \tag{3.42}$$

(for all $x_k \in \mathbb{R}$).

As an immediate corollary of Theorem 3.3.1, we can obtain Theorem 3.0.1.

In addition, the stationary compressible Euler system is reduced to (3.41) with $F_{jk} = \rho v_j v_k + \delta_{jk} p$, $p = a\rho^\gamma$. One can readily obtain a corollary as follows.

Corollary 3.3.1 (Corollary 2 of Chae [5])**.** *If $(\rho, \boldsymbol{v})$ is a continuous weak solution the stationary compressible Euler equations in $\mathbb{R}^N$ with*

$$\int_{\mathbb{R}^N} (\rho|\boldsymbol{v}|^2 + a\rho^\gamma)dx < \infty,$$

then we have

$$\int_{\mathbb{R}^{N-1}} (\rho|v_k|^2 + a\rho^\gamma)dx'_k = 0 \quad \textit{for all } x_k \in \mathbb{R}, k = 1, \ldots, N,$$

and thus $\rho \equiv 0$ (vacuum).

Sketch of the proof of Theorem 3.3.1 in [5]. We use

$$e^{i\xi_k x_k}\sigma_R(x)$$

as a test function in a weak formulation of (3.41) where $\sigma_R(x) = \sigma(x/R)$ and $\sigma(x)$ is a standard cut-off function, which equals to one if $|x| < 1$ and vanishes if $|x| > 2$. Again the integrals involving derivatives of $\sigma_R(x)$ of any order vanishes as $R \to \infty$ whereas the other integrals remain as $R \to \infty$ and we can re-write the resulting integrals as follows:

$$\begin{aligned} 0 &= -\sum_{l,m=1}^{N} \int F_{lm}\partial_{x_l}\partial_{x_m}(e^{i\xi_k x_k})dx \\ &= \xi_k^2 \hat{f}(\xi_k), \qquad f(x_k) = \int_{\mathbb{R}^{N-1}} F_{kk}(x)dx'_k. \end{aligned}$$

Therefore, by using the continuity of $\hat{f}$, it follows that $f(x_m) = 0$ for all $x_m \in \mathbb{R}$. $\square$

By using a test function different from the ones used by Chae in [3–5] and/or by using equations (3.1) directly without taking the divergence of it, we can obtain one of our main results of this thesis, Theorem 3.0.2, which is already stated in the introduction of this chapter. A proof of Theorem 3.0.2 can be found in Section 3.7. Remarks can be found in Remarks 3.0.3, 3.0.4.

In addition to Theorem 3.0.2, we can also prove the following theorem with another type of additional integral identities of $\boldsymbol{v}$ if we make an additional assumption on $\boldsymbol{v}$.

Theorem 3.3.2. *Let* $(\boldsymbol{v}, p) \in L^1(0,T; L^2_{loc}(\mathbb{R}^N)) \times L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ *be a weak solution to the evolutionary Euler or Navier-Stokes equations* (3.30), (3.31). *Suppose* *that for all* $k \in \{1, \dots, N\}$,

$$\begin{gathered}\left(|\boldsymbol{v}(x,t)|^2 + |p(x,t)|\right) \in L^1_{loc,t} L^1_{loc,x_k} L^1_{x_k'} \\ \liminf_{|x_k| \to \infty} \left\| |v_k(x,t)|^2 + |p(x,t)| \right\|_{L^1(\mathbb{R}^{N-1};x_k')} = 0 \qquad a.e.\ t \in (0,T).\end{gathered}$$

In addition, we assume that for all $k \in \{1, \dots, N\}$

$$\boldsymbol{v}(x,t) \in L^1_{loc,t} L^1_{loc,x_k} L^1_{x_k'},$$

and that for all $k \in \{1, \dots, N\}$ *there exists* $t_0 \in (0,T)$ *and* $\delta > 0$ *such that*

$$\liminf_{|x_k| \to \infty} \|v_k(x,t)\|_{L^1(\mathbb{R}^{N-1};x_k')} = 0, \quad a.e.\ t \in (t_0 - \delta, t_0 + \delta).$$

Then it holds that for all $k \in \{1, \dots, N\}$ *and for a.e.* $t \in (0,T)$

$$\int_{\mathbb{R}^{N-1}} \left(v_k^2(x,t) + p(x,t)\right) dx_k' = 0 \quad \textit{for a.e. } x_k \in \mathbb{R}, \tag{3.43}$$

$$\int_{\mathbb{R}^{N-1}} v_k(x,t) dx_k' = 0 \quad \textit{for a.e. } x_k \in \mathbb{R}. \tag{3.44}$$

A proof of Theorem 3.3.2 is written on page 125.

The total flux, $\int_{\mathbb{R}^N} \boldsymbol{v} dx$, of velocity of the Euler equation is a well-known basic conserved quantity. We can prove it by integrating the integral of (3.44) with respect to x_k. However, due to the lack of my knowledge about the evolutionary equations, it is unclear what kind of meaningful application we can make out of (3.44).

As Chae [5] derived integral identities (in the entire plane) for the magnetohydrodynamic equations (see Theorem 3.2.5), we can also apply our idea to these equations to obtain integral identities on a hyperplane.

Theorem 3.3.3. *Let* $(\boldsymbol{v}, \boldsymbol{b}, p) \in [L^1(0,T; L^2_{loc}(\mathbb{R}^N))]^2 \times L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ *be a weak solution to the magnetohydrodynamic equations* (3.33) *with* $\mu, \nu \geq 0$ *on* $\mathbb{R}^N \times (0,T)$.

Assume that for some $t \in (0,T)$ *and for all* $k \in \{1,\dots,N\}$

$$\left(|\boldsymbol{v}(x,t)|^2 + |\boldsymbol{b}(x,t)|^2 + |p(x,t)|\right) \in L^1_{loc,x_k} L^1_{x_k'},$$
$$\liminf_{|x_k|\to\infty} \left\| |\boldsymbol{v}(x,t)|^2 + |\boldsymbol{b}(x,t)|^2 + |p(x,t)| \right\|_{L^1(\mathbb{R}^{N-1};x_k')} = 0.$$

Then for the evolutionary magnetohydrodynamic equations (3.33) *with* $\mu, \nu \geq 0$*, for all* $k \in \{1,\dots,N\}$ *and for a.e.* $t \in (0,T)$

$$\int_{\mathbb{R}^{N-1}} \left(v_k^2(x,t) - b_k^2(x,t) + \left(\frac{1}{2}|\boldsymbol{b}(x,t)|^2 + p(x,t) \right) \right) dx_k' = 0 \qquad a.e.\ x_k \in \mathbb{R}, \quad (3.45)$$

whereas for the stationary magnetohydrodynamic equations (3.33) *with* $\mu, \nu = 0$*, for all* $j, k \in \{1,\dots,N\}$*,*

$$\int_{\mathbb{R}^{N-1}} \left(v_j(x)v_k(x) - b_j(x)b_k(x) + \left(\frac{1}{2}|\boldsymbol{b}(x)|^2 + p(x) \right) \delta_{jk} \right) dx_k' = 0 \quad a.e.\ x_k \in \mathbb{R}, \tag{3.46}$$

$$\int_{\mathbb{R}^{N-1}} \left(v_j(x)b_k(x) - v_k(x)b_j(x) \right) dx_k' = 0 \quad a.e.\ x_k \in \mathbb{R} \tag{3.47}$$

A proof of Theorem 3.3.3 can be found on page 125

3.4 Liouville Type Theorems and Shear Flows

We study Liouville type theorems of the stationary Euler equations as simple applications of the integral identities on a hyperplane in various situations.

Our study of Liouville type theorems of the stationary Euler equations is twofold: 1) one related to Beltrami solutions, 2) one related to shear flows.

In this section, we present Liouville type theorems in two dimensional domains related to shear flows.

For the study related to shear flows, one of our main results of this topic is already stated in Theorem 3.0.10 in the introduction of this chapter. This theorem pertains to the domain $\Omega = (0,1) \times \mathbb{R}$. So in this section, we state our theorems for the other domains $\Omega = \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}^2$ that Hamel and Nadirashvili considered in [30], [31].

To make comparison, we first state Hamel and Nadirahsilvi's results in [30], [31]. We put all of their theorems together for each case of domains $\Omega = (0,1) \times \mathbb{R}, \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}^2$.

Theorem 3.4.1 (Hamel, Nadirashvili [30], [31])**.** *Let $\boldsymbol{v} \in C^2(\overline{\Omega})$ be a solution to the stationary Euler equations in a domain $\Omega = I \times \mathbb{R}$ where I is an interval which will be specified below. Assume*

$$\inf_{x \in \Omega} |\boldsymbol{v}(x)| > 0$$

(for all the cases below). For each case of the following intervals I, we make the following assumptions and obtain the following conclusions:

- Case 1) $I = (0,1)$. *Assume $v_1|_{\partial\Omega} = 0$. Then $\boldsymbol{v}$ is a shear flow, that is, $\boldsymbol{v}(x) = (0, v_2(x_1))$ for all $x \in \Omega$.*

- Case 2) $I = \mathbb{R}_+$. *Assume $v_1|_{\partial\Omega} = 0$ and $\boldsymbol{v}$ is bounded in Ω. Then $\boldsymbol{v}$ is a shear flow, that is, $\boldsymbol{v}(x) = (0, v_2(x_1))$ for all $x \in \Omega$.*

- Case 3) $I = \mathbb{R}$. *Assume $\boldsymbol{v}$ is bounded in Ω. Then $\boldsymbol{v}$ is a shear flow.*

In Case 3, the conclusion does not give information about the direction of the shear flow $\boldsymbol{v}$ as opposed to Case 1 and 2, which make sense because in Case 3 we do not impose any assumptions regarding the direction of $\boldsymbol{v}$ whereas in Case 1 and 2 we impose the boundary conditions, which determine the direction of $\boldsymbol{v}$ on the boundary.

For Case 1 and 2, the actual statements in [30] pertain to the domain $\mathbb{R} \times (0,1), \mathbb{R} \times \mathbb{R}_+$. These changes are made here in order to avoid confusions in comparing it to the results of this thesis.

