CONCEPTS AND EXAMPLES IN RIEMANNIAN GEOMETRY

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THESIS ABSTRACT

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In this paper, we explore various notations and definitions, in Riemannian Geometry, which were created during the past century. Next, we developed a few clear examples from the book *RIEMANNIAN GEOMETRY* written by Do Carmo and explain the article *A THEOREM ON THE AFFINE TRANSFORMATION GROUP OF A RIEMANNIAN MANIFOLD* written by S.Kobayashi in detail.
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List of symbols

\( \mathbb{R} \) the set of real number.

\( \mathbb{C} \) the set of complex number.

\( I \) the interval \([a, b]\), where \( a \leq b \). In most cases, we set \( I = [0,1] \).

\( \mathbb{R}^n \) the n-dimensional Euclidean space.

\( M^n \) differentiable manifold of dimension \( n \).

\( (M^n, g) \) Riemannian manifold \( M^n \) with the metric \( g \).

\( \mathcal{X}(M) \) the set of all vector fields of class \( C^\infty \) on \( M \).

\( \mathcal{D}(M) \) the ring of real-valued functions of class \( C^\infty \) on \( M \).

\( I(M) \) the group of all isometric transformations.

\( A(M) \) the group of all affine transformations.

\( \nabla \) the riemannian connection or Levi-Civita connection.

\( \Gamma^\alpha_{ij} \) the Christoffel symbols.

\( \frac{D}{dt} \) the orthogonal vector called covariant derivative.

\( Hol_x(\nabla) \) the holonomy group of Riemannian connection \( \nabla \) at point \( x \) on manifold \( M \).

\( \langle \cdot, \cdot \rangle \) the inner product on manifold.
CHAPTER 1
BACKGROUND

1.1 The Background of Riemannian Geometry

Geometry is one of the major subjects in mathematics that has broad applications in our daily lives, such as designing and creating tools we may use every day. Shape and size are two core parameters that describe the things we need. Riemannian Geometry was developed by Bernhard Riemann in his lecture *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen* [1] at the end of the 19th century.

The main topic of this paper is to study Riemannian Geometry, so some fundamental concepts and theorems related to algebra or real analysis are used without proof. In particular, we assume the basic knowledge of Calculus, Abstract Algebra, and Linear Algebra. Meanwhile, most concepts in Geometry will be well defined in this paper. To keep the consistent notation, most notations are taken from Manfredo do Carmo [2], who wrote several famous textbooks about Riemannian geometry.

In Chapter 3 and chapter 4, some typical examples will be described and their property being computed in detail. I want to find the general solutions to geodesic system equations by discussing the transformation groups instead of solving them. Such a method can be extended and applied to manifolds of higher dimensions and those with special structures. It is different from the traditional way of computing the first fundamental form from differential geometry, which depends on the line element.

1.2 History of Isometric Transformation in Riemannian Geometry

An isometric transformation is a transformation between two objects that preserves the distance between points. The research of such a concept in Euclidean space originated a
long time ago. One well-known theorem is the classification of isometries in Euclidean space of three dimensions [3] given by Leonhard Euler. In Riemannian Geometry, many mathematicians are interested in the relation between isometric transformations and affine transformations (See definition in Chapter 5) in different metric spaces.

Yano proved in [4] that, If M is compact manifold, then the connected component of the unit of the set of all affine transformations which forms the group $A(M)$ is contained in the set of all isometric transformations $I(M)$.

Kobayashi proved in [5] that, If M is an irreducible and complete Riemannian manifold, then $A(M)$ is equal to $I(M)$, except the case M is the 1-dimensional Euclidean space.

Transformation Groups have been studied by many mathematicians. For instance, Rita Saerens and William R. Zame proved in [6] that, every compact Lie group could be realized as the (full) group of isometries of a compact Riemannian manifold.

These results help us understand the structure of the transformation groups of a Riemannian manifold. In chapter 5, we want to talk about Kobayashi’s theorem on those transformations.
CHAPTER 2
PRELIMINARIES AND NOTATION

2.1 Preliminaries

Definition 1. A differentiable manifold of dimension n is a set M and a family of injective mappings called charts, \( x_\alpha : U_\alpha \subset \mathbb{R}^n \rightarrow M \) where the set \( U_\alpha \) is open in \( \mathbb{R}^n \), where \( \alpha \in I \), such that:

1) \( \bigcup_\alpha x_\alpha(U_\alpha) = M \).
2) For any pair \( \alpha, \beta \in I \), with \( x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset \), the sets \( x_\alpha^{-1}(W) \) and \( x_\beta^{-1}(W) \) are open sets in \( \mathbb{R}^n \) and the mapping \( x_\beta^{-1} \cdot x_\alpha \) is diffeomorphism.
3) The family \( \{(U_\alpha, x_\alpha)\} \) is maximal relative to the conditions (1) and (2), which is called atlas.

Remarks: We can cover a manifold by different combinations of charts if possible. To ensure the uniqueness of atlas, we keep the maximal.

Definition 2. Let M be a differentiable manifold. A differentiable function \( c : I \rightarrow M \) be a curve in M. D be the set of functions on M that are differentiable at \( p = c(t_0) \). Then the tangent vector to the curve c at \( t = t_0 \) is a function \( \frac{dc(t_0)}{dt} \) : \( D \rightarrow \mathbb{R} \) given by:

\[
\frac{dc(t_0)}{dt} f = \left. \frac{d(f \circ c)}{dt} \right|_{t=t_0}
\]

where \( f \in D \). A tangent vector at \( p \) is the tangent vector at \( t = t_0 \) of some curve \( c \) with \( c(t_0) = 0 \). The set of all tangent vectors to M at \( p \) is the tangent space at \( p \), indicated by \( T_pM \).

Definition 3. A vector field \( X : M \rightarrow TM \) on a differentiable manifold M is a correspondence that associates to each point \( p \in M \) a vector \( X(p) \in T_pM \). In terms of
mappings, $X$ is a mapping of $M$ into the tangent bundle $TM$.

**Lemma 2.1. Jacobi Identity.** Let $[X, Y]$ be the Lie bracket of any vector fields $X, Y$ on a differentiable manifold. If $X, Y, Z \in \mathcal{X}(M)$, then

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$$

(2.1)

**Definition 4.** A Riemannian metric on a differentiable manifold $M$ is a correspondence which associates to each point $p$ of $M$ an inner product $\langle \cdot, \cdot \rangle_p$ (that is, a symmetric, bilinear, positive-definite form) on the tangent space $T_pM$, which varies differentiably in the following sense: If $x : U \in \mathbb{R}^n \to M$ is a chart around $p$, with $x(x_1, x_2, ..., x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = dx_q(0, ..., 1, ..., 0)$, then $\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \rangle_q = g_{ij}(x_1, ..., x_n)$ is a differentiable function on $U$.

**Definition 5.** An affine connection $\nabla$ on a differentiable manifold $M$ is a mapping $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ which is denoted by $(X, Y) \to \nabla_X Y$ and which satisfies the following properties:

1. $\nabla_{fX + gY} Z = f \nabla_X Z + g \nabla_Y Z$,
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$,
3. $\nabla_X (fY) = f \nabla_X (Y) + X(f)Y$,

for all $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in D(M)$. Here $\mathcal{X}(M)$ is the set of all vector fields of class $C^\infty$ on $M$. $D(M)$ is the ring of real-valued functions of class $C^\infty$ on $M$.

**Theorem 2.2.** Given a Riemannian manifold $M$, there exists a unique affine connection $\nabla$ (called Levi-Civita connection or Riemannian Connection) on $M$ satisfying the following conditions:

a) $\nabla$ is compatible with the metric, that is,

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$
for $X, Y, Z \in \mathcal{X}(M)$.

b) $\nabla$ is symmetric (torsion-free), for any vector fields $X, Y$, $[X, Y] = \nabla_X Y - \nabla_Y X$, where $[X, Y]$ is the Lie bracket.

