THE CONVERGENCE THEOREM OF DISCRETE UNIFORMIZATION FACTORS ON CLOSED SURFACES

by

XIAOPING ZHU

A dissertation submitted to the

School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Feng Luo

and approved by

New Brunswick, New Jersey

May, 2022

ABSTRACT OF THE DISSERTATION

The Convergence Theorem of Discrete Uniformization Factors on closed surfaces

By Xiaoping Zhu

Dissertation Director:

Feng Luo

In this thesis, we will investigate the convergence of discrete conformal metrics to the classical uniformization metric on Riemannian surfaces. We prove that for a reasonable geodesic triangle mesh on a smooth closed orientable surface, there exists a discrete conformal factor to achieve a surface of constant curvature. And the difference between this discrete conformal factor and the classical uniformization factor is controlled by the maximal edge length of the triangulation. The estimates rely on collections of discrete elliptic estimates and isoperimetric inequalities for triangle meshes. The case for genus $h \geq 1$ is a joint work with Tianqi Wu, the case for genus $h = 0$ is a joint work with Tianqi Wu and Yanwen Luo.

Acknowledgements

This thesis is the result of work for many years and was generously supported by many people from in and outside the department.

First of all, I am deeply indebted to my advisor, Professor Feng Luo, for his support, encouragement and guidance in the last six years. And I am very grateful for his generosity of sharing ideas and the unpublished manuscripts with me. Without his supervision, this dissertation would have never been written. In the same vein, I would like to thank my collobrators, Yan-wen Luo and Tian-qi Wu, for the illuminating discussion and the generosity to allow me to use our results to write this thesis.

I also want to thank all the members in my defense comittee , Professor Hong-bin Sun, Professor Zheng-chao Han, and Professor David Glickenstein, for their comments on my dissertation. I am also grateful to Professor Xiao-chun Rong, Professor Jian Song and Professor Christopher Woodward for their fascinating courses. I would like to thank Professor Joel Hass for his help in my job search. I would like to thank Professor Edward Chien and Professor David Xian-feng Gu, for their guidance for me to get into the field of computer science.

I would like to thank my friends in the department for their help in the years , including but not limited to Ruo-fan Yan, Zhuo-hui Zhang, Fei Qi, Han-long Fang, Xin Fu, Han Lu, Yu-kun Yao, Xin-di Zhang, Andy Huyhn, Ji-kang Wang, Quentin Dubroff, Soham Chanda, Liu-wei Gong.

I would be remiss not to mention the people I get to know in the sports community. Table Tennis is my favorite sports through the years. I would like to thank Yang-shuang Su and Kai-xuan Ma for coaching me table tennis. I will never forget that Yang-shuang rekindled my interest in table tennis during the pandemic. I am also grateful to my training partners, including but not limited to Yu-chung Lin, Han-ning Li, Le-tao Bo and Ryan Crichton, for the wonderful training through the years. As for tennis, I am super grateful to my coach Ming-xiu Du for leading me into the tennis. I am also grateful to Yin-chuan Zhao for supervising my serve and rally. I would like to thank my training partners for letting me get rapid improvement, espeically Zhao-jun Yong, Shi-long Yang and Rui Huang. Another endeavor that has been a constant during these years is Racer, the Chinese football team. I always enjoy playing the matches together through the years.

I am supper grateful to my friends for their love and support, which has been vital in sustaining me through the hardship of doctoral study and living abroad. I would like to thank Xi-he Ouyang for his open mind. It has always been a pleasure to talk with you, from which I get new perspectives in lots of areas. I would like to thank Shi-yu Peng, for sharing the experience in both PhD life and cyptocurrency investment. I would like to thank De-chang Zheng, for the communication in life experience and history.

In the end, I would like to thank my parents, twin brother and grandfather. Their love is often unspoken, but is always clear to me.

Dedication

To my grandfather Zhu, Li-shan, father Zhu, Jun, mother Li, Xiang-ping and my twin brother Zhu, Xiao-fan.

Table of Contents

Chapter 1

Introduction

In this chapter, we will state the main results and outline the structure of the thesis. For a connected surface M, two Riemannian metric g_1 and g_2 on M are *conformally equivalent* if $g_2 = e^{2u} g_1$ for some smooth function u on M. The celebrated uniformization theorem proved by Poincaré and Köebe in 1907 , states the following.

Theorem 1 (Uniformization Theorem)**.** *Any Riemannian metric g on a connected surface M is conformally equivalent to a complete Riemannian metric $\bar{g}$ of constant curvature ± 1 or 0.*

Remark 1.1. *A Riemannianm metric $\bar{g}$ is called flat, hyperbolic or spherical if its Gaussian curvature is 0, -1, or 1 resepectively. For the complete metric $\bar{g} = e^{2\bar{u}} g$ in the uniformization theorem, we call $\bar{u}$ the* **smooth uniformization factor** *of (M, g).*

The discrete analogy of Riemannian metric is the piecewise polyhedral metric. Given a closed surface S and a finite non-empty set $V \subset S$, we call (S, V) a marked surface. A flat (hyperbolic or spherical) *piecewise linear* (or PL) metric on (S, V) is a flat (hyperbolic or spherical) cone metric on S whose cone points are in V. A triangulation T of S with vertex set V is called a triangulation of (S, V). T is always assumed to be a simplicial complex. We use $E(T), V(T)$ and $F(T)$ to denote the set of edges, set of vertices and set of triangles of T. Furthermore, let $|V|, |E|$ be the number of vertices and edges respectively. Furthermore denote $V_i(T)$ as the sets of interior vertices of T, and $V_b(T)$ as the set of boundary vertices of T.

For a triangulation T and a PL metric d, the edge length of an edge $ij \in E(T)$ is defined as the length of geodesic in d that homotopic to ij relative to endpoints i and j. Therefore d is uniquely determined by its length of edges $l : E(T) \to \mathbb{R}_{>0}$. More discussions about PL metric will be given in Section 2.1.

Suppose M is a compact surface, possibly with boudary and T is a triangulation of M. If M is equipped with a smooth Riemannian metric g, we call T a *geodesic triangulation* if any edge in T is a shortest geodesic arc in (M, g).

For a triangulation T of (S, V), an *admissible edge length function* of T is a function

$$l : E(T) \to \mathbb{R}_{>0}, \quad l(ij) = l_{ij}, \quad \text{for any edge } ij \in E(T)$$

such that for every triangle $\triangle ijk \in F(T)$ the triangle inequalities hold, i.e.

$$l_{ij} + l_{jk} > l_{ik}, \quad l_{jk} + l_{ki} > l_{ij}, \quad l_{ki} + l_{ij} > l_{jk}.$$

For a geodesic triangulation T on a Riemannian surface (M, g), we can naturally define an edge length function l by using the geodesic lengths of the edges. Given an admissible edge length function l of a triangulation T, we can construct a flat PL metric $(T, l)_E$ by isometrically gluing the Euclidean triangles with the edge lengths defined by l along pairs of edges. Similarly, a spherical PL metric $(T, l)_S$ can be constructed by replacing Euclidean triangles with spherical triangles of the same edge lengths, provided that $l_{ij} + l_{jk} + l_{ki} < 2\pi$ for any triangle $\triangle ijk$ in $F(T)$. A hyperbolic PL metric $(T, l)_H$ can also be constructed in a similar way.