We present the main idea of Hamel and Nadirashvili's results. For the case of domains with boundary ($I = (0, 1), \mathbb{R}_+$), the main tools of their proofs are the stuy on geometric properties of the streamlines of a flow and one-dimensional symmetry results for solutions of some semilinear elliptic equations. Consider a flow $\boldsymbol{v}$ satisfying the assumptions of Theorem 3.4.1. Then there exists a potential function u such that $\boldsymbol{v} = \nabla^{\perp} u$. They showed all streamlines of $\boldsymbol{v}$ (connected components of level sets of u) go from $-\infty$ to ∞ and showed that the streamlines of $\boldsymbol{v}$ foliate the domain. By using this, they constructed a function f that satisfy a semilinear elliptic equation $\Delta u + f(u) = 0$. And to make a conclusion, they applied one-dimmensional symmetry results for solutions with bounded gradient of such semilinear elliptic equations.

But in the case of the entire plane, $\mathbb{R}^2$, they worked with the argument $\varphi_{\boldsymbol{v}}$ of $\boldsymbol{v}$ and eventually showed that $\varphi_{\boldsymbol{v}}$ is constant. To this end, they showed that the streamlines foliate the domain $\mathbb{R}^2$. By using this, they again constructed a function f satisfying a semilinear elliptic equation, $\Delta u + f(u) = 0$. They used this semilinear elliptic equations to derive an equation of the argument $\varphi_{\boldsymbol{v}}$, which is $\operatorname{div}(|\boldsymbol{v}|^2 \nabla \varphi_{\boldsymbol{v}}) = 0$. Then they proved that the argument $\varphi_{\boldsymbol{v}}$ grows at most as $\ln R$ in balls of large radius R. Lastly, they used a compactness argument and a result of Moser to conclude that the argument $\varphi_{\boldsymbol{v}}$ is constant.

As it was mentioned in the introduction of this chapter, the theorems have one limitation. There are shear flows that do not satisfy the condition $\inf_{x \in \Omega} |\boldsymbol{v}(x)| > 0$. Hence the condition $\inf_{x \in \Omega} |\boldsymbol{v}(x)| > 0$ is not equivalent to being a shear flow.

However, by using the integral identities on a hyperplane, we can find conditions that are equivalent to being a shear flow. Our main results of this topic are Theorem 3.0.10 (in the introduction) and Theorems 3.4.2, 3.4.3, 3.4.4, 3.4.9 below.

Now we present our main results of this topic. In addition to Theorem 3.0.10, which is already stated in the introduction, we can also prove theorems in other domains $\Omega = \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}^2$ as follows.

Theorem 3.4.2. *Let* $\Omega = \mathbb{R}_+ \times \mathbb{R}$ *and let* $(\boldsymbol{v}, p) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ *be a classical solution to the stationary Euler equations in* Ω*. Assume*

$$v_2|_{\partial\Omega} = 0. \tag{3.48}$$

In addition, suppose that

$$\begin{gathered}
v_2^2(x) + |p(x)| \in L^1(\mathbb{R}_+; x_1) \quad \textit{for all } x_2 \in \mathbb{R} \\
\liminf_{|x_2|\to\infty} \int_{\mathbb{R}_+} (v_2^2 + |p|)\, dx_1 = 0, \\
\int_0^s \int_{\mathbb{R}_+} \frac{|v_1(x) v_2(x)|}{1 + |x_1|} dx_1 dx_2 < \infty \quad \textit{for all } s \in \mathbb{R}
\end{gathered}$$

and that

$$\int_{\mathbb{R}_+} p\, dx_1 \geq 0 \quad \textit{for all } x_2 \in \mathbb{R}.$$

Then $\boldsymbol{v}$ *is a shear flow, that is,* $\boldsymbol{v}(x) = (v_1(x_2), 0)$ *for all* $x \in \mathbb{R}_+ \times \mathbb{R}$*.*

This theorem is a simple application of Lemma 3.7.8, whose proof is written on page 130.

Theorem 3.4.3. *Let* $(\boldsymbol{v}, p) \in C^1(\mathbb{R}^2) \times C^1(\mathbb{R}^2)$ *be a solution to the stationary Euler equations in* $\mathbb{R}^2$*. Suppose that*

$$(v_2^2 + |p|), v_1 v_2 \in L^1_{loc, x_2} L^1_{x_1}, \quad \liminf_{|x_2|\to\infty} \int_{\mathbb{R}} (v_2^2 + |p|)\, dx_1 = 0,$$

and that

$$\int_{\mathbb{R}} p\, dx_1 \geq 0 \quad \textit{for all } x_2 \in \mathbb{R}.$$

Then $\boldsymbol{v}$ *is a shear flow, that is,* $\boldsymbol{v}(x) = (v_1(x_2), 0)$ *for all* $x \in \mathbb{R}^2$*.*

This theorem is also a simple application of Lemma 3.7.3, whose proof is written on page 121.

Theorem 3.4.4. *Let $(\boldsymbol{v}, p) \in C^1(\mathbb{R}^2) \times C^1(\mathbb{R}^2)$ be a classical solution to the stationary Euler equations in $\mathbb{R}^2$. Assume that there exists a constant $a > 0$ such that*

$$\boldsymbol{v}(x + a\boldsymbol{e}_1) = \boldsymbol{v}(x) \quad \textit{for all } x \in \mathbb{R}^2.$$

and

$$\liminf_{|x_2| \to \infty} \int_0^a (v_2^2 + |p|)\, dx_1 = 0.$$

If the pressure p satisfies

$$\int_0^a p\, dx_1 \geq 0 \quad \textit{for all } x_2 \in \mathbb{R}, \tag{3.49}$$

then $\boldsymbol{v}$ is a shear flow, that is, $\boldsymbol{v}(x) = (v_1(x_2), 0)$ for all $x \in \mathbb{R}^2$.

This theorem is a simple application of Lemma 3.7.9.

We leave here some remarks about all these theorems above as well as Theorem 3.0.10.

Remark 3.4.5. For Theorems 3.0.10, 3.4.2, 3.4.3, 3.4.4, the condition

$$\int_I p\, dx_1 \geq 0 \quad \text{for all } x_2 \in \mathbb{R} \tag{3.50}$$

is equivalent to being a shear flow under the boundary and integrability assumptions as well as the assumption

$$\liminf_{|x_2| \to \infty} \int_I (v_2^2 + |p|) dx_1 = 0$$

for a corresponding interval $I = (0, 1), \mathbb{R}^+, \mathbb{R}$. Indeed, if $\boldsymbol{v}$ is a shear flow, then by using the stationary Euler equations, it follows that $p = 0$ up to a constant. Therefore, it obviously satisfies (3.50). Consequently, the condition (3.50) is equivalent to being a shear flow.

Remark 3.4.6. For Theorems 3.0.10, 3.4.2, we assume the non-standard boundary condition $v_2|_{\partial\Omega} = 0$ (normal, not tangential, on the boundary). However, as it was mentioned in the introduction, there are even simple flows that satisfy not only the non-standard but also the standard boundary condition. See (3.20) and Example 3.8.4.

Remark 3.4.7. For Theorem 3.0.10, 3.4.2, in fact, even if we simply replace boundary conditions (3.23), (3.48) by

$$v_1(0, x_2)v_2(0, x_2) = v_1(1, x_2)v_2(1, x_2) \quad \text{for all } x_2 \in \mathbb{R} \tag{3.51}$$

$$v_1(0, x_2)v_2(0, x_2) = 0 \quad \text{for all } x_2 \in \mathbb{R} \tag{3.52}$$

respectively. the conclusion is still true. And these boundary condition (3.51), (3.52) are satisfied if we assume the standard boundary condition $v_1|_{\partial\Omega} = 0$. However, in that case, the conclusion becomes $\boldsymbol{v} \equiv \mathbf{0}$, not just that $\boldsymbol{v}$ is a shear flow. So in this case, the theorem is not comparable to Hamel and Nadirashvili's result, Theorem 3.0.9. On the other hand, for Theorem 3.0.10 it is worth to note that the boundary condition (3.51) is satisfied if we assume a periodic boundary condition on $\partial\Omega$, that is, $\boldsymbol{v}(0, x_2) = \boldsymbol{v}(1, x_2)$ for all $x_2 \in \mathbb{R}$. Theorem 3.4.4 pertains to the periodic case.

Remark 3.4.8. Example (3.20) satisfies all the assumptions of Theorems 3.0.10, 3.4.2, 3.4.3 except

$$\int_I p \, dx_1 \geq 0 \quad \text{for all } x_2 \in \mathbb{R} \tag{3.53}$$

where $I = (0,1), \mathbb{R}_+, \mathbb{R}$ respectively; and the example is not a shear flow. Therefore, we cannot simply remove this condition (3.53) from Theorems 3.0.10, 3.4.2, 3.4.3. (See also Examples 3.8.1, 3.8.2, 3.8.3, 3.8.4.)

In the entire plane $\mathbb{R}^2$, we can also prove a Liouville type theorem with no sign condition on the pressure but on $v_1 v_2$ as follows.

Theorem 3.4.9. *Let $(\boldsymbol{v}, p)$ be a continuous weak solution to the stationary Euler equations in $\mathbb{R}^2$. Assume that*

$$v_j v_k + p\delta_{jk} \in L^1_{loc,x_1} L^1_{x_2}, \tag{3.54}$$

for all j, k except $(j,k) = (1,1)$ and that

$$\liminf_{|x_1| \to \infty} \|v_1 v_2\|_{L^1(\mathbb{R};x_2)} = 0.$$

In addition, suppose either

$$v_1 v_2 \geq 0 \ \textit{in} \ \mathbb{R}^2 \qquad \textit{or} \qquad v_1 v_2 \leq 0 \ \textit{in} \ \mathbb{R}^2. \tag{3.55}$$

And assume

$$|\boldsymbol{v}(x)| > 0 \qquad \textit{for all } x \in \mathbb{R}^2. \tag{3.56}$$

Then $\boldsymbol{v}$ is a shear flow, that is, $\boldsymbol{v} = (v_1(x_2), 0)$ in $\mathbb{R}^2$.

A proof of this Theorem 3.4.9 is written on page 132.

It is worth to compare this theorem to a result of Hamel and Nadirashvili's in [31].