Proof. It is easy to see the existence of such connection. To prove the uniqueness, we want to show that the Levi-civita connection satisfying the condition can be determined by the metric and Lie bracket. Take any three vector, $X, Y, Z$, by compatibility, then we have

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

$$Y\langle X, X \rangle = \langle \nabla_Y Z, X \rangle + \langle Z, \nabla_Y X \rangle$$

$$Z\langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$$

Thus we get

$$X\langle Y, Z \rangle + Y\langle X, X \rangle - Z\langle X, Y \rangle = \langle Z, \nabla_Y X \rangle + \langle Z, \nabla_X Y \rangle$$

$$+ \langle Y, \nabla_X Z \rangle - \langle \nabla_Z Y, X \rangle + \langle \nabla_Y Z, X \rangle - \langle X, \nabla_Z Y \rangle$$

Since $\nabla$ is torsion-free,

$$X\langle Y, Z \rangle + Y\langle X, X \rangle - Z\langle X, Y \rangle = \langle Z, \nabla_Y X \rangle - \langle \{X, Y\}, Z \rangle$$

$$+ \langle \{X, Z\}, Y \rangle + \langle \{Y, Z\}, X \rangle$$

And rearranging the formula, it becomes

$$2\langle Z, \nabla_Y X \rangle = \langle \{X, Z\}, Y \rangle + \langle \{Y, Z\}, X \rangle - \langle \{X, Y\}, Z \rangle$$

$$- X\langle Y, Z \rangle - Y\langle X, X \rangle + Z\langle X, Y \rangle,$$

which means that $\langle Z, \nabla_Y X \rangle$ only depends on metric and Lie bracket, thus shows the existence and uniqueness. □
Remark: For the rest of this paper, the connection $\nabla$ we refer to is the Levi-Civita connection defined above.

**Definition 6.** Let $M$ and $\nabla$ defined as above. For any curve $c : I \to M$, any $t_0 \in I$ and a vector $V_0 \in T_{c(t_0)}M$. There exists a unique parallel vector field $V$ (satisfying $\frac{DV}{dt} = 0$) along $c$, such that $V(t_0) = V_0$. Then $V(t)$ is called the parallel transport of $V(t_0)$ along $c$.

**Definition 7.** Let $M$ and $N$ be Riemannian manifolds, with $g_1$ and $g_2$ as their metric and $\nabla X_1$ and $\tilde{\nabla} X_2$ their respective Levi-Civita connection. For any vector field $V$ on $M$, Let $\phi$ be a differentiable homeomorphism of $M$ onto $N$. If $\phi$ commute with the connection,

$$ d\phi(\nabla_X V) = \tilde{\nabla}_{\tilde{X}}(d\phi V)\phi(p) $$

where $X \in TM$, $\tilde{X} \in TN$, $d\phi : T_pM \to T_{\phi(p)}N$ is the differential, then $\phi$ is called an affine transformation.

**Definition 8.** The curvature $R$ of a Riemannian manifold $M$ is a correspondence that associates to every pair $X,Y \in \mathcal{X}(M)$ a mapping $R(X,Y) : \mathcal{X}(M) \to \mathcal{X}(M)$: given by

$$ R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, $$

$Z \in \mathcal{X}(M)$, where $\nabla$ is connection.

**Definition 9.** Isometric Transformation. Let $M$ and $N$ be Riemannian manifolds. A diffeomorphism $f : M \to N$ (that is, $f$ is a differentiable bijection with a differentiable inverse) is called an isometric transformation if:

$$ \langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}, $$

for all $p \in M$, $u, v \in T_pM$. 
Proposition 2.3. Let $M$ be manifold with $\nabla$. There exists unique vector field $\frac{DV}{dt}$ called covariant derivative of $V$ associates to $V$ along the curve $c : I \rightarrow M$ which satisfying:

a) $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$, where $W$ is a vector field along $c$ and $f$ is a differentiable on $I$.

b) $\frac{D(fV)}{dt} = df dt V + f \frac{dV}{dt}$.

c) If $V$ is induced by a vector field $Y \in \mathcal{X}(M)$, that is, $V(t) = Y(c(t))$, then $\frac{DV}{dt} = \nabla \frac{dc}{dt} Y$.

Proof. We need to prove that $\frac{DV}{dt}$ is such unique correspondence to $\nabla \frac{dc}{dt} V$.

For $\nabla_X Y$, where $X, Y \in \mathcal{X}(M)$ at $p$, choose the system of coordinates $(x_1, ..., x_n)$ such that $X = \sum_i a_i X_i, Y = \sum_j b_j X_j$, where $X_i = \frac{\partial}{\partial x_i}, a_i, b_i$ some vector.

We have

$$\nabla_X Y = \sum_i a_i \nabla_X \left( \sum_j y_j X_j \right) = \sum_i a_i b_i \nabla_X X_j + \sum_i a_i X_i (y_j) X_j$$

and

$$\frac{DV}{dt} = \sum_j \left( \frac{dv^j}{dt} Y_j + \sum_i v^j \frac{DX^j}{dt} \right)$$

We know from the definition of connection that,

$$\frac{DY_j}{dt} = \nabla \frac{dc}{dt} Y = \nabla \sum \frac{db_i}{dt} Y_j = \sum_i \frac{db_i}{dt} \nabla Y_j, Y_j$$

Therefore

$$\frac{DV}{dt} = \sum_j \frac{dv^j}{dt} Y_j + \sum_i \frac{db_i}{dt} v^j \nabla Y_j, Y_j = \nabla \frac{dc}{dt} Y$$

Definition 10. A parametrized curve $\gamma : I \rightarrow M$ is a geodesic at $t_0 \in I$ if $\frac{D\gamma}{dt} = 0$ at the point $t_0$. If $\gamma$ is a geodesic at $t$, for all $t \in I$, we say that $\gamma$ is a geodesic.

Definition 11. Let $M$ be manifold, $p$ be a point of $M, v \in T_p M$ be tangent vector. Assume the existence of geodesics. There exists a unique geodesic $\gamma_v$ satisfying $\gamma_v(0) = p$ with $\gamma'_v(0) = v$. Then the exponential map $\exp_p : B_\epsilon(0) \in T_p M \rightarrow M$ can be defined by
\( \exp_p(v) = \gamma_0(1) \), here \( B_\epsilon(0) \) is the open ball centered at 0 with Euclidean radius \( \epsilon \).

Remark: The uniqueness of such local geodesic can be claimed from the uniqueness of solution to ODE with given initial condition since manifold is locally \( \mathbb{R}^n \).

**Definition 12.** Let \( M \) be a Riemannian manifold, a point \( p \), and \( U, V \in T_pM \) two linearly independent tangent vectors. Then the sectional curvature

\[
K(U, V) = \frac{\langle R(U, V)U, V \rangle}{\langle U, U \rangle \langle V, V \rangle - \langle U, V \rangle^2}
\]

where \( R \) is the curvature.

**Definition 13.** A vector field \( J \) along a geodesic \( \gamma \) is said to be a Jacobi field if it satisfies the Jacobi equation.

\[
\frac{D^2 J}{dt^2} + R(\frac{d\gamma}{dt}, J(t)) \frac{d\gamma}{dt} = 0
\]

We denote \( \frac{d\gamma}{dt} = \gamma'(t) \).
CHAPTER 3
THEOREM ABOUT ISOMETRIC TRANSFORMATION AND GEODESIC

3.1 Gauss Lemma

Lemma 3.1. Exponential map is Jacobi field, that is, if we defined $J(t)$ as follows

$$J(t) = (d\exp_p)_\frac{\gamma(t)}{t} (\frac{dJ}{dt})_{t=0}$$

then $J(t)$ is a Jacobi field.