For a given $(T, l)_E$ ($(T, l)_H$ or $(T, l)_S$), we use θ^i_{jk} to denote the inner angle at a vertex i in the triangle $\triangle ijk$. The **discrete curvature** K_i at a vertex $i \in V(T)$ is defined as

$$K_i = \begin{cases} 2\pi - \sum_{jk \in E: ijk \in F} \theta^i_{jk}, & \text{if } i \in V_i(T), \\ \pi - \sum_{jk \in E: ijk \in F} \theta^i_{jk}, & \text{if } i \in V_b(T). \end{cases}$$

A flat (hyperbolic or spherical) PL metric is globally flat (*resp.* globally hyperbolic or spherical) if and only if $K_i = 0$ for any vertex $i \in V_i(T)$.

Definition 1.1. *Given a triangulation T, a* discrete conformal factor *u is a real-valued function on $V(T)$. For the Euclidean case, $(T, l)_E$ and $(T, l')_E$ are* **discrete conformally equivalent** *if for some discrete conformal factor u,*

$$l'_{ij} = e^{\frac{1}{2}(u_i + u_j)} l_{ij} \tag{1.1}$$

for any edge $ij \in E(T)$. For the hyperbolic case, $(T,l)_H$ and $(T,l')_H$ are **discrete conformally equivalent** *if for some discrete conformal factor u,*

$$\sinh\frac{l'_{ij}}{2} = e^{\frac{1}{2}(u_i+u_j)}\sinh\frac{l_{ij}}{2} \tag{1.2}$$

for any edge $ij \in E(T)$. For the spherical case, $(T,l)_S$ and $(T,l')_S$ are **discrete conformally equivalent** *if for some discrete conformal factor u,*

$$\sin\frac{l'_{ij}}{2} = e^{\frac{1}{2}(u_i+u_j)}\sin\frac{l_{ij}}{2} \tag{1.3}$$

for any edge $ij \in E(T)$.

Remark 1.2. *The discrete conformally equivalence defined above requires the fixed triangluation condition. The Euclidean notion was first introduced by Luo [1]. The hyperbolic and spherical notion was first introduced by Bobenko et.al [2].*

We denote $l' = u * l$ if equation (1.1) holds, $l' = u *_h l$ if equation (1.2) holds and $l' = u *_s l$ if equation (1.3) holds. Given a PL metric $(T,l)_E$ ($(T,l)_H$ or $(T,l)_S$), let $\theta^i_{jk}(u)$ and $K_i(u)$ denote the corresponding inner angle at vertex i in triangle $\triangle ijk$ and the discrete curvature at i respectively in $(T, u*l)_E$ ($(T, u *_h l)_H$ or $(T, u *_s l)_S$). For a PL flat metric $(T,l)_E$, the *area* of $(T,l)_E$ is defined as

$$\mathrm{Area}(T,l)_E = \sum_{\triangle ijk \in F(T)} |\triangle ijk|_E,$$

where $|\triangle ijk|_E$ is the area of Euclidean triangle $\triangle ijk$ with edge length l.

If $(T,l)_E$ is a topological torus, u is called the *discrete uniformization factor* for $(T,l)_E$ if $(T, u*l)_E$ is isometric to the flat torus of unit area, which is equivalent to that the discrete curvature $K(u) := [K_i(u)]_{i\in V(T)}$ is zero and $\mathrm{Area}(T, u*l)_E = 1$. If $(T,l)_H$ is topologically a closed surface of genus $g > 1$, u is called the *discrete uniformization factor* for $(T,l)_H$, if $(T, u *_h l)_H$ is isometric to the closed surface of hyperbolic metric with genus g, which is equivalent to that the discrete curvature $K(u) := [K_i(u)]_{i\in V(T)}$ is zero. If $(T,l)_S$ is a topological sphere, u is called the *discrete uniformization factor* for $(T,l)_S$, if $(T, u *_s l)_S$ is isometric to the unit sphere, which is equivalent to that the discrete curvature $K(u) := [K_i(u)]_{i\in V(T)}$ is zero.

Definition 1.2. *A PL metric $(T,l)_E$ ($(T,l)_H$ or $(T,l)_S$) is called* strictly Delaunay *if for any edge ij of T, two adjacent triangles $\triangle ijk, \triangle ijk'$ sharing edge ij satisfy*

$$\theta_{ij}^k + \theta_{ij}^{k'} < \theta_{jk}^i + \theta_{jk'}^i + \theta_{ik}^j + \theta_{ik'}^j. \tag{1.4}$$

Remark 1.3. *This is an equivalent definition for the traditional Delaunay triangulation and first discovered by Leibon [3]. We will discuss more about Delaunay triangulation in Section 2.1.*

For our convergence result, we introduce the $\epsilon-$regularity to quantify the extent of uniformity of a Delaunay triangulation.

Definition 1.3. *A PL metric $(T,l)_E$($(T,l)_H$ or $(T,l)_S$) is called* ϵ-regular *if*

(a) any inner angle $\theta_{jk}^i \geq \epsilon$, and

(b) for any adjacent triangles $\triangle ijk$ and $\triangle ijl$, $\theta_{ij}^k + \theta_{ij}^l \leq \pi - \epsilon$.

Condition (a) requires that any triangle is away from degenerating, and condition (b) requires a PL metric with sufficiently small edge length to be uniformly strictly Delaunay.

Let $||x|| = \max_{i \in I} |x_i|$ denote the maximal norm of a vector $x \in \mathbb{R}^I$ in a finite dimensional vector space.

In the work by Colin de Verdiére [4], a family of strictly acute triangulations on any closed Riemannian surface with explicit lower bounds on angles were constructed, and the maximal edge lengths of these acute triangulations approach zero. This implies the existence of ϵ-regular geodesic triangulations on any closed Riemannian surface for some ϵ.

Our main results prove the existence of discrete uniformization factor for a reasonable PL metric on a closed surface. And also the difference between the discrete uniformization factor and the classical uniformization function is controlled by the $||l||$ of T. More precisely,

Theorem 2. *Suppose (M,g) is a closed orientable smooth Riemannian surface with genus > 1, and $\bar{u} = \bar{u}_{M,g} \in C^\infty(M)$ is the unique uniformization factor of (M,g).*

Assume T is a geodesic triangulation of (M,g) of geodesic edge length l. Then for any $\epsilon>0$, there exist constants $\delta=\delta(M,g,\epsilon)>0$ and $C=C(M,g,\epsilon)>0$ such that if $(T,l)_H$ is ϵ-regular with $||l||<\delta$, then

(a) there exists a discrete uniformization factor $u\in\mathbb{R}^{V(T)}$ for $(T,l)_H$, and

(b) $||u-\bar{u}|_{V(T)}||\le C||l||$.

Theorem 3. *Suppose (M,g) is a closed orientable smooth Riemannian surface of genus 1, and $\bar{u}=\bar{u}_{M,g}\in C^\infty(M)$ is the unique smooth uniformization factor of (M,g) with $Area(M,e^{2\bar{u}}g)=1$. Assume T is a geodesic triangulation of (M,g) of geodesic edge length function l. Then for any $\epsilon>0$, there exist constants $\delta=\delta(M,g,\epsilon)>0$ and $C=C(M,g,\epsilon)$ such that if $(T,l)_E$ is ϵ-regular and $||l||<\delta$, then*

(a) there exists a discrete uniformization factor $u\in\mathbb{R}^{V(T)}$ for $(T,l)_E$, and

(b) $||u-\bar{u}|_{V(T)}||\le C||l||$.