Theorem 3.4.10 (Hamel, Nadirashvili [31])**.** *Let $\boldsymbol{v} \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be a solution* *to the stationary Euler equations. Assume either*

$$\boldsymbol{v} \cdot \boldsymbol{e} > 0 \textit{ in } \mathbb{R}^2 \quad \textit{or} \quad \boldsymbol{v} \cdot \boldsymbol{e} < 0 \textit{ in } \mathbb{R}^2 \tag{3.57}$$

for some unit vector $\boldsymbol{e}$*. And assume*

$$\inf_{\mathbb{R}^2} |\boldsymbol{v}(x)| > 0. \tag{3.58}$$

Then $\boldsymbol{v}$ *is a shear flow.*

As a matter of fact, this Theorem 3.4.10 is not stated as a theorem in Hamel and Nadirashvili [31]. (This Theorem 3.4.10 has one more extra condition, (3.57), compared to their main result, Theorem 1.1 of [31].) But they provided a proof of this statement separately because this statement can be proved more easily compared to their main result. See Remark 1.2 and Section 2.4 of [31].

Now we write remarks about Theorem 3.4.9.

Remark 3.4.11. For Theorem 3.4.9, it does not provide conditions that are equivalent to being a shear flow due to condition (3.56).

Remark 3.4.12. To compare Theorem 3.4.9 with Theorem 3.4.10, condition (3.56) does not exclude the case where $\boldsymbol{v}$ converges to zero at infinity, as opposed to (3.58). However, given condition (3.56), condition (3.55) is stronger than (3.57). This is because if, for example, $v_1 v_2 \geq 0$ in $\mathbb{R}^2$ and condition (3.56) holds, then for $\boldsymbol{e} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$, condition (3.57) holds, too.

Remark 3.4.13. Without conditions (3.55), (3.56), there is a counterexample (3.20). This example satisfies all the assumptions except (3.55), (3.56), and it is not a shear flow.

3.5 Liouville Type Theorems for Beltrami Solutions

In this section, we present Liouville type theorems for Beltrami solutions to the stationary Euler equations in $\mathbb{R}^3$

$$\begin{aligned} (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \\ \mathrm{div}\boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^3. \end{aligned} \tag{3.59}$$

Let us first recall the definition of Beltrami solutions to the stationary Euler equations in $\mathbb{R}^3$.

Definition 3.5.1. A pair of a vector field $\boldsymbol{v}$ and pressure p is called a *Beltrami solution* to the stationary Euler equations in $\mathbb{R}^3$ if

$$p + \frac{1}{2}|\boldsymbol{v}|^2 = c \quad \text{in } \mathbb{R}^3 \tag{3.60}$$

for some constant c and there exists a function $\lambda = \lambda(x)$ such that

$$\mathrm{curl}\boldsymbol{v}(x) = \lambda(x)\boldsymbol{v}(x) \quad \text{in } \mathbb{R}^3. \tag{3.61}$$

The pair satisfying these two conditions is in fact a solution to the stationary Euler equations. Indeed, the first equation of the stationary Euler equations (3.59) can be re-written as

$$\boldsymbol{v} \times \boldsymbol{\omega} = \nabla(p + \frac{1}{2}|\boldsymbol{v}|^2), \quad \boldsymbol{\omega} = \mathrm{curl}\, \boldsymbol{v}.$$

Therefore, if both conditions (3.60), (3.61) are satisfied, then the vector field $\boldsymbol{v}$ and pressure field p is a solution to the stationary Euler equations. (A vector field satisfying the first condition (3.60) is sometimes called a *Beltrami flow*. For more details about the definition of Beltrami solutions, see Chae and Constantin [8].)

As it was mentioned in the introduction, Enciso and Peralta-Salas [13] in 2012 constructed a non-trivial Beltrami flow which decays of $O(1/|x|)$ as $|x| \to \infty$.

On the other hand, Nadirashvili [42] in 2014 proved a Liouville type theorem for Beltrami solutions as follows.

Theorem 3.5.2 (Theorem of Nadirashvili [42]). *Let $\boldsymbol{v} \in C^1(\mathbb{R}^3)$ be a Beltrami solution. Assume that either $\boldsymbol{v} \in L^p(\mathbb{R}^3), 2 \leq p \leq 3$, or $\boldsymbol{v}(x) = o(1/|x|)$ as $|x| \to \infty$. Then $\boldsymbol{v} \equiv \mathbf{0}$.*

At the time when the paper [42] was published, the problem of constructing a smooth compactly-supported solution to the stationary Euler equations in $\mathbb{R}^3$ was open. Hence the theorem [42] provided a negative result for this problem. We present more details about this problem in the next section.

In 2015, Chae and Constantin proved a similar statement with different, simple proofs. Their theorems have already been stated in the introduction of this chapter (Theorem 3.0.5).

They first proved the integral identities (3.27) in the entire space. See Theorem 3.2.1.

Using the integral identities (3.27) in the entire space, they proved the first part of Theorem 3.0.5 as follows.

Proof of the first part of Theorem 3.0.5. As $(\boldsymbol{v}, p)$ is a Beltrami solution to the stationary Euler equations, condition (3.60) is satisfied and thus it implies that $\tilde{p} := p - c = -\frac{1}{2}|\boldsymbol{v}|^2$ is in $L^1(\mathbb{R}^3)$. Therefore, the integral identities (3.27) hold. And using these integral identities along with condition (3.60), we can obtain

$$\int_{\mathbb{R}^3} \tilde{p} dx = -\int_{\mathbb{R}^3} v_j^2(x) dx \quad \text{for all } j = 1, 2, 3.$$

Hence we have

$$\int_{\mathbb{R}^3} \tilde{p} dx = -\frac{1}{3} \int_{\mathbb{R}^3} |\boldsymbol{v}(x)|^2 dx.$$

By using $\tilde{p} = -\frac{1}{2}|\boldsymbol{v}|^2$, it follows that

$$\int_{\mathbb{R}^3} \tilde{p} dx = -\frac{1}{3} \int_{\mathbb{R}^3} |\boldsymbol{v}|^2 dx = -\frac{1}{2} \int_{\mathbb{R}^3} |\boldsymbol{v}|^2 dx.$$

Therefore, $\boldsymbol{v} \equiv \mathbf{0}$. □

For the second part of Theorem 3.0.5, they provided a different proof. We only present sketch of their proof.

Sketch of the proof of the second part of Theorem 3.0.5. Under the assumptions of the second part of Theorem 3.0.5, it holds that

$$\int_{\mathbb{R}^3} |\boldsymbol{v}|^2 |x|^{\mu-2} dx < \infty$$

for some $\mu \in (1, 2)$. As a Beltrami solution $(\boldsymbol{v}, p)$ satisfies the equation

$$\sum_{j,k=1}^{3} \partial_{x_j} \partial_{x_k} (v_j v_k) = \frac{1}{2} \Delta |\boldsymbol{v}|^2$$

in a weak sense, we can obtain

$$(\mu - 1) \int_{\mathbb{R}^3} |\boldsymbol{v}|^2 |x|^{\mu-2} dx = 2(\mu - 2) \int_{\mathbb{R}^3} (\boldsymbol{v} \cdot x)^2 |x|^{\mu-4} dx,$$

which implies $\boldsymbol{v} \equiv \mathbf{0}$. □

Regarding Theorem 3.0.5, the integral identities (3.27) were used to prove the first part of the theorem and could not be used to prove the second part. We can prove a theorem similar to the second part by using the integral identities on a hyperplane as an immediate application. See one of our main result, Theorem 3.0.7. For a remark, see Remark 3.0.8.

Proof of Theorem 3.0.7. If we assume that $\boldsymbol{v} \in L^\infty_{loc}(\mathbb{R}^3)$ and that there exists $\varepsilon > 0$ such that $|\boldsymbol{v}(x)| = O(1/|x|^{1+\varepsilon})$ as $|x| \to \infty$, then the other two conditions (3.17), (3.18) are satisfied. Therefore, we can apply the integral identities on a hyperplane (3.14) according to one of our main results, Theorem 3.0.2. By integrating the integral on a hyperplane in (3.14) with respect to x_k, we obtain the integral identities

$$\int_{\mathbb{R}^N} (v_j(x)v_k + p(x))\,\delta_{jk}dx = 0 \quad \text{for all } j,k = 1,2,3. \tag{3.62}$$

Therefore, as in the proof of the first of Theorem 3.0.5, it follows that $\boldsymbol{v} \equiv \mathbf{0}$. □

For the rest of this section, we introduce another result in this topic briefly. Chae and Wolf [9] in 2016 extended the result of Chae and Constantin [8] by providing a weaker assumption to ensure the triviality of (weak) Beltrami solutions. We first provide a definition of a weak Beltrami solutions to the stationary Euler equations in $\mathbb{R}^3$.

Definition 3.5.3 (Definition 1.1 and 1.2 of Chae, Wolf [9])**.** A pair $(\boldsymbol{v}, p) \in L^2_{loc}(\mathbb{R}^3) \times L^1_{loc}(\mathbb{R}^3)$ is said to be a *weak solution* to the stationary Euler equations in $\mathbb{R}^3$ if and only if

$$\sum_{l,m=1}^{3} \int_{\mathbb{R}^3} v_l(x)v_m(x)\partial_{x_l}\varphi_m(x)\,dx = -\sum_{m=1}^{N} \int_{\mathbb{R}^3} p(x)\partial_{x_m}\varphi_m(x)dx \quad \text{for all } \boldsymbol{\varphi} \in C^\infty_c(\mathbb{R}^3) \tag{3.63}$$

and

$$\sum_{l=1}^{3} \int_{\mathbb{R}^3} v_l(x)\partial_{x_l}\psi(x)dx = 0 \quad \text{for all } \psi \in C^\infty_0(\mathbb{R}^N).$$

A weak solution $(\boldsymbol{v}, p)$ to the stationary Euler equations is said to be a *Beltrami* *solution* [1] if it satisfies

$$p = -\frac{|\boldsymbol{v}|^2}{2} \quad \text{a.e. in } \mathbb{R}^3. \tag{3.64}$$

Theorem 3.5.4 (Theorem 1.3 of Chae, Wolf [9]). *Let $\boldsymbol{v} \in L^2_{loc}$ be a weak Beltrami solution. Then*

$$\int_{B_R} \frac{|\boldsymbol{v}_N|^2}{|x|} dx \leq \frac{1}{2R} \int_{B_R} |\boldsymbol{v}|^2 dx = \frac{1}{2} \int_{\partial B_R} (|\boldsymbol{v}_T|^2 - |\boldsymbol{v}_N|^2) d\sigma \quad \textit{for all } R \in (0, \infty) \tag{3.65}$$

where $\boldsymbol{v}_N := \left(\boldsymbol{v} \cdot \frac{x}{|x|}\right) \frac{x}{|x|}, \boldsymbol{v}_T := \boldsymbol{v} - \boldsymbol{v}_N$. Therefore, if there exists a sequence $R_k \to \infty$ such that

$$\int_{\partial B_{R_k}} |\boldsymbol{v}_T|^2 d\sigma \to 0 \quad \textit{as } k \to \infty, \tag{3.66}$$

then $\boldsymbol{v} \equiv \boldsymbol{0}$.