Proof. It can be derived immediately from the definition of curvature

$$R\left(\frac{d\gamma}{dt}, J(t)\right)\frac{d\gamma}{dt} = \nabla J \nabla \frac{d\gamma}{dt} - \nabla \frac{d\gamma}{dt} \nabla J + \nabla \frac{d\gamma}{dt} \frac{d\gamma}{dt} = -\frac{D^2 J}{dt^2},$$

\[\square\]

Lemma 3.2. Gauss Lemma. Let $p \in M$ and let $v \in T_p M$ such that $exp_p v$ is defined. Let $w \in T_p M \approx T_v(T_p M)$. Then

$$\langle (d\exp_p)_v (v), (d\exp_p)_v (w) \rangle = \langle v, w \rangle,$$

Proof. Here we want to proof gauss lemma by using Jacobi field

Considering the geodesic $\gamma(t) = exp_p tv$, we have $\gamma(0) = p$ and $\frac{d\gamma}{dt} = \gamma'(0) = v$. Let $J(t) = d(exp_{\gamma(0)}(tv, tw))$ be the Jacobi field along $\gamma$, then we get $J'(0) = w$.

$$\langle v, w \rangle = \langle \gamma'(0), J'(0) \rangle$$
And let \( t = 1 \), we have

\[
\langle (d\exp_\gamma(0))_v(v), (d\exp_\gamma(0))_w(w) \rangle = \langle \gamma'(1), J'(1) \rangle
\]

By the definition of Jacobi field, that is

\[
\frac{d^2 \langle J(t), \gamma'(t) \rangle}{dt^2} = 0
\]

It gives out the linearity of \( \langle J(t), \gamma'(t) \rangle \) and thus \( \langle \gamma'(1), J'(1) \rangle = \langle \gamma'(0), J'(0) \rangle \), which finished the proof.

**Theorem 3.3.** The isometric transformation of \( S^n \subset \mathbb{R}^{n+1} \), with the induced metric, are restrictions to \( S^n \) of the linear orthogonal transformations of \( \mathbb{R}^{n+1} \).

**Proof.** Let \( f : S^n \rightarrow S^n \) be the isometric transformation, that is, \( \langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)} \) for all \( p \in S^n, u, v \in T_p S^n \). We construct \( F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \) by \( F(x) = f(\frac{x}{|x|}) \cdot |x| \), where \( x = (x_1, x_2, ... x_{n+1}) \) to be the restrictions of linear orthogonal transformation. We need to show that \( F \) is linear orthogonal transformation and isometric transformation. First,

\[
\langle dF_p(u), dF_p(v) \rangle_{F(p)} = \langle df_p(u) + f(\frac{P}{|P|})d|P|, df_p(v) + f(\frac{P}{|P|})d|P| \rangle_{f(p)} = \langle u, v \rangle_p
\]

which gives out that \( F \) is isometric transformation. Therefore \( F \) also preserves inner product in Euclidean space. Secondly,

\[
\langle F(Ap + Bq) - AF(p) - BF(q), F(r) \rangle = \langle Ap + Bq - Ap - Bq, r \rangle = 0
\]

, where \( A, B \in \mathbb{R} \) be some real number, which shows \( F \) is linear orthogonal transformation.

**Definition 14.** Christoffel symbols. The Christoffel symbols \( \Gamma^k_{ij} \) can be defined on a coordinate system \((U, x)\) by the connection \( \nabla X_i X_j = \sum_k \Gamma^k_{ij} X_k \), where \( X_i = \frac{\partial}{\partial x_i} \).
Proposition 3.4. Let $M^2 \subset \mathbb{R}^3$ be a surface in $\mathbb{R}^3$ with the induced Riemannian metric. Let $c:I \rightarrow M$ be a differentiable curve on $M$ and let $V$ be vector field tangent to $M$ along $c$; $V$ can be thought of as a smooth function $V:I \rightarrow \mathbb{R}^3$, with $V(t) \in T_{c(t)}M$. If $S^2 \subset \mathbb{R}^3$ is the unit sphere of $\mathbb{R}^3$, then the velocity field along great circles, parametrized by arc length, is a parallel field.

Proof. The definition of that $V$ is parallel, is $DV/dt=0$. First we want to compare $\nabla V$ and $DV/dt=0$ to show that $V$ is parallel if and only if $dV/dt$ is perpendicular to $T_{c(t)}M \subset \mathbb{R}^3$.

Using Christoffel symbols, let $c(t)$ be the curve, $\{x^i\}$ be the local expression of $c(t)$, $V = \sum_i V^i \frac{\partial}{\partial X^i} = \sum_i V^i \partial_i$ the vector tangent to $M$ along $c$. $DV$ can be expressed from definition:

$$\frac{DV}{dt} = \sum_k \left( \frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx^i}{dt} \Gamma_{ij}^k \right) X_k.$$

Let $x^i = \frac{\partial}{\partial x^i}$, we also have

$$\nabla_X V = \nabla_X (V_a x^a) = (\nabla_X V_a) x^a + V_a \nabla_X x^a = (XV_a)x^a - V_a (\gamma_{ij} x^j) = (XV_j - V_a \gamma_{ij}) x^j$$

where $\Gamma$ is Christoffel symbols, $X \in T_{c(t)}M$ be some vector.

Since for any $X \in T_{c(t)}M$, any $W \in M$, we have $\nabla V(W,X) = (\nabla_X V)W$, we get $\nabla V = 0$ if and only if $\nabla_{dc(t)/dt} V = 0$.

Therefore, $V$ is parallel if and only if $dV/dt$ is perpendicular to $T_{c(t)}M \subset \mathbb{R}^3$.

Next, we want to show the velocity field $v(t)$ is tangent to sphere, then $dv/dt$ must be perpendicular to $T_{c(t)}S^2$. Secondly we use a) to conclude that $v$ is parallel.

Given the position vector $r(t)$ be a differentiable great circle, with $|r(t)| = r$, where $r$ is the radius of great circle.

Then $r(t) \cdot r(t) = |r(t)|^2 = r^2$.

We have velocity field $v(t) = r'(t)$. Since $0 = dr(t)^2/dt = 2v(t) \cdot r(t)$, $v(t)$ is tangent to
sphere, then dV/dt must be perpendicular to $T_{c(t)}S^2$.

Since $V$ is parallel if and only if $dV/dt$ is perpendicular to $T_{c(t)}M \subset \mathbb{R}^3$, we proof that $V$ is parallel field.

3.2 The 2nd Bianchi Identity

**Theorem 3.5.** We know the relation among $\nabla$ and curvature as following:

$$\nabla R(X_1, X_2, X_3, X_4, X_5)$$

$$= X_5(R(X_1, X_2, X_3, X_4)) - R(\nabla_{X_5}X_1, ..., X_4) - ... - R(X_1, ..., \nabla_{X_5}X_4) \tag{3.1}$$

and

$$R(X_1, X_2, X_3, X_4) = \langle R(X_1, X_2)X_3, X_4 \rangle \tag{3.2}$$

for any five vectors $X_1, X_2, X_3, X_4, X_5 \in \mathcal{X}(M)$. Then

$$\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + \nabla R(X, Y, T, Z, W) = 0, \tag{3.3}$$

for all $X, Y, Z, W, T \in \mathcal{X}(M)$. Here $\nabla R(X, Y, Z, W, T) : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow R$.

**Proof.** First, choose a point $p \in M$ and a geodesic frame (that is parallel transporting the orthonormal frame along a geodesic to satisfying $\nabla_{e_i}e_j(p) = 0$, where $e_i$ is basis) at $p$. Here we denote $\nabla R(e_i, e_j, e_k, e_t, e_h)$ as $\nabla 1$, because there are too much lower indices. By definition of curvature and the compatible of metric, since $\nabla_{e_i}e_j(p) = 0$, where $e_i = \frac{\partial}{\partial X^i}$.
at $p$, be the basis, we get

$$\nabla 1 = \nabla R(e_i, e_j, e_k, e_l, e_h)$$

$$= e_h < R(e_i, e_j)e_k, e_l >$$

$$= e_h < R(e_k, e_l)e_i, e_j >$$

$$= < \nabla e_h \nabla e_l \nabla e_k e_i - \nabla e_h \nabla e_k \nabla e_l e_i + \nabla e_h \nabla [e_k, e_l]e_i, e_j > .$$

Next, comparing with

$$R(e_l, e_h, \nabla e_k e_i, e_j) = < \nabla e_h \nabla e_l \nabla e_k e_i - \nabla e_h \nabla e_k \nabla e_l e_i + \nabla e_h \nabla e_i, e_j >,$$

we add some terms and subtract it for $\nabla 1$,

$$\nabla 1 = < \nabla e_h \nabla e_l \nabla e_k e_i - \nabla e_h \nabla e_k \nabla e_l e_i + \nabla e_h \nabla [e_k, e_l]e_i$$

$$+ \nabla [e_l, e_h] \nabla e_k e_i - \nabla [e_l, e_h] \nabla e_k e_i, e_j > .$$