Let $\hat{\mathbb{C}}$ be the standard Riemann sphere, which can be identified with the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ by the stereographic projection, the theorem for spherical PL metric is

Theorem 4. *Suppose (M,g) is a closed smooth Riemannian surface of genus zero with three marked points X,Y,Z, and $\bar{u}\in C^\infty(M)$ is the unique uniformization conformal factor such that $(M,e^{2\bar{u}}g)$ is isometric to the unit sphere $\mathbb{S}^2\cong\hat{\mathbb{C}}$ via isometric map ϕ, where $\phi(Z)=0$, $\phi(Y)=1$, $\phi(X)=\infty$. Assume T is a geodesic triangulation of (M,g) of geodesic edge length l such that its one-skeleton is a 4-vertex-connected graph. Then for any $\epsilon>0$, there exist constants $\delta=\delta(M,g,X,Y,Z,\epsilon)>0$ and $C=C(M,g,X,Y,Z,\epsilon)>0$ such that if $(T,l)_S$ is ϵ-regular and $||l||\le\delta$, then*

(a) there exists a unique discrete conformal factor u on $V(T)$, such that $(T,u*_s l)_S$ is strictly Delaunay and isometric to the unit sphere via an isometric map ψ such that $\psi(Z)=0$, $\psi(Y)=1$, and $\psi(X)=\infty$, and

(b) $||u-\bar{u}|_{V(T)}||\le C||l||$.

Remark 1.4. *The uniqueness part of the theorem is already known as a consequence of Springborn's Theorem 10.5 in [5], which is equivalent to Rivin's earlier result on hyperbolic polyhedral realization in [6].*

The organization of the thesis is as follows.

In chapter 2, we will briefly introduce basic background for this dissertation.

In chapter 3, we will cover the key ingredients for the proof of our main theorems. The *C-isoperimetric* condition and discrete elliptic esimates will be introduced.

In chapters 4,5,6, we will prove convergence results for flat, hyperbolic and spherical PL metric.

Chapter 2

Preliminaries

2.1 Some conventions and Delaunay triangulations

The following is a more detailed explanation for the definition of PL metric in the Chapter 1.

Take a finite disjoint union I of triangles and identify its edges in pairs by homeomorphisms. The quotient space S is a compact surface with a triangulation T whose simplices are the quotients of the simplices in the disjoint union I. We call T a triangulation of the marked surface (S, V). If each triangle in the disjoint union I is Euclidean and the identification maps are isometries, then the quotient metric d on (S, V) is called a *flat piecewise linear (or PL) metric*. The *hyperbolic PL metric* and *spherical PL metric* are defined in a similar way. Let (S, V, d) denote this polyhedral surface, i.e. the marked surface (S, V) equipped with the PL metric d. Given a flat (hyperbolic or spherical) PL metric d and a triangulation T on (S, V), if each triangle in T (in metric d) is isometric to a Euclidean(hyperbolic or spherical) triangle, we say T is geodesic in d.

Suppose e is an edge in T adjacent to two distinct triangles t_1, t_2. Then the *diagonal switch* on T is a new triangulation T' obtained from T by replaces an edge e by the other diagonal in the quadrilateral $t_1 \cup_e t_2$. See Figure 2.1. The diagonal switch changes the set of edges and faces of T but preserves the set of vertices.

A geodesic triangulation T of a (S, V, d) is called *Delaunay* if for each edge e in T with two adjacent triangle t_1 and t_2 the interior of the circumscribled circle of t_1 does not contain the vertices of t_2 after we lift t_1 and t_2 to the universal cover. In [3], Leibon gave an algebraic description for the Delaunay condition for hyperbolic PL surface as

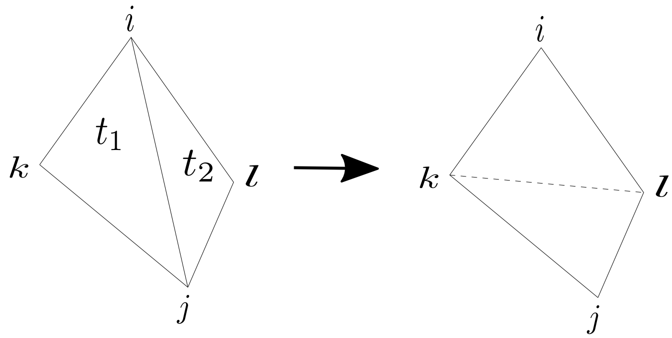

Figure 2.1: Diagonal switch - replace the diagonal ij by the other diagonal kl

in Definition 1.2. The criterion (1.4) also applies for flat PL surface and spherical PL surface. We say an edge ij in T is a *Delaunay edge* if the inequality 1.4 holds. Therefore we can operate a diagonal switch on T to make a non-Delaunay edge to be Delaunay.

We present two related results for Delaunay triangulations on PL surfaces. See Bobenko-Springborn [7] and Gu et.al. [8] for proof.

Proposition 1. *[7]*

(a) Each flat PL metric on (S, V) has a Delaunay triangulation.

(b) If T and T' are Delaunay triangulations of a flat PL metric d, then there exists a sequence of Delaunay triangulations $T_1 = T, T_2, \cdots, T_k = T'$ of d so that T_{i+1} is obtained from T_i by a diagonal switch.

Proposition 2. *[8]*

(a) Each hyperbolic PL metric on (S, V) has a Delaunay triangulation.

(b) If T and T' are Delaunay triangulations of a hyperbolic polyhedral metric d on a closed marked surface (S, V), then there exists a sequence of Delaunay triangulations $T_1 = T, T_2, \cdots, T_k = T'$ of d so that T_{i+1} is obtained from T_i by a diagonal switch.

(c) Suppose T is a Delaunay triangulation of a compact hyperbolic polyhedral surface (S, V, d) whose diameter is D. Then the length of each edge e in T is at most $2D$. In particular, there exists an algorithm to find all Delaunay triangulations of a hyperbolic polyhedral surface.

2.2 Discrete Uniformization Theorems

2.2.1 Related works

The smooth unifomization theorem shows that there exists a complete uniformization metric of the constant curvature conformally equivalent to a given Riemannian metric. The natural analogy of this theorem in discrete setting is for a prescribled discrete curvature condition K^* and given polyhedral metric d on a marked surface (S, V), can we find a PL metric d^* realizes the prescribled curvature condition in the discrete conformal class of the given polyhedral metric d? Unfortunately, the solution to the prescribled curvature problem in the discrete conformal class defined in Definition 1.1 is not guaranteed to solve the precribled curvature problem. But this problem can be fixed by requiring diagonal switch for the triangulation in the definition of discrete conformal class.

The precise definition of the discrete conformal class mentioned above, introduced by Gu et.al.[9], is as follows.