Remark 3.5.5. Theorem 3.5.4 extends the results of Chae, Constantin [8]. Indeed, if a weak Beltrami solution $\boldsymbol{v} \in L^2_{loc}$ satisfy one of the following conditions:

- $|\boldsymbol{v}_T(x)| = o(|x|^{-1})$ as $|x| \to \infty$
- $\boldsymbol{v}_T \in L^q(\mathbb{R}^3)$ for some $q \in [2, 3]$
- $\frac{|\boldsymbol{v}_T|^2}{|x|^\mu} \in L^1(\mathbb{R}^3)$ for some $\mu \in (-\infty, 1]$,

then there exists a sequence $R_k \to \infty$ such that condition (3.66) holds. For more details, see Remark 1.4 of Chae, Wolf [9].

The main tool of their proof is the mean value formulas for weak solutions to the stationary Euler equations given below.

[1] In fact, they call it a Beltrami flow. But to be consistent with the definition given in Chae and Constantin, we call it a Beltrami solution.

Lemma 3.5.6 (Lemma 2.1 of Chae, Wolf [9]). *Let $\boldsymbol{v}, p$ be a weak solution to the stationary Euler equations. Then there holds*

$$\int_{\partial B_R} (p + |\boldsymbol{v}_N|^2)\, d\sigma = \frac{1}{R}\int_{B_R} (3p + |\boldsymbol{v}|^2) dx \quad \textit{for a.e. } 0 < R < \infty. \tag{3.67}$$

3.6 Smooth Compactly-Supported Solutions to the Stationary Euler Equations

In this section, we briefly present history of the topic of construction of smooth compactly-supported solutions to the stationary Euler equations. For any even-dimension, we can easily construct an example as already mentioned in the introduction of this chapter. But for readers' convenience, choose $\varphi \in C_c^\infty[0,\infty)$ such that $\varphi(r) = 0$ for $r \leq r_0$ with some $r_0 > 0$. Then a pair of vector and pressure given by

$$v_{2j-1}(x) = -\varphi(|x|)x_{2j},\ v_{2j}(x) = \varphi(|x|)x_{2j-1}, \quad j = 1, \ldots, N, \quad p(x) = -\int_{|x|}^\infty s\varphi^2(s)ds$$

is a smooth, compactly-supported solution to the stationary Euler equations in $\mathbb{R}^{2N}$, $N \in \mathbb{N}$.

Construction of such a solution in odd dimensions has been a big open problem. As was already mentioned in the previous section, Nadirashilvi [42] in 2014 proved a Liouville type theorem for a Beltrami solution in $\mathbb{R}^3$, which furnishes a negative result for the open problem at least for Beltrami solutions.

Recently, Gavrilov [26] in 2019 established a remarkable result by constructing a smooth, compactly-supported solution to the stationary Euler equations in $\mathbb{R}^3$ resolving the open problem in case of three dimension. Gavrilov's construction is explicit but somewhat obscure. And Constantin, La, Vicol in 2019 [11] provided a proof of Gavrilov's result, which provides a general method applicable to many other hydrodynamic equations, such as the incompressible 2D Boussinesq system, the incompressible porous medium equation. We present their main ideas briefly step by step: 1) we seek for an axisymmetric solution $\boldsymbol{v}$ to the stationary Euler equations via the Grad-Shfranov ansatz:

$$\boldsymbol{v} = \frac{1}{r}(\partial_z \psi)e_r - \frac{1}{r}(\partial_r \psi)e_z + \frac{1}{r}F(\psi)e_\varphi;$$

2) Step 1 leads to the Grad-Shafranov equation augmented by a localizability condition

$$\frac{|\boldsymbol{v}|^2}{2} = A(\psi);$$

3) we apply a hodograph transformation to construct a solution to the equations above; 4) we localize the solution by using the localizability condition.

3.7 Equations with Divergence Structure

In this section, we prove most of our main theorems, Theorems 3.0.2, 3.3.2, 3.3.3, 3.0.10, 3.4.2, 3.4.3, 3.4.4, 3.4.9.

Recall that we are mainly interested in deriving the integral identities on a hyperplane

$$\int_{\mathbb{R}^{N-1}} (v_j v_k + p\delta_{jk})\, dx_k' = 0 \quad \text{for a.e. } x_k \in \mathbb{R}, \text{ for all } j, k \tag{3.68}$$

for a solution $(\boldsymbol{v}, p)$ to the Euler or the Navier-Stokes equations in $\mathbb{R}^N$. But the main idea of our proof (as well as many other proofs in [3], [4]) is in fact only related to divergence structure of the Euler and the Navier-Stokes equations. Hence, we first work on several types of equations with divergence structure in this section, and later we will re-write the Euler and the Navier-Stokes equations in the form of these types of equations.

Type 1. A double divergence free equation

$$\sum_{l,m=1}^{N} \partial_{x_l}\partial_{x_m} F_{lm}(x) = 0 \quad \text{in } \mathbb{R}^N, \tag{3.69}$$

for a tensor field $\boldsymbol{F}(x) = \{F_{jk}(x)\}$ of order 2 $(j, k = 1, \ldots, N)$.

Type 2. A divergence free equation

$$\sum_{l=1}^{N} \partial_{x_l} F_{lk}(x) = 0 \quad \text{in } \mathbb{R}^N, \quad k = 1, \ldots, N \tag{3.70}$$

for a tensor field $\boldsymbol{F}(x) = \{F_{jk}(x)\}$ of order 2 $(j, k = 1, \ldots, N)$.

Type 3.

$$\partial_t v_k(x,t) + \sum_{l=1}^{N} \partial_{x_l} F_{lk}(x,t) = \nu \Delta v_k(x,t) \quad \text{in } \mathbb{R}^N, \quad k = 1, \ldots, N. \tag{3.71}$$

for a tensor field $\boldsymbol{F}(x) = \{F_{jk}(x)\}$ of order 2 $(j, k = 1, \ldots, N)$ and divergence free vector field $\boldsymbol{v}$ and a parameter $\nu \geq 0$.

In order to see how the Euler and the Navier-Stokes equations are related to the types of equations given above, first we can formally take the divergence of the Euler or the Navier-Stokes equations, which leads to

$$\sum_{l,m=1}^{N} \partial_{x_l}\partial_{x_m} \left(v_l(x,t)v_m(x,t) + p(x,t)\delta_{lm}\right) = 0 \quad \text{in } \mathbb{R}^N \times (0,T). \tag{3.72}$$

Note that in taking divergence of the Euler and the Navier-Stokes equations, both the time derivative $\partial_t v$ and the Laplace term Δv disappear because $\boldsymbol{v}$ is assumed to be divergence free. Fixing $t \in (0,T)$, setting $F_{jk}(x) = v_j(x,t)v_k(x,t) + p(x,t)\delta_{jk}$ in $\mathbb{R}^N$, the Euler and the Navier-Stokes equations fall into the form of Type 1, a double divergence free equations. This reformulation of the Euler and the Navier-Stokes equations is well known especially to study pressure p and it has been a main tool to study the integral identities (3.3) for the Euler and the Navier-Stokes equations in various papers, such as Chae [3], [4], Chae and Constantin [8], Brandolese and Meyer [2].

On the other hand, when it comes to the stationary Euler equations, we can write the equations into

$$\sum_{l=1}^{N} \partial_{x_l} \left(v_l(x) v_k(x) + p(x)\delta_{lk} \right) = 0. \tag{3.73}$$

which apparently falls into Type 2, a divergence free equation.

We now provide a definition of a weak solution to each type of equations.

Definition 3.7.1. Type 1. A tensor field $\boldsymbol{F}(x) = \{F_{jk}(x)\}$ in $L^1_{loc}(\mathbb{R}^N; M_{N\times N}(\mathbb{R}))$ is called a *weak solution* to the double divergence free equations (3.69) if and only if

$$\sum_{l,m=1}^{N} \int_{\mathbb{R}^N} F_{lm}(x) \partial_{x_l} \partial_{x_m} \psi(x) dx = 0 \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N). \tag{3.74}$$

Type 2. A pair of a tensor field $\boldsymbol{F}(x) = \{F_{jk}(x)\}$ in $L^1_{loc}(\mathbb{R}^N)$ is called a *weak solution* to the divergence free equation (3.70) if and only if for all $k \in \{1, \dots, N\}$

$$-\sum_{l=1}^{N} \int_{\mathbb{R}^N} F_{lk}(x) \partial_{x_l} \psi(x) dx = 0 \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N). \tag{3.75}$$

Type 3. A triplet of a tensor field $\boldsymbol{F}(x,t) = \{F_{jk}(x,t)\}$ in $L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ and weakly divergence free vector field $\boldsymbol{v}(x,t)$ in $L^1(0,T; L^1_{loc}(\mathbb{R}^N))$ is called a *weak solution* to equation (3.71) of Type 3 if and only if for all $k \in \{1, \dots, N\}$

$$-\int_0^T \int_{\mathbb{R}^N} v_k(x,t)\psi(x)\partial_t\phi(t)dxdt - \sum_{l=1}^{N} \int_0^T \int_{\mathbb{R}^N} F_{jk}(x,t)\partial_{x_j}\psi(x)\phi(t)dxdt$$
$$= \int_0^T \int_{\mathbb{R}^N} v_k(x,t)\Delta\psi(x,t)\phi(t)dxdt \quad (3.76)$$

for all $\psi \in C_c^\infty(\mathbb{R}^N)$ and all $\phi \in C_c^\infty(0,T)$.