Similarly,

$$\nabla 2 = \nabla R(e_i, e_j, e_l, e_h, e_k)$$

$$= < \nabla e_k \nabla e_l \nabla e_h e_i - \nabla e_k \nabla e_h \nabla e_l e_i + \nabla e_k \nabla [e_l, e_h]e_i$$

$$+ \nabla [e_l, e_h] \nabla e_k e_i - \nabla [e_l, e_h] \nabla e_k e_i, e_j >$$

$$\nabla 3 = \nabla R(e_i, e_j, e_h, e_k, e_l)$$

$$= < \nabla e_l \nabla e_h \nabla e_k e_i - \nabla e_l \nabla e_h \nabla e_k e_i + \nabla e_l \nabla [e_h, e_k]e_i$$

$$+ \nabla [e_h, e_k] \nabla e_l e_i - \nabla [e_h, e_k] \nabla e_l e_i, e_j >$$
Then we have

\[ \nabla 1 + \nabla 2 + \nabla 3 \]

\[ = \langle \nabla_{e_h} \nabla_{e_l} \nabla_{e_k} e_i - \nabla_{e_k} \nabla_{e_l} \nabla_{e_h} e_i + \nabla_{e_h} \nabla_{[e_k,e_l]} e_i \\
+ \nabla_{[e_l,e_h]} \nabla_{e_k} e_i - \nabla_{[e_k,e_l]} \nabla_{e_h} e_i \\
+ \nabla_{e_k} \nabla_{e_h} \nabla_{e_l} e_i - \nabla_{e_k} \nabla_{e_l} \nabla_{e_h} e_i + \nabla_{e_k} \nabla_{[e_l,e_h]} e_i \\
+ \nabla_{[e_h,e_k]} \nabla_{e_l} e_i - \nabla_{[e_k,e_l]} \nabla_{e_h} e_i \\
+ \nabla_{e_l} \nabla_{e_k} \nabla_{e_h} e_i - \nabla_{e_l} \nabla_{e_h} \nabla_{e_k} e_i + \nabla_{e_l} \nabla_{[e_h,e_k]} e_i \\
+ \nabla_{[e_k,e_l]} \nabla_{e_h} e_i - \nabla_{[e_l,e_h]} \nabla_{e_k} e_i \rangle \]

Rearranging the formula,

\[ \nabla 1 + \nabla 2 + \nabla 3 \]

\[ = < \nabla_{e_h} \nabla_{e_l} \nabla_{e_k} e_i - \nabla_{e_k} \nabla_{e_l} \nabla_{e_h} e_i + \nabla_{e_h} \nabla_{[e_k,e_l]} e_i \\
+ \nabla_{e_k} \nabla_{e_h} \nabla_{e_l} e_i - \nabla_{e_k} \nabla_{e_l} \nabla_{e_h} e_i + \nabla_{e_k} \nabla_{[e_l,e_h]} e_i \\
+ \nabla_{e_l} \nabla_{e_k} \nabla_{e_h} e_i - \nabla_{e_l} \nabla_{e_k} \nabla_{e_h} e_i + \nabla_{e_l} \nabla_{[e_h,e_k]} e_i \\
+ \nabla_{e_h} \nabla_{[e_k,e_l]} e_i - \nabla_{[e_l,e_h]} \nabla_{e_k} e_i + \nabla_{e_h} \nabla_{[e_l,e_k]} e_i \\
- \nabla_{[e_h,e_k]} \nabla_{e_l} e_i + \nabla_{e_l} \nabla_{[e_h,e_k]} e_i - \nabla_{[e_k,e_l]} \nabla_{e_h} e_i, e_j > \]

\[ = < R(e_l, e_h) \nabla_{e_k} e_i + R(e_h, e_k) \nabla_{e_l} e_i + R(e_k, e_l) \nabla_{e_h} e_i \\
+ \nabla_{e_k} \nabla_{[e_k,e_l]} e_i - \nabla_{[e_l,e_h]} \nabla_{e_k} e_i + \nabla_{e_k} \nabla_{[e_l,e_h]} e_i \\
- \nabla_{[e_h,e_k]} \nabla_{e_l} e_i + \nabla_{e_l} \nabla_{[e_h,e_k]} e_i - \nabla_{[e_k,e_l]} \nabla_{e_h} e_i, e_j > \]

Using the Jacobi identity to cancel the bracket, and since \( \nabla_{e_i} e_j(p) = 0 \), therefore we get

\[ \nabla 1 + \nabla 2 + \nabla 3 = 0. \]

\[ \square \]

**Theorem 3.6.** Let \( M \) be a Riemannian manifold with sectional curvature identically zero, for every \( p \in M \), the mapping \( \text{exp}_B : B_\epsilon(0) \subset T_pM \rightarrow B_\epsilon(P) \) is an isometry, where \( B_\epsilon(P) \) is an open ball at \( p \).

**Proof.** Let \( J(t) = (d\text{exp}_p)_{tw}(tu) \) be the Jacobi field along the geodesic \( \gamma(t) = \text{exp}_p(tw) \),
$t \in I$ on the normal ball at $p$, where $w \in T_p M$ and any $u \in T_w(T_p M)$. Then we have the Jacobi equation

$$\frac{D^2 J}{dt^2} + KJ = 0,$$

(3.4)

where $K$ is the sectional curvature satisfying $R(\gamma', J)\gamma' = KJ$. Since the sectional curvature is identically zero, that is $K = 0$, the Jacobi equation can be deduced into

$$\frac{D^2 J}{dt^2} = 0$$

(3.5)

with initial condition $J(0) = 0$, $J'(0) = u(0)$. Then one solution to this equation is $J(t) = tu(t)$. Similarly, for any $v \in T_w(T_p M)$, we have $(d\exp_p)_{tw}(tv) = tv(t)$. Therefore,

$$\langle u, v \rangle_p = \langle d\exp_p(u), d\exp_p(v) \rangle_{\exp(p)}$$

(3.6)

indicates that $\exp$ is isometric transformation, and finished the proof.

From these theorem, we can see a strong link between isometric transformation and Jacobi field.

### 3.3 Isometry Preserve Geodesics

**Theorem 3.7.** Isometry preserve geodesics, that is, let $M$ be manifold with affine connection $\nabla$, $N$ with connection $\tilde{\nabla}$. If $\gamma : I \to M$ is a geodesic in $M$, and $\phi : M \to N$ an isometric transformation, then $\phi(\gamma)$ is a geodesic in $N$.

**Proof.** Let $\phi : M \to N$ be the isometric transformation defined as above.

For any vector $V \in T_{\gamma(t_0)} M$ and $\tilde{V}$ in $T_{\phi(\gamma(t_0))} N$,

$$d\phi(\nabla_X Y)V = \tilde{\nabla}_{d\phi(X)}d\phi(Y)\tilde{V}$$

(3.7)
and by the chain rule, we have

$$\frac{d(\phi(\gamma))}{dt} = d\phi(\frac{d\gamma}{dt})$$  \hspace{1cm} (3.8)

Therefore we get

$$\tilde{\nabla}_{d/dt}\phi(\gamma)' = \tilde{\nabla}_{d/dt}\phi(\gamma') = \phi(\nabla_{d/dt}\gamma') = 0$$  \hspace{1cm} (3.9)

It shows that isometric transformation preserve geodesics.
CHAPTER 4
EXAMPLES

4.1 Euclidean space

We start with the Euclidean space to illustrate how our new symbol can be applied to manifolds trivial and non-trivial.

Example 1. Euclidean space of 3 dimensions. The Euclidean space of 3 dimensions \( \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\} \) can be seen as a manifold equipped with Euclidean metric and Euclidean distance.