Definition 2.1. *Let (T, l) and (T', l') be two piecewise flat (hyperbolic or spherical) polyhedral metrics on marked surface (S, V). We call (T, l) and (T', l') are discrete conformal if there exists a sequence of PL metrics $l_1 = d_1, d_2, \cdots, d_m = l'$ and triangulations $T = T_1, T_2, \cdots, T_m = T'$ on (S, V) satisfying*

(1) (Delaunay condition) each T_i is Delaunay in l_i

(2) (Vertex scaling condition) if $T_i = T_{i+1}$, then for any edge $e \in E(T)$

$$\begin{aligned} &l_{i+1}(e) = e^{\frac{u_i+u_j}{2}} l_i(e) && \textit{for flat PL metric} \\ &\sinh(l_{i+1}(e)/2) = e^{\frac{u_i+u_j}{2}} \sinh(l_i(e)/2) && \textit{for hyperbolic PL metric} \\ &\sin(l_{i+1}(e)/2) = e^{\frac{u_i+u_j}{2}} \sin(l_i(e)/2) && \textit{for spherical PL metric} \end{aligned}$$

for some vertex function $u : V \to \mathbb{R}$.

(3) If $T_i \neq T_{i+1}$, then (T_i, l_i) is isometric to (T_{i+1}, l_{i+1}) by an isometry homotopic to an identity in (S, V).

Under this definition, Gu et.al [9] solved the prescribled curvature problem for piecewise flat polyhedral metric on closed surfaces. Specifically, their theorem (Thorem 1.2 in [9]) is as in the following.

Theorem 5. *Suppose (S,V) is a closed connected marked surface and d is any PL metric on (S,V). Then for any $K^*: V \to (-\infty, 2\pi)$ with $\sum_{v\in V} K^*(v) = 2\pi\chi(S)$, there exists a PL metric d_0 ,unique up to scaling,on (S,V)so that d_0 is discrete conformal to d and the discrete curvature of d_0 is K^*.*

The prescribled curvature problem for piecewise hyperbolic metric was solved by Gu et.al in [8].

Theorem 6. *Suppose (S,V) is a closed connected marked surface and d is any PL hyperbolic metric on (S,V). Then for any $K^*: V \to (-\infty, 2\pi)$ with $\sum_{v\in V} K^*(v) > 2\pi\chi(S)$, there exists an unique PL metric d_0 ,on (S,V)so that d_0 is discrete conformal to d and the discrete curvature of d_0 is K^*.*

Remark 2.1. *The case $K^* \equiv 0$ was first proved by Fillastre [10].*

Remark 2.2. *In our convergence theorems, the discrete conformality between PL metrics requires the fixed triangulation without diagonal switch.*

The discrete uniformization for spherical PL metric is derived from the following theorem by Rivin [6].

Theorem 7 ([6])**.** *Every complete hyperbolic surface S of finite area that is homeomorphic to a punctured sphere can be realized as a convex ideal polyhedron in three-dimensional hyperbolic space $\mathbb{H}^3$. The realization is unique up to isometries of $\mathbb{H}^3$.*

In [5], Springborn gave an equivalent description to Theorem 7.

Theorem 8. *(discrete uniformization of spheres). For every piecewise euclidean metric d on the marked 2-sphere (S_0,V), there is a realization of (S_0,V) as a convex euclidean polyhedron P with vertex set V, such that all vertices lie on the unit sphere and the induced piecewise euclidean metric is discretely conformally equivalent to d. The polyhedron P is unique up to projective transformations of $RP^3 \supset \mathbb{R}^3$ mapping the unit sphere to itself.*

Remark 2.3. *The global spherical metric we achieved in theorem 4 actually induces the polyhedral metric in this theorem.*

2.2.2 Relation to hyperbolic geometry

In [2], Bobenko et.al introduced a nice way to connect discrete conformality between flat PL metrics with the hyperbolic geometry.

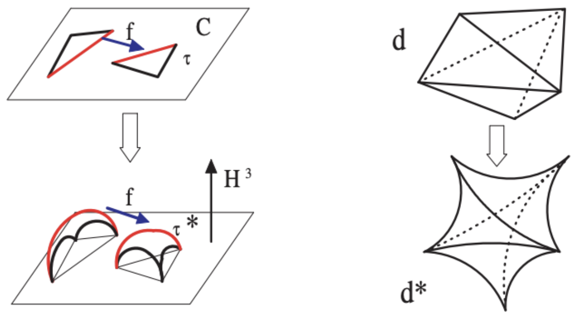

Figure 2.2: Discrete conformality in terms of hyperbolic geometry

For a flat PL metric $(T, l)_E$ of (S, V), and for each $\tau = \triangle ijk$ in $F(T)$, replace τ by an ideal hyperbolic triangle τ^* in the hyperbolic $3-$space $\mathbb{H}^3$ such that τ^* and τ has the same set of vertices $\{i, j, k\}$ in the complex plane $\mathbb{C}$. Here we consider $\mathbb{H}^3$ as $\mathbb{C} \times \mathbb{R}_{>0}$. If two triangles τ and σ are glued along their sharing edge e by a Euclidean isometry f, then we glue two ideal hyperbolic triangles τ^* and σ^* along the correponding edges e^* by the same isometry f, where f is being considered as a rigid motion in $\mathbb{H}^3$. Therefore we can construct a hyperbolic metric d^* on $S \backslash V$. See Figure 2.2.

Their definition for the discrete conformal class of flat polyhedral metric with same combinatorics is as follows.

Definition 2.2. *Two flat PL metric $(T, l)_E$ and $(T, \bar{l})_E$ with the same combinatorics are discretely conformally equivalent if and only if the hyperbolic metrics with cusps induced by the circumcircles are isometric.*

From Proposition 1, every flat PL metric d on (S, V) has an associated Delaunay triangulation. For a given flat PL metric d, we can take a Delaunay triangulation T of d and construct a hyperbolic metric d^* on $S \backslash V$ as the process shown in Figure 2.2.

The d^* is independent of the choice of triangulation. In [9], Gu et.al. proved that two flat PL metrics d_1 and d_2 on (S, V) are discrete conformal if d_1^* is isometric to d_2^* by an isometry homotopic to the identity on (S, V).

Chapter 3

Estimates on polyhedral surfaces

Our main purpose in this chapter is to prove the C-isoperimetric condition for graphs on closed surfaces and the discrete elliptic estimate (theorem 9). In the section 3.2, we will also prove several geometric lemmas that will be used in the proof of convergence theorems.

3.1 Calculus on Graph

In this section, we will introduce some basic operators and lemmas about the graph.

Assume $G = (V, E)$ is an undirected connected graph. Let ij denote an edge $e \in E$ with endpoints i and j and $i \sim j$ denote i are connected by an edge $ij \in E$. Let $\mathbb{R}^E$ and $\mathbb{R}^E_A$ be vector spaces of dimension $|E|$ such that

(a) a vector $x \in \mathbb{R}^E$ is represented symmetrically, i.e., $x_{ij} = x_{ji}$, and

(b) a vector $x \in \mathbb{R}^E_A$ is represented anti-symmetrically, i.e., $x_{ij} = -x_{ji}$, which can also be called a *flow* on G.