Now for each type of equations, we can prove an analogue of the integral identities (3.8) on a hyperplane.

Lemma 3.7.2. *Assume a symmetric tensor field $\boldsymbol{F}(x) = \{F_{jk}(x)\}$ of order 2 is a weak solution to the double divergence free equation* (3.69) *in* $\mathbb{R}^N$. *Fix* $k \in \{1, \dots, N\}$. *Suppose that*

$$F_{kk}(x) \in L^1_{loc,x_k} L^1_{x'_k}. \quad (3.77)$$

$$\frac{F_{km}(x)}{1+|x'_k|} \in L^1_{loc,x_k} L^1_{x'_k} \quad \forall m \neq k \quad (3.78)$$

$$\frac{F_{lm}(x)}{(1+|x'_k|)^2} \in L^1_{loc,x_k} L^1_{x'_k} \quad \forall m \neq k, l \neq k \quad (3.79)$$

Then there exist constant $A_k, B_k \in \mathbb{R}$ *such that*

$$\int_{\mathbb{R}^{N-1}} F_{kk}(x)dx'_k = A_k x_k + B_k \quad \textit{for a.e. } x_k \in \mathbb{R}. \quad (3.80)$$

Therefore, if we additionally assume either

$$\begin{aligned} \|F_{kk}(x)\|_{L^1(\mathbb{R}^{N-1};x'_k)} &\in L^1_{weak}(\mathbb{R}), \ \textit{or} \\ \liminf_{|x_k|\to\infty} \|F_{kk}(x)\|_{L^1(\mathbb{R}^{N-1};x'_k)} &= 0, \end{aligned} \quad (3.81)$$

then

$$\int_{\mathbb{R}^{N-1}} F_{kk}(x)dx_k' = 0 \quad \textit{for a.e. } x_k \in \mathbb{R}.$$

Proof of Lemma 3.7.2. Fix $k \in \{1, \ldots, N\}$. The product $\xi(x_k)\sigma(x_k')$ is smooth with compact support in $\mathbb{R}^N$ so we use this product function in the weak formulation (3.74) of Type 1 to obtain

$$\sum_{l,m=1}^{N} \int_{\mathbb{R}^N} F_{lm}(x)\partial_l\partial_m \left(\xi(x_k)\sigma(x_k')\right) dx = 0, \tag{3.82}$$

which leads to

$$\sum_{l=1}^{N} \int_{\mathbb{R}^N} F_{lk}(x)\partial_l \left(\partial_k\xi(x_k)\sigma(x_k')\right)\, dx \\ + \sum_{l=1}^{N}\sum_{m\neq k} \int_{\mathbb{R}^N} F_{lm}(x)\partial_l \left(\xi(x_k)\partial_m\sigma(x_k')\right)\, dx = 0 \tag{3.83}$$

where we have just applied the derivative ∂_m to the product $\xi\sigma$. Furthermore, we apply the derivative ∂_l to the various products to obtain

$$\int_{\mathbb{R}^N} F_{kk}(x)\partial_k^2\xi(x_k)\sigma(x_k')\, dx + \sum_{l\neq k} \int_{\mathbb{R}^N} \big(F_{lk}(x) + F_{kl}(x)\big)\partial_k\xi(x_k)\partial_l\sigma(x_k')\, dx \\ + \sum_{l\neq k}\sum_{m\neq k} \int_{\mathbb{R}^N} F_{lm}\xi(x_k)\partial_l\partial_m\sigma(x_k')\, dx = 0. \tag{3.84}$$

Now we set $\sigma_R(x_k') = \sigma(x_k'/R)$ and simply replace σ in (3.84) by σ_R. Denoting each term in (3.84) with σ_R by I_1, I_2, I_3, we can compute the limit of each term as $R \to \infty$.

First, note that

$$I_1 = \int_{\mathbb{R}} \partial_k^2 \xi(x_k) \left(\int_{\mathbb{R}^{N-1}} F_{kk}(x) \sigma_R(x_k') dx_k' \right) dx_k \\ \longrightarrow \int_{\mathbb{R}} \partial_k^2 \xi(x_k) \left(\int_{\mathbb{R}^{N-1}} F_{kk}(x) dx_k' \right) dx_k \quad (3.85)$$

where we have applied the dominated convergence theorem by using (3.77).

Next, we claim that $I_2 \to 0$ as $R \to \infty$. Indeed, note that

$$I_2 = \sum_{l \neq k} \int_{\mathbb{R}} \partial_k \xi(x_k) \left(\int_{\mathbb{R}^{N-1}} \left(F_{lk}(x) + F_{kl}(x) \right) \partial_l \sigma_R(x_k') dx_k' \right) dx_k.$$

Here the inner integral converges to 0 as $R \to \infty$ because

$$\int_{\mathbb{R}^{N-1}} \left(|F_{lk}(x)| + |F_{kl}(x)| \right) |\partial_l \sigma_R(x_k')| dx_k' \leq \frac{C}{R} \int_{\{R < |x_k'| < 2R\}} \left(|F_{lk}(x)| + |F_{kl}(x)| \right) dx_k'$$

and we can use (3.78). Therefore, we can apply the dominated convergence theorem to prove that I_2 converges to 0 as $R \to \infty$.

Similarly, we can prove that I_3 converges to 0 as R tends to ∞. We can rewrite the integrals of I_3 as follows:

$$I_3 = \sum_{l \neq k} \sum_{m \neq k} \int_{\mathbb{R}} \xi(x_k) \left(\int_{\mathbb{R}^{N-1}} F_{lm}(x) \partial_l \partial_m \sigma_R(x_k') dx_k' \right) dx_k.$$

Again with the help of assumptions (3.77), we can apply the dominated convergence theorem to establish the claim.

Therefore, given all the limits of I_1, I_2, I_3, we can conclude that

$$\int_{\mathbb{R}} \partial_k^2 \xi(x_k) \left(\int_{\mathbb{R}^{N-1}} F_{kk}(x) dx_k' \right) dx_k = 0$$

Note that this is true for all $\xi(x_k) \in C_c^\infty(\mathbb{R})$. Then due to (3.78), it follows that

$$\int_{\mathbb{R}^{N-1}} F_{kk}(x)dx_k' = Ax_k + B \quad \text{for a.e. } x_k \in \mathbb{R}.$$

Finally, by using (3.81), it follows that $A = 0, B = 0$, which completes the proof of Lemma 3.7.2. $\square$

Now we can work on the divergence equation (3.70) (Type 2) by using a similar idea. To the best of my knowledge, this type of equations have not been investigated yet in order to study the integral identities (3.14) on a hyperplane.

Lemma 3.7.3. *Let a tensor field* $\boldsymbol{F} = \{F_{jk}\} \in L^1_{loc}(\mathbb{R}^N)$ *of order 2 (which are not necessarily symmetric) satisfy the weak formulation of the divergence free equation, that is, satisfying* (3.75)*. Fix* $j, k \in \{1, \ldots, N\}$*. Assume that*

$$F_{jk}(x) \in L^1_{loc,x_j} L^1_{x_j'}, \quad \frac{F_{lk}(x)}{1+|x_j'|} \in L^1_{loc,x_j} L^1_{x_j'}, \quad \forall\, l \neq j \tag{3.86}$$

Then there exists a constant $A_{jk} \in \mathbb{R}$ *such that*

$$\int_{\mathbb{R}^{N-1}} F_{jk}(x)dx_j' = A_{jk} \quad \textit{for a.e. } x_j \in \mathbb{R}.$$

Therefore, if we additionally assume either

$$\|F_{jk}(x)\|_{L^1(\mathbb{R}^{N-1};x_j')} \in L^1_{weak}(\mathbb{R}), \ \textit{or}$$

$$\liminf_{|x_j|\to\infty} \|F_{jk}(x)\|_{L^1(\mathbb{R}^{N-1};x_j')} = 0,$$

Then it holds that

$$\int_{\mathbb{R}^{N-1}} F_{jk}(x)dx_j' = 0 \quad \textit{for a.e. } x_j \in \mathbb{R}.$$

Proof of Lemma 3.7.3. We fix $j, k \in \{1, \ldots, N\}$ and let $\xi(x_j), \sigma(x_j')$ be cut-off function defined as in Section 3.1 but now with the variables x_j, x_j'. We use the product $\xi(x_j)\sigma(x_j')$ as a test function in the weak formulation (3.75) of the equation of Type 2 to obtain

$$
\begin{aligned}
-\int_{\mathbb{R}} \partial_j \xi(x_j) \left(\int_{\mathbb{R}^{N-1}} F_{jk}(x)\sigma(x_j') dx_j' \right) dx_j \\
- \sum_{l \neq j} \int_{\mathbb{R}} \xi(x_j) \left(\int_{\mathbb{R}^{N-1}} F_{lk}(x) \partial_l \sigma(x_j') dx_j' \right) dx_j = 0.
\end{aligned}
$$

Replacing $\sigma(x_j')$ by $\sigma_R(x_j')$ and letting $R \to \infty$, we obtain

$$
\int_{\mathbb{R}} \partial_j \xi(x_j) \left(\int_{\mathbb{R}^{N-1}} F_{jk}(x) dx_j' \right) dx_j = 0.
$$

Consequently, it follows that

$$
\int_{\mathbb{R}^{N-1}} F_{jk}(x) dx_j' = 0 \quad \text{for a.e. } x_j \in \mathbb{R}.
$$

This finishes the proof. □

Finally we can work on equation (3.71) of Type 3. If we take the divergence of (3.71), then equation (3.71) turns into the double divergence free equation (3.69). Hence we may apply 3.7.2. Not only that, but also we can apply Lemma 3.7.3 to $\boldsymbol{v}$ because $\boldsymbol{v}$ is divergence free. Moreover, we can apply our main idea of deriving integral identities on a hyperplane to equation (3.71) directly to obtain something that we would not obtain if we only work with the divergence of (3.71).