![Figure 4.1: Euclidean space of 3 dimensions and geodesics](image)

4.1.1 Metric

Euclidean metric is constant that follows the idea of flatten space. Flatten space is a space with zero curvature and the inner product for two vectors \((a_1, b_1, c_1), (a_2, b_2, c_2) \in \mathbb{R}^3\)
\[(a_1, b_1, c_1), (a_2, b_2, c_2) = a_1a_2 + b_1b_2 + c_1c_2.\]

Therefore we have metric
\[
(g_{ij}) = \begin{pmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

where \(i, j = 1, 2, 3\) for 3 dimension.

\[
(g^{ij}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

### 4.1.2 Geodesic Equations

\[
\frac{d^2x}{dt^2} = 0,
\]

\[
\frac{d^2y}{dt^2} = 0,
\]

\[
\frac{d^2z}{dt^2} = 0.
\]

And we know the solution is and should be the straight line, which gives the parameterization of straight line.

\[
\gamma(t) = (x(t), y(t), z(t)) = (pt, qt, rt) + (k, m, l)
\]

where \(p, q, r, k, m, l \in \mathbb{R}\) are some constant, \(t\) go through real number.

### 4.2 Unit Sphere

Here we want to use three different examples to show the diversity of Riemannian manifolds. Two of these examples are regular surfaces which are often referred in many subjects and classical research. The third one is not regular surface as a specific example in geome-
Try.

**Definition 15.** Regular surface. A subset \( M^k \subset \mathbb{R}^n \) is called a regular surface of dimension \( k \) in \( \mathbb{R}^n \), if for every \( p \in M^k \) there exists a neighborhood \( W \) of \( p \) in \( \mathbb{R}^n \) and a mapping \( x : U \subset \mathbb{R}^k \to M \cap W \) of an open set \( U \in \mathbb{R}^k \) onto \( M \cap W \) such that:

1. \( x \) is differentiable homeomorphism.

2. \((dx)_q : \mathbb{R}^k \to \mathbb{R}^n \) is injective for all \( q \in U \).

**Example 2.** Sphere. the unit sphere \( S^2 \subset \mathbb{R}^3 \) given by
\[
S^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \},
\]
is a regular surface of dimension 2.

![Figure 4.2: sphere and geodesics](image)

Since \( S^2 \) is a regular surface, we may show some properties of riemannian manifold.

**Chart**

There are six charts covering the sphere.

**Parameterization**

Take a Cartesian coordinate system and change it into polar coordinate. \((\xi_1, \xi_2, \xi_3) = f(\theta, \psi) = (\sin(\theta)\cos(\psi), \sin(\theta)\sin(\psi), \cos(\theta))\), with \((\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 - 1 = 0\) and \(-\pi < \theta \leq \pi \) and \(-\pi < \psi \leq \pi \).

The partial derivative of parameterization are
\[
\frac{\partial f(\theta, \psi)}{\partial \theta} = (\cos(\theta)\cos(\psi), \cos(\theta)\sin(\psi), -\sin(\theta))
\]
\[
\frac{\partial f(\theta, \psi)}{\partial \psi} = (-\sin(\theta)\sin(\psi), \sin(\theta)\cos(\psi), 0)
\]

### 4.2.1 Metric

Since the component of the metric $g$ related to such a basis is

\[
g_{ij} = \sum_k \frac{\partial \xi_k}{\partial x^i} \frac{\partial \xi_k}{\partial x^j},
\]

(4.1)

then we $g_{ij}$ compute for each combination $\theta, \psi$, which are

- $g_{\theta\theta} = \frac{\partial \xi_1}{\partial \theta} \frac{\partial \xi_1}{\partial \theta} + \frac{\partial \xi_2}{\partial \theta} \frac{\partial \xi_2}{\partial \theta} + \frac{\partial \xi_3}{\partial \theta} \frac{\partial \xi_3}{\partial \theta} = (\cos(\theta)\cos(\psi))^2 + (\cos(\theta)\sin(\psi))^2 + (-\sin(\theta))^2 = 1$,
- $g_{\theta\psi} = \frac{\partial \xi_1}{\partial \theta} \frac{\partial \xi_1}{\partial \psi} + \frac{\partial \xi_2}{\partial \theta} \frac{\partial \xi_2}{\partial \psi} + \frac{\partial \xi_3}{\partial \theta} \frac{\partial \xi_3}{\partial \psi} = \cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi) + \cos(\theta)\sin(\psi)\sin(\theta)\cos(\psi) + 0 = 0$,
- $g_{\psi\theta} = \frac{\partial \xi_1}{\partial \psi} \frac{\partial \xi_1}{\partial \theta} + \frac{\partial \xi_2}{\partial \psi} \frac{\partial \xi_2}{\partial \theta} + \frac{\partial \xi_3}{\partial \psi} \frac{\partial \xi_3}{\partial \theta} = \cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi) + \cos(\theta)\sin(\psi)\sin(\theta)\cos(\psi) + 0 = g_{\theta\psi} = 0$,
- $g_{\psi\psi} = \frac{\partial \xi_1}{\partial \psi} \frac{\partial \xi_1}{\partial \psi} + \frac{\partial \xi_2}{\partial \psi} \frac{\partial \xi_2}{\partial \psi} + \frac{\partial \xi_3}{\partial \psi} \frac{\partial \xi_3}{\partial \psi} = (-\sin(\theta)\sin(\psi))^2 + (\sin(\theta)\cos(\psi))^2 = (\sin(\theta))^2$.

Thus the covariance and contravariant metric tensor could be written as

\[
\begin{pmatrix}
g_{ij}
\end{pmatrix}
= \begin{pmatrix}
g_{\theta\theta} & g_{\theta\psi} \\
g_{\psi\theta} & g_{\psi\psi}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & (\sin(\theta))^2
\end{pmatrix}
\]

\[
\begin{pmatrix}
g^{ij}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & \frac{1}{(\sin(\theta))^2}
\end{pmatrix}
\]

since $g^{ij} = 1/g_{ij}$ if $g_{ij} \neq 0$.

### 4.2.2 Christoffel Symbols

We have the equation about Christoffel symbols and metric,

\[
\Gamma^m_{ij} = \frac{1}{2} \sum_k \left( g_{jk,i} + g_{ki,j} - g_{ij,k} \right) g^{km}
\]
where $g_{jk,i} = \frac{\partial g_{jk}}{\partial x^i}$ for any $j, k, i$. Then we can expand Christoffel symbols as following,

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2} \{g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta}\} g^{\theta\theta} + \frac{1}{2} \{g_{\theta\psi,\theta} + g_{\psi\theta,\theta} - g_{\theta\psi,\theta}\} g^{\psi\theta} = 0.$$  

$g_{\theta\theta,\theta} = 0,$

$g_{\theta\theta,\psi} = 0,$

$g_{\theta\psi,\theta} = g_{\theta\psi,\psi} = g_{\psi\theta,\theta} = g_{\psi\theta,\psi} = 0,$

$g_{\psi\psi,\theta} = 2 \sin(\theta) \cos(\theta),$  

$g_{\psi\psi,\psi} = 0.$

We notice that here $g^{km} = 0$ if $k \neq m$. Then the equation can be simplify to

$$\Gamma^m_{ij} = \frac{1}{2} \{g_{jm,i} + g_{mi,j} - g_{ij,m}\} g^{mm},$$  

$$\Gamma^\theta_{\psi\theta} = \Gamma^\theta_{\psi\theta} = \frac{1}{2}(g_{\psi\theta,\theta} + g_{\theta\psi,\theta} - g_{\theta\psi,\theta}) g^{\theta\theta} = 0.$$  

$$\Gamma^\psi_{\psi\theta} = \Gamma^\psi_{\psi\theta} = \frac{1}{2}(g_{\psi\theta,\theta} + g_{\theta\psi,\theta} - g_{\theta\psi,\theta}) g^{\psi\theta} = 0.$$  

$$\Gamma^\psi_{\psi\psi} = \Gamma^\psi_{\psi\psi} = \frac{1}{2}(g_{\psi\psi,\theta} + g_{\psi\psi,\theta} - g_{\psi\psi,\theta}) g^{\psi\psi} = 0.$$  

Therefore the christoffel symbol in matrix form can be written as

$$\Gamma^\theta = \begin{pmatrix} \Gamma^\theta_{\theta\theta} & \Gamma^\theta_{\theta\psi} \\ \Gamma^\theta_{\psi\theta} & \Gamma^\theta_{\psi\psi} \end{pmatrix} = \begin{pmatrix} 0 & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & 0 \end{pmatrix}$$  

$$\Gamma^\psi = \begin{pmatrix} \Gamma^\psi_{\theta\theta} & \Gamma^\psi_{\theta\psi} \\ \Gamma^\psi_{\psi\theta} & \Gamma^\psi_{\psi\psi} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{pmatrix}$$