An *edge weight* η on G is a vector in $\mathbb{R}^E$. Given an edge weight η, the *gradient* $\nabla f = \nabla_\eta f$ of a vector $f \in \mathbb{R}^V$ is a flow in $\mathbb{R}^E_A$ defined as:

$$(\nabla f)_{ij} = \eta_{ij}(f_j - f_i).$$

Given a flow $x \in \mathbb{R}^E_A$, its *divergence* $div(x)$ is a vector in $\mathbb{R}^V$ such that

$$div(x)_i = \sum_{j \sim i} x_{ij}.$$

Given an edge weight η, the associated *Laplacian* $\Delta = \Delta_\eta : \mathbb{R}^V \to \mathbb{R}^V$ is defined as

$\Delta f = div(\nabla_\eta f)$, i.e.,

$$(\Delta f)_i = \sum_{j\sim i} \eta_{ij}(f_j - f_i).$$

Similar to the Green's identity in PDE, we have a discrete Green's identity on graphs.

Proposition 3. *(Green's identity) Given $x, y \in \mathbb{R}^V$,*

$$\sum_{i\in V} x_i(\Delta y)_i = \sum_{i\in V} y_i(\Delta x)_i.$$

Proof.

$$\sum_{i\in V} x_i(\Delta y)_i = \sum_{i\in V} x_i \sum_{j\sim i} \eta_{ij}(y_j - y_i) = \sum_{ij\in E} \eta_{ij} x_i y_j - \sum_{i\in V} x_i y_i \sum_{j\sim i} \eta_{ij}.$$

Then Green's identity holds by symmetry. □

A Laplacian Δ_η is a linear transformation on $\mathbb{R}^V$, and can be identified as a $|V|\times|V|$ symmetric matrix. By definition, $\mathbf{\Delta 1} = 0$ where $\mathbf{1} = (1, 1, ..., 1) \in \mathbb{R}^V$. On the other hand, for a nonegative edge weight $\eta \in \mathbb{R}^E_{>0}$, $ker(\Delta) = \mathbb{R}\mathbf{1}$ by the connectedness of the graph G.

In the rest of this section, we always assume $\eta \in \mathbb{R}^E_{>0}$. In this case, Δ is invertible on the subspace $\mathbf{1}^\perp = \{x \in \mathbb{R}^V : \sum_{i\in V} x_i = 0\}$. We denote Δ^{-1} by the inverse of Δ on $\mathbf{1}^\perp$. The following regularity property is necessary in the proof of our convergence theorem.

Lemma 3.1. *$(\eta, y) \mapsto \Delta_\eta^{-1} y$ is a smooth map from $\mathbb{R}^V_{>0} \times \mathbf{1}^\perp$ to $\mathbf{1}^\perp$.*

Proof. It is equivalent to show that $\Phi(\eta, y) = (\eta, \Delta_\eta^{-1} y)$ is a smooth mapping from $\mathbb{R}^V_{>0} \times \mathbf{1}^\perp$ to itself. By the inverse function theorem, it suffices to show that $\Phi^{-1}(\eta, x) = (\eta, \Delta_\eta x)$ is smooth and $D(\Phi^{-1})$ is non-degenerate. The smoothness is obvious, and

$$D(\Phi^{-1}) = \begin{pmatrix} id & \partial\eta/\partial x \\ \partial(\Delta_\eta x)/\partial\eta & \partial(\Delta_\eta x)/\partial x \end{pmatrix} = \begin{pmatrix} id & 0 \\ \partial(\Delta_\eta x)/\partial\eta & \Delta_\eta \end{pmatrix}$$

is indeed nondegenerate, since Δ_η is invertible on $\mathbf{1}^\perp$. □

Analogous to the isoperimetric condition on a Riemannian surface, we introduce the notion of C-isoperimetric conditions for a graph $G = (V, E)$ with edge length $l \in \mathbb{R}^E_{>0}$.

Given $V_0 \subset V$, we denote

$$\partial V_0 = \{ij \in E : i \in V_0, \text{ and } j \notin V_0\}$$

(See Figure 3.1 below), and define the *l-perimeter* of V_0 and the *l-area* of V_0 as

$$|\partial V_0|_l = \sum_{ij \in \partial V_0} l_{ij}, \quad \text{and} \quad |V_0|_l = \sum_{ij \in E: i,j \in V_0} l_{ij}^2.$$

Given a constant $C > 0$, a pair (G, l) is called *C-isoperimetric* if for any $V_0 \subset V$,

$$\min\{|V_0|_l, |V|_l - |V_0|_l\} \leq C|\partial V_0|_l^2.$$

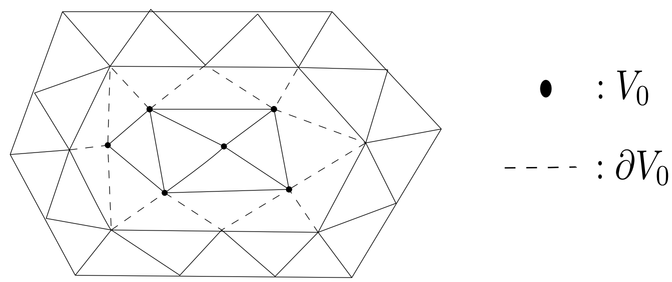

Figure 3.1: The C-isoperimetric condition on graphs.

For a triangulation T of a marked surface (S, V), let T^1 be its one skeleton. For a given edge length function l on $E(T)$, (T^1, l) can be seen as a graph with edge length l. We say a PL metric $(T, l)_E((T, l)_H)$ is $C-$ isoperimetric if (T^1, l) is $C-$isoperimetric.

We will see, from part (b) of Theorem 10, that a uniform C-isoperimetric condition is satisfied by regular polyhedral surfaces approximating a closed smooth surface.

Also, the following discrete elliptic estimate plays an important role in proving our main theorems. The proof will be covered in Chapter 3.

Theorem 9. *Assume (G, l) is C_1-isoperimetric, and $x \in \mathbb{R}_A^E, \eta \in \mathbb{R}_{>0}^E, C_2 > 0, C_3 > 0$ such that*

1. *$|x_{ij}| \leq C_2 l_{ij}^2$ for any $ij \in E$, and*

2. *$\eta_{ij} \geq C_3$ for any $ij \in E$.*

Then

$$||\Delta_\eta^{-1} \circ div(x))|| \leq \frac{4C_2\sqrt{C_1+1}}{C_3}||l|| \cdot |V|_l^{1/2}.$$

Furthermore if $y \in \mathbb{R}^V$, $C_4 > 0$ *and a diagonal matrix* $D \in \mathbb{R}^{V\times V}$ *satisfying*

$$|y_i| < C_4 D_{ii}||l|| \cdot |V|_l^{1/2}$$

for any $i \in V$, *then*

$$||(D-\Delta_\eta)^{-1}(div(x)+y)|| \leq \left(C_4 + \frac{8C_2\sqrt{C_1+1}}{C_3}\right)||l|| \cdot |V|_l^{1/2}.$$

3.2 Geometric lemmas

For a triangle $\triangle ABC$, we will use A to represent the vertex A or the angle $\angle A$. Furthermore, $\angle_E A, \angle_H A$ and $\angle_S A$ represents the angle at vertex A in a flat, hyperbolic and spherical geodesic triangle respectively. Also, let $|\triangle ABC|$ be the area of $\triangle ABC$.

Lemma 3.2. *Suppose* $\triangle_E ABC$, $\triangle_H ABC$ *and* $\triangle_S ABC$ *are Euclidean, hyperbolic and spherical triangles respectively, with the same edge lengths* $a, b, c < 0.1$.