Lemma 3.7.4. *Let $\boldsymbol{v}(x,t) \in L^1_{loc}(0,T; L^1_{loc}(\mathbb{R}^N))$ be a weakly divergence free vector field and let $\boldsymbol{F} = \{F_{jk}\}$ be a symmetric tensor field of order 2 in $L^1_{loc}(0,T; L^1_{loc}(\mathbb{R}^N))$. Assume these fields $(\boldsymbol{v}, \boldsymbol{F})$ form a weak solution to the equation of Type 3, that is,* *satisfying* (3.76). *Fix* $k \in \{1, \ldots, N\}$. *Assume that*

$$\left.\begin{aligned} F_{kk}(x,t) &\in L^1_{loc,t} L^1_{loc,x_k} L^1_{x'_k} \\ \frac{F_{km}(x,t)}{1+|x'_k|} &\in L^1_{loc,t} L^1_{loc,x_k} L^1_{x'_k}, \quad m \neq k \\ \frac{F_{lm}(x,t)}{(1+|x'_k|)^2} &\in L^1_{loc,x_k} L^1_{x'_k}, \quad a.e.\ t \in (0,T), \quad l \neq k, m \neq k, \\ \liminf_{|x_k| \to \infty} \|F_{kk}(x,t)\|_{L^1(\mathbb{R}^{N-1};x'_k)} &= 0 \quad a.e.\ t \in (0,T) \\ v_k(x,t) &\in L^1_{loc,t} L^1_{loc,x_k} L^1_{x'_k}, \\ \frac{v_l(x,t)}{1+|x'_k|} &\in L^1_{loc,x_k} L^1_{x'_k}, \quad a.e.\ t \in (0,T), \quad l \neq k \end{aligned}\right\} \tag{3.87}$$

Then for a.e. $t \in (0,T)$

$$\int_{\mathbb{R}^{N-1}} F_{kk}(x,t)dx'_k = 0, \quad \int_{\mathbb{R}^{N-1}} v_k(x,t)dx'_k = A_k, \quad a.e.\ x_k \in \mathbb{R} \tag{3.88}$$

for some constant $A_k \in \mathbb{R}$. *Hence if we additionally assume*

$$\liminf_{|x_k| \to \infty} \|v_k(x,t)\|_{L^1(\mathbb{R}^{N-1};x'_k)} = 0, \quad a.e.\ t \in (t_0 - \delta, t_0 + \delta) \tag{3.89}$$

for some $t_0 \in (0,T)$ *and* $\delta > 0$, *then for a.e.* $x_k \in \mathbb{R}, t \in (0,T)$

$$\int_{\mathbb{R}^{N-1}} v_k(x,t)dx'_k = 0. \tag{3.90}$$

Proof of Lemma 3.7.4. If we take the divergence of equation (3.71) of Type 3, then it turns into the double divergence free equation (3.69) . Hence according to Lemma 3.7.2, the first integral identity of (3.88) follows.

As $\boldsymbol{v}$ is divergence free for a.e. $t \in (0,T)$, according to Lemma 3.7.3, there exists a constant $A_k(t) \in \mathbb{R}$, which is, in fact, a function of t, such that for a.e.

$x_k \in \mathbb{R}, t \in (0,T)$

$$\int_{\mathbb{R}^{N-1}} v_k(x,t)dx'_k = A_k(t) \tag{3.91}$$

Now we go back to the weak formulation of equation (3.76) of Type 3. Use $\xi(x_k)\sigma(x'_k)$ as a test function in the weak formulation to obtain

$$\begin{aligned}
&-\int_0^T \int_{\mathbb{R}^N} v_k(x,t)\xi(x_k)\sigma(x'_k)\partial_t\phi(t)dxdt \\
&-\int_0^T \int_{\mathbb{R}^N} F_{kk}(x,t)\partial_k\xi(x_k)\sigma(x'_k)\phi(t)dxdt - \sum_{l\neq k}\int_0^T \int_{\mathbb{R}^N} F_{lk}(x,t)\xi(x_k)\partial_l\sigma(x'_k)\phi(t)dxdt \\
&\qquad = \nu\int_0^T \int_{\mathbb{R}^N} v_k(x,t)\partial_k^2\xi(x_k)\sigma(x'_k)\phi(t)dxdt \\
&\qquad\qquad + \sum_{l\neq k}\nu\int_0^T \int_{\mathbb{R}^N} v_k(x,t)\xi(x_k)\partial_l^2\sigma(x'_k)\phi(t)dxdt
\end{aligned} \tag{3.92}$$

Now we replace $\sigma(x'_k)$ by $\sigma_R(x'_k)$ and let $R \to \infty$. Then by using the integral identity of $\boldsymbol{F}$ and the one of v_k proved above, we get

$$\int_0^T \int_{\mathbb{R}^N} v_k(x,t)\xi(x_k)\partial_t\phi(t)dxdt = 0.$$

This holds for all $\xi \in C_c^\infty(\mathbb{R}), \phi \in C_c^\infty(0,T)$. Hence it implies $A_k(t)$ is actually a constant for a.e. $t \in (0,T)$. The rest of Theorem 3.7.4 follows straightforwardly. $\square$

Now using all these lemmas, we can prove our main theorems, Theorem 3.0.2, 3.3.2, 3.3.3, 3.4.3.

Proof of Theorem 3.0.2. For $k \in \{1,\dots,N\}$ and $\psi \in C_0^\infty(\mathbb{R}^N)$, we put $\partial_{x_k}\psi$ into the weak formulation (3.30) as a test function. And taking the sum of the resulting equations over $k \in \{1,\dots,N\}$, by using the divergence free condition (3.31), we can obtain

$$\sum_{l,m=1}^{N} \int_0^T \left(\int_{\mathbb{R}^N} \left(v_l(x,t) v_m(x,t) + p(x,t)\delta_{lm} \right) \partial_{x_l} \partial_{x_m} \psi(x)\, dx \right) \phi(t)\, dt = 0.$$

This holds for all $\phi \in C_0^\infty(\mathbb{R})$, and thus for a.e. $t \in (0,T)$

$$\sum_{l,m=1}^{N} \int_{\mathbb{R}^N} \left(v_l(x,t) v_m(x,t) + p(x,t)\delta_{lm} \right) \partial_{x_l} \partial_{x_m} \psi(x)\, dx = 0,$$

which is the weak formulation (3.74) of the double divergence free equation with $F_{lm} = v_l v_m + p\delta_{lm}$. Let $t \in (0,T)$ be the time value such that assumptions (3.11), (3.12) are satisfied. Then applying Lemma 3.7.2, we obtain (3.13) for the evolutionary Euler or Navier-Stokes equations.

Now in the case of the stationary Euler equations ($\nu = 0$), the weak formulation (3.30) can be directly re-written as

$$\sum_{l=1}^{N} \int_{\mathbb{R}^N} \left(v_l(x) v_k(x) + p(x)\delta_{lk} \right) \partial_{x_k} \psi(x) dx = 0. \tag{3.93}$$

Hence we can apply now Lemma 3.7.3 to obtain (3.14). This finishes the proof. □

Proof of Theorem 3.3.2. By using the fact that $\boldsymbol{v}$ is divergence free, it is straightforward to see that the evolutionary Euler or Navier-Stokes equation is in the form of equation (3.71) of Type 3 with $F_{lm} = v_l v_m + p\delta_{lm}$. Hence Theorem 3.3.2 directly follows from Lemma 3.7.4. □

Proof of Theorem 3.3.3. For $k \in \{1, \ldots, N\}$ and $\psi \in C_0^\infty(\mathbb{R}^N)$, we can put $\partial_{x_k}\psi(x)$ into the weak formulation (3.34) as a test function and take the sum of the resulting equations over $k \in \{1, \ldots, N\}$. Then by using the weakly divergence free condition of $\boldsymbol{v}$, it leads to

$$\sum_{l,m=1}^{N}\int_0^T\left[\int_{\mathbb{R}^N}\left\{v_lv_m+b_lb_m-\left(p+\frac{1}{2}|\boldsymbol{b}|^2\right)\right\}\partial_{x_l}\partial_{x_m}\psi(x)\,dx\right]\phi(t)\,dt=0.$$

As it holds for all $\phi\in C_0^\infty(\mathbb{R})$,

$$\sum_{l,m=1}^{N}\int_{\mathbb{R}^N}\left\{v_lv_m+b_lb_m-\left(p+\frac{1}{2}|\boldsymbol{b}|^2\delta_{lm}\right)\right\}\partial_{x_l}\partial_{x_m}\psi(x)dx=0.$$

This is in the form of the double divergence equation (3.69) with $F_{lm}=v_lv_m+b_lb_m-(p+\frac{1}{2}|\boldsymbol{b}|^2)\delta_{lm}$. Therefore, applying Lemma 3.7.2, we can obtain (3.45).

Now for the stationary MHD equation with $\mu,\nu=0$, the first equation (3.33) is in the form of a divergence free equation (3.70) with the same tensor field F_{lm} as above. Hence we can apply Lemma 3.7.3 to obtain (3.46).

Similarly we can work with the second equation of (3.33), which can be written in the form of Type 2 with $F_{lm}=v_lb_m-b_lv_m$ in the stationary case. Therefore, we also obtain (3.47). It finishes the proof. □

Remark 3.7.5. In addition to the MHD equations, we can also find divergence structure out of equation (3.4). Take dot product of equation (3.4) with $\partial_{x_k}\boldsymbol{v}$ for a fixed k. After a simple calculation, the resulting equation turns into

$$\sum_{l\neq k}\partial_{x_l}(\partial_{x_l}\boldsymbol{v}\cdot\partial_{x_k}\boldsymbol{v})+\partial_{x_k}\left\{\left(\sum_{m\neq k}\frac{|\partial_{x_m}\boldsymbol{v}|^2}{2}\right)-\frac{|\partial_{x_k}\boldsymbol{v}|}{2}+H(\boldsymbol{v})\right\}=0.$$

This equation can be written in the form of the divergence free equation (3.70) with F_{lk} given by

$$F_{lk}=\begin{cases}\partial_{x_l}\boldsymbol{v}\cdot\partial_{x_k}\boldsymbol{v} & \text{if } l\neq k\\ \left(\sum_{m\neq k}\frac{|\partial_{x_m}\boldsymbol{v}|^2}{2}\right)-\frac{|\partial_{x_k}\boldsymbol{v}|}{2}+H(\boldsymbol{v}) & \text{if } l=k\end{cases}$$

Hence it explains why integral identities on a hyperplane could be found in two totally different contexts, the stationary Euler equations (3.1) and the vector-valued Allen-Cahn equations (3.4). See (3.6). This is because both equations can be re-written in the form of the divergence free equation (3.70), which enjoy integral identities on a hyperplane. Rewriting equation (3.4) in the form of the divergence free equation (3.70) is implicitly used in Gui [28] to prove integral identities on a hyperplane for equation (3.4).