4.2.3 Geodesic Equations

General geodesic equation, given out by the solution to system of equation in the system of coordinates about geodesic $\gamma(t)$.

$$\frac{d^2 u}{dt^2} + \frac{du}{dt} \Gamma^u_{uu} + 2 \frac{du}{dt} \frac{dv}{dt} \Gamma^u_{uv} + \frac{dv}{dt} \frac{dv}{dt} \Gamma^u_{vv} = 0$$
\[
\frac{d^2 v}{dt^2} + \frac{du}{dt} \Gamma^v_{uw} + 2 \frac{du}{dt} \frac{dv}{dt} \Gamma^v_{uw} + \frac{dv}{dt} \Gamma^v_{vv} = 0
\]

For the unit sphere, that is,
\[
\frac{d^2 \theta}{dt^2} - \sin(\theta) \cos(\theta) \frac{d^2 \psi}{dt^2} = 0
\]

\[
\frac{d^2 \psi}{dt^2} + 2 \frac{\cos(\theta)}{\sin(\theta)} \frac{d\theta}{dt} \frac{d\psi}{dt} = 0
\]

The solution gives out the general great circle on sphere.

\[-P = (Q \cdot \cos(\psi) + R \cdot \sin(\psi)) \cdot \tan(\theta)\]

where P,Q,R are non zero constant. Isometric transformation The isometric transformation on sphere is the linear orthogonal transformation.

### 4.3 Surface of Revolution

**Example 3. Surface of revolution.** A surface if called surface of revolution if the surface is generated by the rotation of the curve \((f(v), g(v))\) around the axis 0z, where \(f\) and \(g\) are differentiable functions, with \(f'(v)^2 + g'(v)^2\) not zero and \(f(v)\) not zero. Specifically, given a function \(\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^3\), by \(\phi(u,v) = (f(v) \cos u, f(v) \sin u, g(v))\), \(U = \{(u,v) \in \mathbb{R}^2, u_0 < u < u_1, v_0 < v < v_1\}\), is a parameterization and an immersion.

We can see that surface of revolution is a regular surface of dimension 2. Parameterization

Take a Cartesian coordinate system and change it into polar coordinate. \((\xi_1, \xi_2, \xi_3) = \phi(u, v) = (f(v) \cos u, f(v) \sin u, g(v))\), with \(u_0 < u < u_1, v_0 < v < v_1\).
The partial derivative of parameterization are
\[
\frac{\partial f}{\partial u} = (-f(v) \sin u, f(v) \cos u, 0)
\]
\[
\frac{\partial f}{\partial v} = (f(v)' \cos u, f(v)' \sin u, g(v)')
\]

### 4.3.1 Metric

Since the component of the metric g related to such a basis is
\[
g_{ij} = \sum_k \frac{\partial \xi_k}{\partial x^i} \frac{\partial \xi_k}{\partial x^j},
\]

then we compute $g_{ij}$ for each combination $\theta, \psi$, which are
\[
g_{uu} = \frac{\partial \xi_1}{\partial u} \frac{\partial \xi_1}{\partial u} + \frac{\partial \xi_2}{\partial u} \frac{\partial \xi_2}{\partial u} + \frac{\partial \xi_3}{\partial u} \frac{\partial \xi_3}{\partial u} = (-f(v) \sin u)^2 + (f(v) \cos u)^2 = f^2,
\]
\[
g_{uv} = g_{vu} = \frac{\partial \xi_1}{\partial u} \frac{\partial \xi_1}{\partial v} + \frac{\partial \xi_2}{\partial u} \frac{\partial \xi_2}{\partial v} + \frac{\partial \xi_3}{\partial u} \frac{\partial \xi_3}{\partial v} = -f(v) \sin u f(v)' \cos u + f(v) \cos u f(v)' \sin u = 0,
\]
\[
g_{vv} = \frac{\partial \xi_1}{\partial v} \frac{\partial \xi_1}{\partial v} + \frac{\partial \xi_2}{\partial v} \frac{\partial \xi_2}{\partial v} + \frac{\partial \xi_3}{\partial v} \frac{\partial \xi_3}{\partial v} = (-f'(v) \sin u)^2 + (f'(v) \cos u)^2 + (g(v))'^2 = (f')^2 + (g')^2.
\]
Thus the metric tensor could be written as
\[
(g_{ij}) = \begin{pmatrix}
g_{uu} & g_{uv} \\
g_{vu} & g_{vv}
\end{pmatrix} = \begin{pmatrix}f^2 & 0 \\
0 & (f')^2 + (g')^2\end{pmatrix}
\]
\[(g^{ij}) = \begin{pmatrix} \frac{1}{f^2} & 0 \\ 0 & \frac{1}{(f')^2 + (g')^2} \end{pmatrix}\]

### 4.3.2 Christoffel Symbols

We also have the equation \(\Gamma^m_{ij} = \frac{1}{2} \sum_k (g_{jk,i} + g_{ki,j} - g_{ij,k}) g^{km}\), where \(g_{jk,i} = \frac{\partial g_{jk}}{\partial x^i}\) for any \(j, k, i\).

\(g_{uv,u} = g_{uv,v} = g_{vu,u} = g_{vu,v} = 0\), \(g_{uv,u} = 2ff'\), \(g_{uv,v} = 0\), \(g_{vv,v} = 2f'f'' + 2g'g''\)

\(\Gamma^u_{uu} = \frac{1}{2} g^{uu}(g_{uu,u} + g_{uu,u} - g_{uu,u}) = 0\),

\(\Gamma^v_{uu} = \Gamma^v_{uv} = \frac{1}{2} g^{uu}(g_{uv,u} + g_{uv,u} - g_{uv,u}) = \frac{1}{2} \frac{1}{f^2}(2ff')\),

\(\Gamma^v_{vv} = \Gamma^v_{vu} = \frac{1}{2} g^{vv}(g_{vu,v} + g_{vu,v} - g_{vu,v}) = 0\),

\(\Gamma^u_{uv} = \frac{1}{2} g^{uv}(g_{vu,u} + g_{vu,u} - g_{vu,v}) = \frac{1}{2} \frac{1}{(f')^2 + (g')^2} (-2ff')\),

\(\Gamma^u_{vv} = \frac{1}{2} g^{vv}(g_{vv,u} + g_{vv,u} - g_{vv,v}) = \frac{1}{2} \frac{1}{(f')^2 + (g')^2} (2f'f'' + 2g'g'')\).

Therefore the christoffel symbol in matrix form can be written as

\[
\Gamma^u = \begin{pmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} \\ \Gamma^v_{vu} & \Gamma^v_{vv} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \frac{1}{f^2} (2ff') \\ \frac{1}{2} \frac{1}{f^2} (2ff') & 0 \end{pmatrix}
\]

\[
\Gamma^v = \begin{pmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} \\ \Gamma^v_{vu} & \Gamma^v_{vv} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \frac{1}{(f')^2 + (g')^2} (-2ff') & 0 \\ 0 & \frac{1}{2} \frac{1}{(f')^2 + (g')^2} (2f'f'' + 2g'g'') \end{pmatrix}
\]

### 4.3.3 Geodesic Equation

System of equation in the system of coordinates about geodesic \(\gamma(t)\) can be written as,

\[
\frac{d^2 u}{dt^2} + \frac{2ff'}{f^2} \frac{du}{dt} \frac{dv}{dt} = 0
\]

\[
\frac{d^2 v}{dt^2} - \frac{ff'}{f'^2 + g'^2} \left(\frac{dv}{dt}\right)^2 + \frac{f'f'' + g'g''}{f'^2 + g'^2} \left(\frac{dv}{dt}\right)^2 = 0
\]
Isometric transformation The rotations around z-axis are isometric transformation on surface of revolution. If for \(f(u,v), g(u,v)\), the variables \(u\) is fixed, we rotate such trajectory which is a geodesic and it forms another geodesic on surface of revolution (if \(v\) is fixed, not geodesic).