(a) If all the inner angles in $\triangle_E ABC$ *are at least* $\epsilon > 0$, *then for any* $P \in \{E, H, S\}$,

$$\frac{\epsilon}{8}a^2 \leq |\triangle_P ABC| \leq \frac{1}{\epsilon}a^2.$$

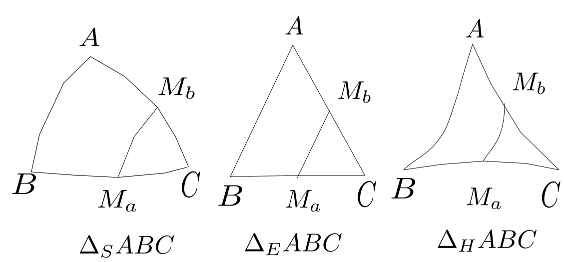

Figure 3.2: Triangles in Lemma 3.2

(b) Assume M_a *is the middle point of* BC, *and* M_b *is the middle point of* AC, *and* $\triangle_P CM_aM_b$ *is the geodesic triangle in* $\triangle_P ABC$ *with vertiecs* C, M_a, M_b, *where* $P \in \{E, H, S\}$. *Then*

$$|\triangle_P CM_aM_b| \geq \frac{1}{5}|\triangle_P ABC|$$

for any $P \in \{E, H, S\}$.

Remark 3.1. *By the well-known Toponogov comparison theorem (see Lemma 3.4), the assumption in part (a) of Lemma 3.2 can be replaced by that all the inner angles in* $\triangle_H ABC$ *are at least* $\epsilon > 0$*.*

Proof of (a). We begin with three well-known Heron's formulae for Euclidean, hyperbolic and spherical triangles.

$$|\triangle_E ABC|^2 = s(s-a)(s-b)(s-c),$$

$$\tan^2 \frac{|\triangle_H ABC|}{4} = \tanh\frac{s}{2}\tanh\frac{s-a}{2}\tanh\frac{s-b}{2}\tanh\frac{s-c}{2}, \tag{3.1}$$

$$\tan^2 \frac{|\triangle_S ABC|}{4} = \tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}, \tag{3.2}$$

where $s = \frac{a+b+c}{2}$.

The hyperbolic Heron's formula can be found in Theorem 1.1 in [12], The spherical one is also called L'Huilier's Theorem and can be found in Section 4.19.2 in [13].

Notice that $|\triangle_E ABC| \leq a^2 + b^2 + c^2 \leq 0.03$, and for $x \in [0, 0.1]$,

$$\frac{\tanh x}{x} \in (0.99, 1) \quad \text{and} \quad \frac{\tan x}{x} \in (1, 1.01).$$

So by the three parallel Heron's formulae and simple approximation estimates, we only need to show the following stronger estimates (3.3) and (3.4) for the Euclidean case. By the law of sines in the Euclidean triangle $\triangle_E ABC$,

$$b = \frac{a \sin\angle_E B}{\sin\angle_E A} \leq \frac{a}{\sin\epsilon} \leq \frac{\pi}{2\epsilon}a.$$

So

$$|\triangle_E ABC| = \frac{1}{2}ab\sin C \leq \frac{1}{2}a \cdot \frac{\pi}{2\epsilon}a = \frac{\pi}{4}\frac{a^2}{\epsilon}. \tag{3.3}$$

By the triangle inequality, we may assume $b \geq a/2$ without loss of generality, and then

$$|\triangle_E ABC| = \frac{1}{2}ab\sin C \geq \frac{1}{2}a \cdot \frac{a}{2} \cdot \sin\epsilon \geq \frac{\epsilon}{2\pi}a^2. \tag{3.4}$$

$\square$

Proof of (b). The Euclidean case is obvious. To prove the hyperbolic and spherical cases, we use the following two formulae

$$\cot\frac{|\triangle_H ABC|}{2} = \frac{\coth\frac{a}{2}\coth\frac{b}{2} - \cos\angle_H C}{\sin\angle_H C}, \tag{3.5}$$

$$\cot\frac{|\triangle_S ABC|}{2}=\frac{\cot\frac{a}{2}\cot\frac{b}{2}+\cos\angle_S C}{\sin\angle_S C},\tag{3.6}$$

where equation (3.5) was developed in Theorem 6 of [14]. The equation (3.6) can be obtained by

$$\cot\frac{|\triangle_S ABC|}{2}=\cot\frac{\angle_S A+\angle_S B+\angle_S C-\pi}{2}=-\tan(\frac{\angle_S A+\angle_S B}{2}+\frac{\angle_S C}{2})$$

and the well-known Napier's analogies

$$\tan\frac{\angle_S A+\angle_S B}{2}=\cot\frac{C}{2}\cdot\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}.$$

Here we only prove the hyperbolic case using equation (5.4) and the proof for the spherical case is very similar. Firstly we apply the formula (3.5) to $\triangle_H CM_aM_b$ and get

$$\cot\frac{|\triangle_H CM_aM_b|}{2}=\frac{\coth\frac{a}{4}\coth\frac{b}{4}-\cos\angle_H C}{\sin\angle_H C}.$$

Then

$$\frac{\tan\frac{\triangle_H CM_aM_b}{2}}{\tan\frac{\triangle_H ABC}{2}}=\frac{\coth\frac{a}{2}\coth\frac{b}{2}-\cos\angle_H C}{\coth\frac{a}{4}\coth\frac{b}{4}-\cos\angle_H C}\geq\frac{(2/a)(2/b)-1}{(4/a)(4/b)/0.99^2+1}=\frac{4-ab}{16/0.99^2+ab}\geq\frac{1}{5}.$$

Since $|\triangle_H CM_aM_b|\leq|\triangle_H ABC|$ and $\frac{\tan x}{x}$ is increasing on $(0,\infty)$,

$$\frac{|\triangle_H CM_aM_b|}{|\triangle_H ABC|}\geq\frac{\tan\frac{\triangle_H CM_aM_b}{2}}{\tan\frac{\triangle_H ABC}{2}}\geq\frac{1}{5}.$$

□

The following Lemma 3.3, first proved by Gu-Luo-Wu (see Proposition 5.2 in [15]), indicates that the discrete conformal change $l'_{ij}=e^{(u_i+u_j)/2}l_{ij}$ is close to the continuous conformal change with the error of cubic terms. For a Riemannian surface (M,g), we use $d_g(x,y)$ to denote the geodesic distance between x and y under metric g.

Lemma 3.3. *Suppose (M,g) is a closed Riemannian surface, an $u\in C^\infty(M)$ is a conformal factor. Then there exists $C=C(M,g,u)>0$ such that for any $x,y\in M$,*

$$|d_{e^{2u}g}(x,y)-e^{\frac{1}{2}(u(x)+u(y))}d_g(x,y)|\leq Cd_g(x,y)^3.$$

The following lemma states the angle of a geodesic triangle with bounded curvature can be bounded by the angle of triangle in the model space with same edge length.

Lemma 3.4. *Assume $\triangle ABC, \triangle A'B'C', \triangle A''B''C''$ are geodesic Riemannian triangles with the same edge lengths a, b, c such that $\triangle A'B'C'$ has negative constant curvature $-K$ and $\triangle A''B''C''$ has positive constant curvature K and the curvature of $\triangle ABC$ is always in $[-K, K]$. If*

$$\max\{a,b,c\} < \frac{\pi}{2\sqrt{K}},$$

then

$$A' \leq A \leq A''.$$

This lemma is a combination of the well-known Toponogov comparison theorem and the CAT(K) Theorem. See Theorem 79 on page 339 in [16] for the Toponogov comparison theorem, and Characterization Theorem on page 704 in [17] or Theorem 1A.6 on page 173 in [18] for the CAT(K) Theorem.