Now for the rest of this section, we consider domains with boundary.

Lemma 3.7.6. *Let $\Omega = \Omega' \times I$ where I is a nonempty (possibly unbounded) open interval in $\mathbb{R}$ and Ω' is a bounded locally Lipschitz domain in $\mathbb{R}^{N-1}, N \geq 2$. Let $(\boldsymbol{v}, p) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a solution to the stationary Euler equations in Ω. Then for all $j \in \{1, \cdots, N\}$,*

$$\frac{d}{dx_N}\int_{\Omega'}(v_j(x)v_N(x) + \delta_{Nj}p(x))dx'_N + \int_{\partial\Omega'} \boldsymbol{n} \cdot (v_j(x)\boldsymbol{v}(x) + p(x)\boldsymbol{e}_j)\, d\sigma(x'_N) = 0 \quad (3.94)$$

for all $x_N \in I$ where $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega$.

Proof of Lemma 3.7.6. Fix $j \in \{1, \cdots, N\}$. We can re-write j-th component of the stationary Euler equations as

$$\nabla \cdot (v_j\boldsymbol{v} + p\boldsymbol{e}_j) = 0$$

by using the divergence free condition of $\boldsymbol{v}$. Separating the derivative with respect to x_N from the others, it follows that

$$\partial_{x_N}(v_jv_N + \delta_{Nj}p) + \sum_{l \neq N} \partial_{x_l}(v_jv_l + p\delta_{jl}) = 0.$$

Integrating this equation in Ω', we can obtain that for all $x_N \in I$

$$\frac{d}{dx_N}\int_{\Omega'}(v_j v_N + \delta_{Nj} p) dx'_N + \int_{\Omega'} \sum_{l \neq N} \partial_{x_l}(v_j v'_l + p\delta_{jl})\, dx'_N = 0$$

Now we can apply the divergence theorem to the second integral above. As $n_N = 0$ on $\partial\Omega'$ where $\boldsymbol{n} = (n_1, \ldots, n_N)$, we can obtain the conclusion (3.94). $\square$

We can apply this Lemma 3.7.6 with a boundary condition.

Lemma 3.7.7. *Let $\Omega = \Omega' \times \mathbb{R}$ where Ω' is a bounded locally Lipschitz domain in $\mathbb{R}^{N-1}, N \geq 2$. Let $(\boldsymbol{v}, p) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a classical solution to the stationary Euler equations in Ω. Assume that*

$$(\boldsymbol{v} \cdot \boldsymbol{n}) v_N|_{\partial\Omega' \times \mathbb{R}} = 0 \tag{3.95}$$

where $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega$. Then there exists a constant c_N

$$\int_{\Omega'} (v_N^2 + p) dx'_N = c_N \quad \textit{for all } x_n \in \mathbb{R}.$$

Furthermore, if one additionally assumes

$$\liminf_{|x_N| \to \infty} \int_{\Omega'} (v_N^2 + p)\, dx'_N = 0 \tag{3.96}$$

$$\int_{\Omega'} p\, dx'_N \geq 0 \qquad \textit{for all } x_N \in \mathbb{R}, \tag{3.97}$$

then $v_N \equiv 0$ in Ω.

Proof of Lemma 3.7.7. Applying Lemma 3.7.6 with $j = N$, we can obtain (3.94) with $j = N$. From the boundary assumptions (3.95), the boundary integral in (3.94) with

$j = N$ vanishes. Therefore, we can get

$$\frac{d}{dx_N} \int_{\Omega'} (v_N^2 + p)\, dx'_N = 0 \quad \text{for all } x_N \in \mathbb{R}.$$

Therefore, for some constant $c_N \in \mathbb{R}$

$$\int_{\Omega'} (v_N^2 + p)\, dx'_N = c_N \quad \text{for all } x_N \in \mathbb{R}.$$

Now from assumption (3.96), it holds that $c_N = 0$. The last conclusion of Theorem 3.7.7 follows directly from (3.97). □

We can also apply Lemma 3.7.6 to the half space, $\mathbb{R}_+^{N-1} \times \mathbb{R}$.

Lemma 3.7.8. *Let $(\boldsymbol{v}, p) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a classical solution to the stationary Euler equations in $\Omega = \mathbb{R}_+^{N-1} \times \mathbb{R}$. On the boundary $(\partial\mathbb{R}_+^{N-1}) \times \mathbb{R}$ assume*

$$(\boldsymbol{v} \cdot \boldsymbol{n}) v_N|_{(\partial\mathbb{R}_+^{N-1}) \times \mathbb{R}} = 0. \tag{3.98}$$

where $\boldsymbol{n}$ is the unit outward normal vector on $(\partial\mathbb{R}_+^{N-1}) \times \mathbb{R}$. Suppose

$$v_N^2(x) + |p(x)| \in L^1(\mathbb{R}_+^{N-1}; x'_N) \quad \text{for all } x_N \in \mathbb{R}, \text{ and} \tag{3.99}$$

$$\sum_{l=1}^{N-1} \int_0^s \int_{\mathbb{R}_+^{N-1}} \frac{|v_N(x) v_l(x)|}{1 + |x'_N|}\, dx'_N dx_N < \infty \quad \text{for every } s \in \mathbb{R}. \tag{3.100}$$

Then there exists a constant $c_N \in \mathbb{R}$ such that

$$\int_{\mathbb{R}_+^{N-1}} (v_N^2 + p)\, dx'_N = c_N \quad \text{for all } x_N \in \mathbb{R}.$$

Furthermore, if one additionally assumes that

$$\liminf_{|x_N|\to\infty} \int_{\mathbb{R}_+^{N-1}} (v_N^2 + |p|)\, dx_N' = 0, \ \ \text{and}$$

$$\int_{\mathbb{R}_+^{N-1}} p\, dx_N' \geq 0 \qquad \text{for all } x_N \in \mathbb{R},$$

then $v_N \equiv 0$ *in* $\mathbb{R}^N$.

Proof of Lemma 3.7.8. Let B_R' denote the ball of radius R centered at the origin in $\mathbb{R}_+^{N-1}$. Apply Lemma 3.7.6 with $\Omega' = B_R' \cap \mathbb{R}_+^{N-1}, j = N$, we can obtain that for all $x_N \in \mathbb{R}$

$$\frac{d}{dx_N} \int_{B_R' \cap \mathbb{R}_+^{N-1}} (v_N^2(x) + p(x)) dx_N' \\ + \sum_{l=1}^{N-1} \int_{\partial(B_R' \cap \mathbb{R}_+^{N-1})} n_l(x_N') v_N(x) v_l(x) d\sigma(x_N') = 0$$

where $\boldsymbol{n} = (n_1, \dots, n_{N-1})$ is the unit outward normal vector on $\partial(B_R' \cap \mathbb{R}_+^{N-1})$. Note the boundary integral on $\partial(B_R' \cap \mathbb{R}_+^{N-1})$ above can be written as the sum of the one on $(\partial B_R') \cap \mathbb{R}_+^{N-1}$ and the one on $B_R' \cap \partial\mathbb{R}_+^{N-1}$. The latter one vanishes due to the boundary condition (3.98). Therefore, it follows that

$$\frac{d}{dx_N} \int_{B_R' \cap \mathbb{R}_+^{N-1}} (v_N^2 + p) dx_N' + \sum_{l=1}^{N-1} \int_{(\partial B_R') \cap \mathbb{R}_+^{N-1}} n_l(x_N') v_N(x) v_l(x) d\sigma(x_N') = 0$$

For $s \in \mathbb{R}$, integrating the equation above with respect to x_N from $x_N = 0$ to $x_N = s$,

$$\int_{B_R' \cap \mathbb{R}_+^{N-1}} (v_N^2 + p)|_{x_N=0}^{x_N=s} dx_N' + \sum_{l=1}^{N-1} \int_0^s \int_{(\partial B_R') \cap \mathbb{R}_+^{N-1}} n_l(x_N') v_N(x) v_l(x) d\sigma(x_N')\, dx_N$$

We claim that there exists a sequence R_k that converges to infinity such that the second integral over $(\partial B_{R_k}') \cap \mathbb{R}_+^{N-1}$ converges to zero as $k \to \infty$. Fixing $s \in \mathbb{R}$, we can re-write the following integral

$$\sum_{l\neq N}\int_0^s\int_{\mathbb{R}_+^{N-1}}\frac{|v_N(x)v_l(x)|}{1+|x'_N|}dx'_N dx_N$$
$$=\sum_{l\neq N}\int_0^\infty\left(\int_0^s\int_{\partial B'_r\cap\mathbb{R}_+^{N-1}}\frac{|v_N(x)v_l(x)|}{1+|x'_N|}d\sigma(x'_N)dx_N\right)dr$$

And this whole integral is finite due to assumption (3.100). Therefore, the claim follows.

Hence it holds that there exists a constant $c_N\in\mathbb{R}$ such that

$$\int_{\mathbb{R}_+^{N-1}}(v_N^2+p)dx'_N=c_N\quad\text{for all } x_N\in\mathbb{R}.$$

And the very last part of Lemma 3.7.8 directly follows from this conclusion. □

This proof is motivated by the proof of Theorem 1.2 of Changfeng Gui [28].

Now, we impose periodic boundary conditions.

Lemma 3.7.9. *Let $(\boldsymbol{v},p)\in C^1(\mathbb{R}^N)\times C^1(\mathbb{R}^N)$ be a classical solution to the stationary Euler equations in $\mathbb{R}^N$. Assume that there exist constant $a_j>0, j=1,\ldots,N-1$ such that $\boldsymbol{v}(x+a_j\boldsymbol{e}_j)=\boldsymbol{v}(x)$ for all $x\in\mathbb{R}^N$. Then there exists a constant c_0 such that*

$$\int_{\Omega'}(v_N^2(x)+p(x))dx'_N=c_0\quad\textit{for all } x_N\in\mathbb{R}.\tag{3.101}$$

where $\Omega'=(0,a_1)\times\cdots\times(0,a_{N-1})$. Therefore, if we additionally assume either

$$\liminf_{|x_N|\to\infty}\int_{\Omega'}(v_N^2(x)+p(x))\,dx'_N=0\ \textit{and}\ \int_{\Omega'}p\,dx'_N\geq 0\quad\textit{for all } x_N\in\mathbb{R},$$

or

$$\int_{\Omega'} p\, dx'_N \geq c_0 \quad \textit{for all } x_N \in \mathbb{R},$$

then $v_N \equiv 0$.