### 4.4 Lobatchevski Plane

**Example 4.** Lobatchevski plane/Hyperbolic plane. Let \(H\) be the upper half plane, that is, \(H = \{(x, y) \in \mathbb{R}^2 : y > 0\}\) with the Riemannian metric \(g_{11} = g_{22} = \frac{1}{y^2}, g_{12} = g_{21} = 0\). \(H\) is called Lobatchevski plane.

![Figure 4.4: Lobatchevski plane and geodesics](image)

**4.4.1 Metric**

The metric tensor could be written as

\[
(g_{ij}) = \begin{pmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{y^2} & 0 \\
0 & \frac{1}{y^2}
\end{pmatrix}
\]

\[
(g^{ij}) = \begin{pmatrix}
y^2 & 0 \\
0 & y^2
\end{pmatrix}
\]
### 4.4.2 Christoffel Symbols

\[ g_{12,1} = g_{12,2} = g_{21,1} = g_{21,2} = 0, \ g_{11,1} = 0 = g_{22,1} = 0, \ g_{22,2} = g_{11,2} = -\frac{2}{y^2}. \]

\[ \Gamma^1_{11} = \frac{1}{2} g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) = 0, \]

\[ \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2} g^{11}(g_{11,2} + g_{12,1} - g_{12,1}) = -\frac{1}{y}, \]

\[ \Gamma^1_{22} = \frac{1}{2} g^{11}(g_{12,2} + g_{12,2} - g_{22,1}) = 0, \]

\[ \Gamma^2_{11} = \frac{1}{2} g^{vv}(g_{21,1} + g_{21,1} - g_{11,2}) = 0, \]

\[ \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22}(g_{12,2} + g_{22,1} - g_{12,2}) = 0, \]

\[ \Gamma^2_{22} = \frac{1}{2} g^{22}(g_{22,2} + g_{22,2} - g_{22,2}) = -\frac{1}{y}. \]

Therefore the Christoffel symbol in matrix form can be written as

\[
\Gamma^1 = \begin{pmatrix}
\Gamma^1_{11} & \Gamma^1_{12} \\
\Gamma^1_{21} & \Gamma^1_{22}
\end{pmatrix} = \begin{pmatrix}
0 & -\frac{1}{y} \\
-\frac{1}{y} & 0
\end{pmatrix}
\]

\[
\Gamma^2 = \begin{pmatrix}
\Gamma^2_{11} & \Gamma^2_{12} \\
\Gamma^2_{21} & \Gamma^2_{22}
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & -\frac{1}{y}
\end{pmatrix}
\]

### 4.4.3 Geodesic Equations

System of equations in the system of coordinates about geodesic \( \gamma(t) \) can be written as,

\[
\frac{d^2x}{dt^2} - 2\frac{1}{y} \frac{dx}{dt} \frac{dy}{dt} = 0
\]

\[
\frac{d^2y}{dt^2} + \frac{1}{y} \left( \frac{dx}{dt} \right)^2 - \frac{1}{y^2} \left( \frac{dy}{dt} \right)^2 = 0
\]

It is not easy to solve the system of equations, but we can easily see that The vertical line and upper semicircle satisfying above system of equations are geodesics on the Lobatchevski plane.

**Proposition 4.1.** Geodesic on Lobatchevski plane are vertical lines and upper semicircles.
Proof. Let \( c(t) \) be the curve of vertical line pass the origin, that is, \( c(t) = (0, t) \), \( t > 0 \). Thus \( c(t) \) satisfying the system, which is geodesic. Given the mobius transformation on complex plane.

\( f : \mathbb{C} \to \mathbb{C} \), by \( z \to \frac{az+b}{cz+d} \), with \( ad-bc=1 \).

Here the Lobatchevski plane is defined on \( \mathbb{R}^2 \), so we use x-y plane which is easier to see the geodesic instead of complex plane. Thus \( z \) is replaced by \( x+yi \).

\( (x, y) = f(c(t)) = f(0, t) = \frac{at+b}{ct+d} = \frac{a(t^2+bd)}{c(t^2+d)} + i \frac{t}{c(t^2+d)} \),

We know the equation of a semicircle on \( \mathbb{R}^2 \) with center at \( (p,0) \) on x-axis, that is, \( x^2 + y^2 − 2px = q \), where \( p, q \in \mathbb{R} \) are two constant real number, \( y > 0 \).

Thus the LHS of the equation of circle becomes

\[
x^2 + y^2 − 2px = \left( \frac{act^2+bd}{c^2t^2+d^2} \right)^2 + \left( \frac{t}{c^2t^2+d^2} \right)^2 - 2\frac{act^2+bd}{c^2t^2+d^2}.
\]

If we go thorough the curve \( f(c(t)) \), we have \( \lim_{t \to 0} (x, y) = (\frac{b}{a}, 0), \lim_{t \to \infty} (x, y) = (\frac{c}{a}, 0) \).

For the case \( d \neq 0 \) and \( c \neq 0 \),

Choose \( p \) be the x-axis component of the middle point of these two limitation, that is ,

\[
p = \frac{1}{2}(\frac{a}{c} + \frac{b}{d}) = \frac{ad+bc}{2dc}.
\]

We also notice that \( 1 = ad − bc = a^2d^2 − 2abcd + b^2c^2 \).

Therefore

\[
x^2 + y^2 − 2px = \left( \frac{act^2+bd}{c^2t^2+d^2} \right)^2 + \left( \frac{t}{c^2t^2+d^2} \right)^2 - 2\frac{act^2+bd}{c^2t^2+d^2}.
\]

For the case \( d = 0 \) or \( c = 0 \), it is clear that the isometry degenerates to linear translation.

Since isometry is invariant under translation, dilation. And the inverse of isometry is still an isometry. We prove that isometric transformation maps geodesic to another geodesic on Lobatchevski plane. \( \square \)
CHAPTER 5  
AFFINE TRANSFORMATION AND ISOMETRICAL TRANSFORMATION

5.1 Kobayashi’s theorem

**Definition 16.** A Riemannian manifold $M$ is (geodesically) complete if for all $p \in M$, the exponential map $\exp_p : T_pM \to M$, is defined for all $v \in T_pM$.

**Definition 17.** Let $M$ and $N$ be Riemannian manifolds, with $g_1$ and $g_2$ as their metric and Levi-Civita connection $\nabla_1 X_1$ and $\nabla_2 X_2$. For any vector field $V$ on $M$, Let $\phi$ be a differentiable homeomorphism of $M$ onto $N$. If $\phi$ commute with the connection,

$$\phi(\nabla_1 V) = \nabla_2 (\phi V) \phi$$

$\phi$ is called an affine transformation. Equivalently, by Nomizu[7], let $\tau$ be any curve from $p$ to $q$ and let $\phi(\tau)$ be the curve which is the image of $\tau$ by $\phi$. Denoting the parallel displacement(or we called parallel transport) along a curve by the same letter as the curve, $\phi$ is called affine if

$$d\phi \tau X = \phi(\tau) d\phi X$$

for every $X \in T_pM$. $\phi$ is an affine transformation of $M$ if it is affine at every point of $M$.

**Theorem 5.1.** Hopf and Rinow Theorem[8]. (1) $M$ is complete as a metric space. (2) $M$ is geodesically complete. (3) Every cauchy sequence are converged. These three statement are equivalent.

Remark. The manifold $M$ is not necessarily equipped with some metrics $g$, here we assume these manifolds are equipped.
**Proposition 5.2.** If $M$ is complete or (geodesically) complete, for any $q \in M$, there exists a geodesic $\gamma$ joining $p$ to $q$ with $\text{len}(\gamma) = d(p,q)$, where $d(p,q)$ is the distance in metric space.

**Definition 18.** Let $M$ manifold, $\nabla$ Levi-Civita connection, and $\alpha(t)$ be loop, consider the mapping $P: T_{\alpha(t_0)}M \rightarrow T_{\alpha(t)}M$ be the parallel transport map along $\alpha$. Let $GL(T_{\alpha(t_0)}M)$ be a group of linear transformation of $\alpha(t_0)T_{\alpha(t_0)}M$. Then the **holonomy group** of connection $\nabla$ at point $x$ on manifold $M$ is defined as $\text{Hol}_x(\nabla) = \{ P_{\alpha} \in GL(T_{\alpha(t_0)}M) : \alpha \text{ is a loop based at } x \}$.