The following Lemma gives an explicit bound for the difference between the corresponding angles for two geodesic triangles with same edge length and same curvature bound.

Lemma 3.5. *Assume $\triangle ABC$ and $\triangle A'B'C'$ are two geodesic Riemannian triangles with the same edge lengths a, b, c such that Gaussian curvature on $\triangle ABC$ and $\triangle A'B'C'$ are both bounded in $(-K, K)$. If $\max\{a,b,c\} < \frac{\pi}{3\sqrt{K}}$, then*

$$|A' - A| \leq 2(a+b+c)^2 K.$$

Proof. By Lemma 3.4, without loss of generality, we may assume that $\triangle ABC$ has constant curvature $-K$ and $\triangle A'B'C'$ has constant curvature K. Then

$$A' - A > 0, \quad B' - B > 0, \quad C' - C > 0,$$

and by the Gauss-Bonnet theorem

$$0 < A' - A \leq (A' + B' + C') - (A + B + C) = K \cdot \big(|\triangle A'B'C'| + |\triangle ABC|\big).$$

By a simple scaling, the Heron's formulae can be generalized to the following

$$\tan^2 \frac{|\triangle ABC| \cdot K}{4} = \tanh \frac{s\sqrt{K}}{2} \tanh \frac{(s-a)\sqrt{K}}{2} \tanh \frac{(s-b)\sqrt{K}}{2} \tanh \frac{(s-c)\sqrt{K}}{2},$$

$$\tan^2\frac{|\triangle A'B'C'|\cdot K}{4}=\tan\frac{s\sqrt{K}}{2}\tan\frac{(s-a)\sqrt{K}}{2}\tan\frac{(s-b)\sqrt{K}}{2}\tan\frac{(s-c)\sqrt{K}}{2}\leq s^4K^2,$$

where $s=(a+b+c)/2$. So

$$|\triangle ABC|\leq|\triangle A'B'C'|\leq\frac{4}{K}\tan\frac{|\triangle A'B'C'|\cdot K}{4}\leq\frac{4}{K}\cdot s^2K=(a+b+c)^2$$

and we are done. □

3.3 C-isoperimetric condition

In this section, we will prove the following theorem.

Theorem 10. *Suppose (M,g) is a closed Riemannian surface, and T is a geodesic triangulation of (M,g) with geodesic edge length l. Assume $(T,l)_E$ or $(T,l)_H$ is ϵ-regular. Given a conformal factor $u\in C^\infty(M)$, there exist constants $\delta=\delta(M,g,u,\epsilon)>0$ and $C=C(M,g,\epsilon)$, such that if $||l||<\delta$, then*

(a) there exists a geodesic triangulation T' in $(M,e^{2u}g)$ such that $V(T')=V(T)$, and T' is homotopic to T relative to $V(T)$. Further $(T,\bar{l})_E$ and $(T,\bar{l})_H$ are $\frac{1}{2}\epsilon$-regular where $\bar{l}\in\mathbb{R}^{E(T)}\cong\mathbb{R}^{E(T')}$ denotes the geodesic lengths of the edges of T' in $(M,e^{2u}g)$.

(b) (T,l) is C-isoperimetric.

First, we will prove the existence of geodesic representative of T in $(M,e^{2u}g)$, which is included in the following lemma.

Lemma 3.6. *Suppose (M,g) is a closed Riemannian surface, and $u\in C^\infty(M)$ is a conformal factor, then for any $\epsilon>0$, there exists $\delta=\delta(M,g,u)>0$ such that for any $x,y\in M$ with $d_g(x,y)<\delta$,*

(a) there exists a unique shortest geodesic segment l in (M,g), and l' in $(M,e^{2u}g)$, connecting x and y, and

(b) the angle between l and l' at x, measured in (M,g), is less or equal to ϵ.

Proof. Assume $K(x)$ is the Gaussian curvature of (M,g) at x, and $\|K\|_\infty=\max_{x\in M}|K(x)|$. Let $B_g(x,\delta):=\{y\in M|d_g(x,y)\leq\delta\}$ and $|B_g(x,\delta)|_g$ be its area under metric g. It is

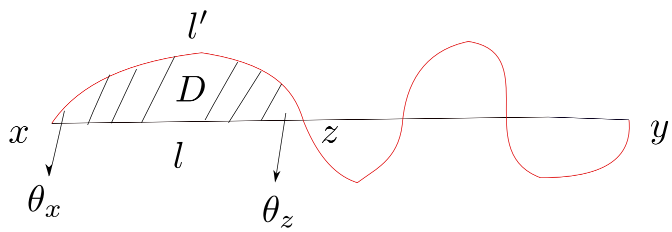

Figure 3.3: Geodesics in the proof of Lemma 3.6

easy to find a sufficiently small δ such that (a) is satisfied, and for any $x \in M$,

$$|B_g(x,\delta)|_g \cdot \|K\|_\infty < \epsilon/2.$$

For the unit circle bundle

$$A = \{(x, \vec{a}) \in TM : x \in M, \|\vec{a}\|_{e^{2u}g} = 1\},$$

under the local coordinates (v_1, v_2), let Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$ are Christoffel symbols for g and$e^{2u}g$ respectively. Then for any geodesic $\gamma(t) = (v_1(t), v_2(t))$ in $(M, e^{2u}g)$ with $\gamma(0) = x$ and $\gamma'(0) = \vec{a}$, the geodesic curvature k_g of γ in (M, g) at point x is

$$\frac{-\sqrt{g_{11}g_{22} - g_{12}^2}(-\Gamma^2_{11}\dot{v}_1^3 + \Gamma^1_{22}\dot{v}_2^3 - (2\Gamma^2_{11} - \Gamma^1_{11})\dot{v}_1^2\dot{v}_2 + (2\Gamma^1_{12} - \Gamma^2_{22})\dot{v}_1\dot{v}_2^2 + \ddot{v}_1\dot{v}_2 - \ddot{v}_2\dot{v}_1)}{\|\vec{a}\|_g^3}$$

(see Theorem 17.19 in [19] for a proof). Here $(\dot{v}_1, \dot{v}_2) = \vec{a}$, and $\ddot{v}_1, \ddot{v}_2$ are determined by $(\dot{v}_1, \dot{v}_2)$ through the geodesic equations

$$\ddot{v}_i + \sum_{j,k} \tilde{\Gamma}^i_{jk}\dot{v}_j\dot{v}_k = 0.$$

By this way k_g can be viewed as a smooth function of $(x, \vec{a})$ defined on the compact manifold A, and thus is bounded by $[-C, C]$ for some constant $C = C(M, g, u)$.