Proof of Lemma 3.7.9. Applying Lemma 3.7.6, due to the periodic boundary condition, it follows that

$$\frac{d}{dx_N} \int_{\Omega'} (v_N^2(x) + p(x)) dx'_N = 0, \quad \text{for all } x_N \in \mathbb{R},$$

which proves (3.101). The rest of Lemma (3.7.9) is straightforward. □

Finally we can prove Theorem 3.4.9.

Proof of Theorem 3.4.9. Applying Lemma 3.7.3, it follows that

$$\int_{\mathbb{R}} v_1(x) v_2(x)\, dx_2 = 0 \quad \text{for all } x_2 \in \mathbb{R}.$$

Hence due to (3.55), it follows that $v_1(x)v_2(x) = 0$ for all $x \in \mathbb{R}^2$. Therefore, the sets S_1, S_2 given by

$$S_1 = \{x \in \mathbb{R}^2 : v_1(x) = 0\}, \quad S_2 = \{x \in \mathbb{R}^2 : v_2(x) = 0\}$$

satisfy $S_1 \cup S_2 = \mathbb{R}^2$. And the two sets S_1, S_2 are disjoint because of (3.56). As $\boldsymbol{v}$ is assumed to be continuous, the two sets S_1, S_2 are closed.

We claim S_1 is open. If it is not, then there exists a sequence $\{X_j\}_{j=1}^{\infty}$ in S_2 and $X_0 \in S_1$ such that $X_j \to X_0$ as $j \to \infty$. However, $v_2(X_j) = 0$ for all j and v_2 is continuous, and thus $v_2(X_0) = 0$. So $X_0 \in S_2$, which contradicts to the fact that S_1 and S_2 are disjoint. Therefore, S_1 is open.

As the set S_1 is open and closed, it is either empty or the entire space $\mathbb{R}^2$. We claim S_1 is empty. If it is not, then $S_1 = \mathbb{R}^2$, and by the divergence free condition of $\boldsymbol{v}$, it holds that $v_2 = v_2(x_1)$. Then by the assumption (3.54), $v_2 \equiv 0$, which implies $\boldsymbol{v} \equiv 0$, a contradiction to (3.56) again. Therefore, S_1 is empty.

This implies $S_2 = \mathbb{R}^2$. By the divergence free condition of $\boldsymbol{v}$ again, it follows that $v_1 = v_1(x_2)$, which completes the proof. □

3.8 Examples

This section is devoted to the study of some simple examples. There are various basic example flows that enjoy the integral identities on a hyperplane (or on a section of a domain). These flows are not shear flows and they do not satisfy the condition involving sign of an integral of pressure from our main theorems. All the examples here are well-known, but not in this context to the best of my knowledge.

Example 3.8.1. In $\mathbb{R}^2$, define

$$\begin{aligned}
\boldsymbol{v}(x) &= (\,\sin(2\pi x_1)\sin(2\pi x_2),\ \cos(2\pi x_1)\cos(2\pi x_2)\,)\,,\\
p(x) &= \frac{1}{4}\{\cos(4\pi x_1) - \cos(4\pi x_2)\} - \frac{1}{4}.
\end{aligned}$$

This is a solution to the Euler equations. The vector field $\boldsymbol{v}$ and pressure field p are periodic in terms of both x_1 and x_2. A simple computation yields

$$\begin{aligned}
&\int_0^1 v_1^2 + p\,dx_1 = 0, && \int_0^1 v_2^2 + p\,dx_2 = 0\\
&\int_0^1 p\,dx_1 = -\frac{1}{4}\cos(4\pi x_2) - \frac{1}{4} && \int_0^1 p\,dx_2 = \frac{1}{4}\cos(4\pi x_1) - \frac{1}{4}\\
&\int_0^1 v_1 v_2\,dx_1 = 0, && \int_0^1 v_1 v_2\,dx_2 = 0.
\end{aligned}$$

This solution satisfies all the assumptions of Theorem 3.4.4 except (3.49) and is not a shear flow. Therefore, if we simply remove assumption (3.49) in Theorem 3.4.4, then the conclusion of the theorem does not hold any more.

In addition, this solution in $(0,1)\times\mathbb{R}$ satisfies the standard boundary condition whereas this solution in $(-1/4,1/4)\times\mathbb{R}$ satisfies the non-standard boundary condition, $v_2|_{\partial\Omega}=0$, of Theorem 3.0.10.

Example 3.8.2. Define

$$\boldsymbol{v}(x) = \left(\sin(\pi x_1)e^{x_2}, -\pi\cos(\pi x_1)e^{x_2}\right), \qquad p(x) = -\frac{\pi^2}{2}e^{2x_2}.$$

This example is a solution to the stationary Euler equations in $\mathbb{R}^2$. But if we restrict our attention to $(0,1)\times\mathbb{R}$, then it satisfies the standard boundary condition on the boundary $\partial\left((0,1)\times\mathbb{R}\right)$, that is, $v_1(0,x_2)=0, v_1(1,x_2)=0$ for all $x_2\in\mathbb{R}$. Moreover, $\boldsymbol{v}, p$ are periodic in terms of x_1; $\boldsymbol{v}(x+\boldsymbol{e}_1)=\boldsymbol{v}(x), p(x+\boldsymbol{e}_1)=p(x)$ for all $x\in\mathbb{R}^2$. A simple calculation yields

$$\int_0^1 v_2^2 + p\,dx_1 = 0, \qquad \int_0^1 p\,dx_1 = -\frac{\pi^2}{2}e^{2x_2}, \qquad \int_0^1 v_1v_2\,dx_1 = 0.$$

In this case, this example satisfies all the assumptions of Lemma 3.7.7 ($N=2$) except (3.97) and is not a shear flow. Hence if we remove (3.97) in Lemma 3.7.7, then the conclusion of Lemma 3.7.7 does not hold.

Now if we consider this solution in $(-1/2,1/2)\times\mathbb{R}$, then the vector field $\boldsymbol{v}$ has only normal component on the boundary $\partial((-1/2,1/2)\times\mathbb{R})$, that is, $v_2(-1/2,x_2)=0, v_2(1/2,x_2)=0$ for all $x_2\in\mathbb{R}$. And again a simple calculation yields

$$\int_{-1/2}^{1/2} v_2^2 + p\,dx_1 = 0, \qquad \int_{-1/2}^{1/2} p\,dx_1 = -\frac{\pi^2}{2}e^{2x_2}, \qquad \int_{-1/2}^{1/2} v_1v_2\,dx_1 = 0.$$

Hence in the second case, this example satisfies all the assumptions of Theorem 3.0.10 except (3.24) and is not a shear flow. Therefore, it also shows that we cannot simply remove (3.24) in Theorem 3.0.10.

Example 3.8.3. Define $\Omega = B_1 \times \mathbb{R}$ where $B_1 \subset \mathbb{R}^2$ is the unit ball centered at $(0,0)$, and define

$$\boldsymbol{v}(x) = (-x_2, x_1, 1), \qquad p(x) = \frac{1}{2}(x_1^2 + x_2^2) - \frac{9}{4}.$$

This example is a solution to the stationary Euler equations in Ω. The standard boundary condition holds on the boundary $\partial\Omega$;

$$\boldsymbol{n} \cdot \boldsymbol{v} = 0 \qquad \text{on } \partial B_1 \times \mathbb{R}.$$

(This example comes from Hamel and Nadirashvili [30].) A simple calculation yields

$$\int_{B_1} v_3^2 + p \, dx_{1,2} = 0, \qquad \int_{B_1} p \, dx_{1,2} = -2\pi.$$

Therefore, this example satisfies all the assumptions of Lemma 3.7.7 except (3.97), which again shows that we cannot simply remove (3.97) from Lemma 3.7.7.

Example 3.8.4. Let $\varphi \in C_0^\infty[0,\infty)$, and in $\mathbb{R}^2$, define

$$v_1(x) = -\varphi(|x|)x_2, \quad v_2(x) = \varphi(|x|)x_1,$$

and

$$p(x) = -\int_{|x|}^\infty s\varphi^2(s)ds.$$

This vector field $\boldsymbol{v} = (v_1, v_2)$ is a solution to the stationary Euler equations in $\mathbb{R}^2$ with pressure field p. Moreover, this example $(\boldsymbol{v}, p)$ has compact support in $\mathbb{R}^2$. (This example is very likely well-known. It comes from [48].) Note

$$\int_{\mathbb{R}} p\, dx_2 = -\int_{\mathbb{R}}\int_{|x|}^{\infty} s\varphi^2(s)\, ds dx_2$$

By changing the order of the integrations, the above integral equals to

$$-\int_{|x_1|}^{\infty}\int_{-\sqrt{s^2-x_1^2}}^{\sqrt{s^2-x_1^2}} s\varphi^2(s)\, dx_2 ds$$

A simple calculation show that the above integral equals to

$$\begin{aligned}
-\int_{|x_1|}^{\infty} 2s\varphi^2(s)\sqrt{s^2-x_1^2}\, ds &= -\int_0^{\infty} 2\varphi^2(|x|)x_2^2 dx_2 \\
&= -\int_{\mathbb{R}} \varphi^2(|x|)x_2^2 dx_2 \\
&= -\int_{\mathbb{R}} v_1^2\, dx_2.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} (v_1^2 + p)\, dx_2 = 0 \qquad \text{for all } x_1 \in \mathbb{R}.$$

Similarly,

$$\int_{\mathbb{R}} (v_2^2 + p)\, dx_1 = 0 \qquad \text{for all } x_2 \in \mathbb{R}.$$

Here the right hand side being zero is even expected from the beginning due to the fact that this solution has compact support. This example satisfies all the assumptions of Theorem 3.4.3 except the positivity condition of the integral of p with respect to one variable and is not a shear flow.

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