**Definition 19.** Irreducible manifold. A Riemannian manifold $M$ is irreducible if any embedded sphere of $M$ bounds an embedded ball. or equivalent definition: $M$ is said reducible or irreducible if the restricted homogeneous holonomy group of $M$ is reducible or not.

**Definition 20.** A point of a function is called **fixed point** if for the function $f: X \rightarrow X$, and the point $p \in X$, $f(p) = p$.

**Theorem 5.3.** Banach Fixed Point Theorem. Let $(M,d)$ be a compete metric space with with a contraction mapping $T: X \rightarrow X$, that is, there exists $Q \in I$, such that $d(T(x),T(y)) \leq Qd(x,y)$ for all $x, y$ in $M$, here $d(x,y)$ is the distance. Then $T$ admits a fixed point in $M$.

**Lemma 5.4.** If $M$ is irreducible, then for affine transformation $\phi: (M, g) \rightarrow (M, g)$, we have $d\phi(g) = c \cdot g$, where $c$ is a positive constant, $g$ is the metric.

**Lemma 5.5.** Let $M,N$ be Riemannian manifold. $f: M \rightarrow N$ be any differentiable mapping. The differential of $f$, $df: T_pM \rightarrow T_{fp}N$ is linear mapping.

**Lemma 5.6.** Let $M$ be Riemannian manifold with affine connection $\nabla$. $\phi$ be the affine transformation, then $\text{Hol}$ are the subset of general linear group of $T_xM$, which is subset of automorphism group at $T_xM$. And we can obtain such commutative diagram.

$$
\begin{array}{ccc}
T_xM & \xrightarrow{\gamma} & T_yM \\
\downarrow^{d\phi} & & \downarrow^{d\phi} \\
T_{\phi(x)}M & \xrightarrow{\phi(\gamma)} & T_{\phi(y)}M
\end{array}
$$
Proof. Levi-Civita connection is torsion-free and thus we can find an abelian group \( GL(n) \) admits \( \gamma = d\phi^{-1}\phi(\gamma)d\phi \). the holonomy group is the subset of \( GL(n) \).

\[ \nabla = d\phi^{-1}\phi(\gamma)d\phi \]

\[ \gamma = \frac{d\phi^{-1}\phi(\gamma)d\phi}{d\phi} \]

The holonomy group is the subset of \( GL(n) \).

Lemma 5.7. Similar matrices have the same characteristic polynomial \( \det(\lambda E - A) \), that is, \( \det(\lambda E - B^{-1}AB) = \det(\lambda E - A) \), where \( E \) is \( n \times n \) identity matrix, \( A, B \) are some \( n \times n \) matrix.

Proof. Since

\[ \lambda E = B^{-1}\lambda EB \]

then we get

\[ \det(\lambda E - B^{-1}AB) \]

\[ = \det(B^{-1}\lambda EB - B^{-1}AB) \]

\[ = \det(B^{-1})\det(\lambda E - A)\det(B) = \det(\lambda E - A). \]

Theorem 5.8. (Nomizu prove in [9]) Let \( M \) be a complete Riemannian manifold. If the restricted homogeneous holonomy group \( \text{Hol} \) is contained in the linear isotropy group at each point \( p \) of \( M \), then \( M \) is Riemannian symmetric, that is, the covariant derivatives of the curvature tensor field are zero.

Theorem 5.9. Kobayashi’s theorem.

If \( M \) is an irreducible and complete Riemannian manifold, then \( \text{A}(M) \) is equal to \( \text{I}(M) \), except the case \( M \) is the 1-dimensional Euclidean space.

Proof. Since \( d\phi(g) = g \) implies \( \text{A}(M) = \text{I}(M) \), we want to show that \( c = 1 \) in Lemma 5.3 by contradiction. Assume that the constant \( c \neq 1 \).

1 The first case. If the affine transformation \( \phi \) with the unique Levi-Civita connection has no fixed point, \( \phi^{-1} \) dose not have fixed point, either.

1.1 Here we consider \( c < 1, \ c \in (0, 1) \). Take any point \( x \in M \) and a geodesic \( \gamma \), such that \( x \in \gamma \) and \( \phi(x) \in \gamma \). It is possible because the manifold is (geodesically) complete. Since
we have the existence but not uniqueness of local geodesic, $\phi^k(\gamma)$ is one of the geodesic joining $\phi^k(x)$ and $\phi^{k+1}(x)$. If $\text{len}(\gamma)$ is the length of the geodesic $\gamma$, use induction to show the length of geodesic $\text{len}(\phi^k(\gamma))$ is $c^k \cdot \text{len}(\gamma)$. The case $k = 1$ is true. Suppose it is true for $k = n$. Then for $k = n + 1$, $\text{len}(\phi^{k+1}(\gamma)) = c^{k+1} \cdot \text{len}(\gamma)$, which finished the induction. $c^k \cdot \text{len}(\gamma)$ tend to zero as k tends to infinity, therefore the sequences of points $\{\phi^k(x)\}_k$ forms a Cauchy sequence. Since M is complete, by Hopf and Rinow Theorem, this sequence must converges to a point $x^*$. By Banach Fixed Point Theorem, since $\phi$ is contraction mapping, $\phi$ has unique fixed point $x^*$. Therefore $\phi^{-1}$ has a fixed point. It contradict to the assumption. We have proved that, if $\phi$ has no fixed point, $c \geq 1$.

1.2 For $c > 1$, consider the inverse map $\phi^{-1}$, therefore $1/c$ is the radius of convergence. Similarly, $\phi^{-1}$ has a fixed point. It contradict to the assumption. We claim that, if $\phi$ has no fixed point, then we have $c \leq 1$, and $c = 1$.

2. The second case. If $\phi$ has at least one fixed point. $\phi^{-1}$ has at least one fixed point, either.

2.1 Here we consider $c < 1$. Let $\tau$ be a piecewise differentiable closed curve in M starting from $x$. Since $c < 1$, the sequence $\{\phi^k(\tau)\}$ converges to the point $x$. Then $\frac{D\phi^k(\tau)}{dt}$ tend to 0, because $\phi^k(\tau)$ itself converges uniformly when $k$ tends to infinity. Let $h_k$ be the element of $\text{Hol}_x(\nabla)$, by the definition of irreducible manifold. Let $d\phi$ be the linear transformation of tangent space $T_x$ given by $\phi$. Using induction, we want to show $h_0 = d\phi^{-k}h_kd\phi^k$. For the first case $k = 1$, $h_0 = d\phi^{-1}h_1d\phi$ by the properties of general linear group $\text{GL}(n)$, that is, linear transformation and affine transformation $\delta\phi$, commute with parallel transport $h_0$. We assume that $h_0 = d\phi^{-k}h_kd\phi^k$ is true for $k = n$. Thus for $k = n + 1$, we apply $h_k = d\phi^{-1}h_{k+1}d\phi$ to $h_0 = d\phi^{-k}h_kd\phi^k$ again and gives out $h_0 = d\phi^{-k-1}h_{k+1}d\phi^{k+1}$. Hence it is true for every $n$.

Let $P_k$ be the characteristic polynomial of the transformation $d\phi^{-k}h_kd\phi^k$, which is equal to the characteristic polynomial of $h_k$, since Similar matrices have the same characteristic polynomial. As $h_k$ tends to identity transformation, the $P_k$ tends to $(1 - \lambda)^n$. Since $h_0 = d\phi^{-k}h_kd\phi^k$, the characteristic polynomial of $h_0$ is $(1 - \lambda)^n$. Hence $h_0$ must be the
identity transformation of the tangent space $T_x$. The restricted homogeneous holonomy group $\text{Hol}$ is contained in the linear isotropy group at each point $p$ of $M$ because $\tau$ is arbitrary differentiable closed curve starting from $x$. If $h_0$ is always the identity transformation, and $M$ is irreducible and complete, it is Riemannian symmetric. Thus $M$ is 1-dimensional torus or 1-dimensional Euclidean space. In this case, if $\phi$ has at least one fixed point, $A(M) = I(M)$. \qed
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