As shown in Figure 3.3, assume l and l' start at x and first meet at a point z. Let l_0 (*resp.* l_0') be the part of l (*resp.* l') between x and z, and then by the Jordan-Schoenflies theorem $l_0 \cup l_0'$ bounds a small closed disk D. Let θ_x (*resp.* θ_z) be the intersecting angle of l_0 and l_0' at x (*resp.* at z). Then by the Gauss-Bonnet theorem

$$\int_D K dA_g + \int_{l_0} k_g ds_g + \int_{l_0'} k_g ds_g + (\pi - \theta_x) + (\pi - \theta_z) = 2\pi$$

and

$$\theta_x \leq \theta_x + \theta_z = \int_D K dA_g + \int_{l_0'} k_g ds_g \leq \|K\|_\infty \cdot |D|_g + C \cdot s_g(l') \leq \frac{\epsilon}{2} + C \cdot s_g(l')$$

where

$$s_g(l') \le e^{\|u\|_\infty} \cdot s_{e^{2u}g}(l') \le e^{\|u\|_\infty} \cdot s_{e^{2u}g}(l) \le e^{2\|u\|_\infty} \cdot s_g(l) \le e^{2\|u\|_\infty} \cdot \delta.$$

So $\theta_x \le \epsilon$ if $\delta \le \epsilon/(2Ce^{2\|u\|_\infty})$. □

3.3.1 Proof of Part (a) of C-isoperimetric Condition Theorem

Recall that

Theorem (Part (a) of Theorem 10)**.** *Suppose (M, g) is a closed Riemannian surface, and T is a geodesic triangulation of (M, g) with geodesic edge length l, such that $(T, l)_E$ or $(T, l)_H$ is ϵ-regular. Given a conformal factor $u \in C^\infty(M)$, there exist constants $\delta = \delta(M, g, u, \epsilon) > 0$ and $C = C(M, g, \epsilon)$, such that if $||l|| < \delta$, then*

(a) there exists a geodesic triangulation T' in $(M, e^{2u}g)$ such that $V(T') = V(T)$, and T' is homotopic to T relative to $V(T)$. Further $(T, \bar{l})_E$ and $(T, \bar{l})_H$ are $\frac{1}{2}\epsilon$-regular where $\bar{l} \in \mathbb{R}^{E(T)} \cong \mathbb{R}^{E(T')}$ denotes the geodesic lengths of the edges of T' in $(M, e^{2u}g)$.

Proof of Part (a) of Theorem 10. Denote

(a) $\theta^i_{jk}(M)$ as the inner angle of the geodesic triangle in $F(T)$ in (M, g),

(b) $\theta^i_{jk}(E)$ as the inner angle in $(T, l)_E$

(c) $\theta^i_{jk}(H)$ as the inner angle in $(T, l)_H$

(d) $\bar{\theta}^i_{jk}(M)$ as the inner angle of the geodesic triangle in $F(T')$ in $(M, e^{2u}g)$

(e) $\bar{\theta}^i_{jk}(E)$ as the inner angle in $(T, \bar{l})_E$

(f) $\bar{\theta}^i_{jk}(H)$ as the inner angle in $(T, \bar{l})_H$.

By Lemma 3.5 and 3.6, if $\delta(M, g, u, \epsilon)$ is sufficiently small, then

(a) $|\theta^i_{jk}(M) - \theta^i_{jk}(E)| \le \epsilon/12$ and $|\theta^i_{jk}(M) - \theta^i_{jk}(H)| \le \epsilon/12$

(b) for any edge $ij \in E(T)$ there exists a unique shortest geodesic e_{ij} in $(M, e^{2u}g)$ connecting i, j

(c) for each $\triangle ijk \in F(T)$, e_{ij}, e_{ik}, e_{jk} bounds a geodesic triangle F_{ijk} in $(M, e^{2u}g)$

(d) $|\bar{\theta}^i_{jk}(M) - \theta^i_{jk}(M)| \leq \epsilon/12$, $|\bar{\theta}^i_{jk}(E) - \bar{\theta}^i_{jk}(M)| \leq \epsilon/12$, $|\bar{\theta}^i_{jk}(H) - \bar{\theta}^i_{jk}(M)| \leq \epsilon/12$, and therefore both $(T, \bar{l})_E$ and $(T, \bar{l})_H$ are $\epsilon/2$-regular,

(e) for any vertex $i \in V(T) = V(T')$, its adjacent edges $\{ij\}_{j\sim i}$ in T are placed in the same order as the adjacent edges $\{e_{ij}\}_{j\sim i}$ in T', i.e., there are no folding triangles in T'.

We can define a continuous map $f : M \to M$ such that

(1) $f(i) = i$ for any $i \in V$

(2) for any edge $ij \in E(T)$, f is a homeomorphism from ij to e_{ij}

(3) for any $\triangle ijk \in F(T)$, f is a homeomorphism from $\triangle ijk$ to F_{ijk}.

Then by the property (e) above, f is a local homeomorphism. Furthermore if δ is sufficiently small, f is homotopic to the identity. Therefore f is a global homeomorphism and $T' = (V, \{e_{ij}\}, \{F_{ijk}\})$ is a triangulation of M. $\square$

3.3.2 Proof of Part (b) of C-isoperimetric Condition Theorem

Recall that

Theorem (Part (b) of Theorem 10)**.** *Suppose (M, g) is a closed Riemannian surface, and T is a geodesic triangulation of (M, g) with geodesic edge length l such that $(T, l)_E$ or $(T, l)_H$ is ϵ-regular. Then there exist constants $\delta = \delta(M, g, \epsilon)$ and $C = C(M, g, \epsilon)$ such that if $||l|| < \delta$, then*
(b) (T, l) is C-isoperimetric.

For a finite union of curves γ, we denote its length measured in g as $s(\gamma)$, or $s_g(\gamma)$. We first prove a continuous version.

Lemma 3.7. *Suppose (M, g) is a closed Riemannian surface, and $\Omega \subset M$ is an open subset with $\partial\Omega$ being a finite disjoint union of piecewise smooth Jordan curves, then there exists a constant $C = C(M, g) > 0$ such that*

$$\min\{|\Omega|, |M - \Omega|\} \leq CL^2$$

where $L = s(\partial\Omega)$ denotes the length of $\partial\Omega$ in (M, g).

Proof. If Ω is simply connected, then it is well known (See Theorem 4.3 in [20]) that

$$L^2 \geq |\Omega|(4\pi - 2\int_\Omega K^+)$$

where $K^+(p) = \max\{0, K(p)\}$.

For a point x on the Riemannian surface (M, g) and a $r > 0$, let $B(x, r) = B_g(x, r) = \{y \in M : d_g(x, y) < r\}$ and $|B(x, r)|$ be its area measured in metric g.

Pick $r = r(M, g) > 0$ to be smaller than the injectivity radius of (M, g), such that

$$|B(p, r)| \cdot \|K\|_\infty \leq \pi$$

for any $p \in M$.

Now we pick our constant

$$C(M, g) = \max\{\frac{2}{\pi}, \frac{|M|}{r^2}\}.$$

If $L \geq r$, then $CL^2 \geq |M|$ and we are done. If $\Omega \subset B(p, r)$ for some $p \in M$ and is connected, then without loss of generality we may assume Ω is simply connected by filling up the holes. Therefore we have

$$CL^2 \geq \frac{2}{\pi} \cdot |\Omega|(4\pi - 2\int_\Omega K^+) \geq |\Omega|\big(8 - \frac{4}{\pi} \cdot |B(p, r)| \cdot \|K\|_\infty\big) \geq 4|\Omega| \tag{3.7}$$

and we are done.

If Ω has multiple connected components $\Omega_1, ..., \Omega_n$ with the boundary lengths $L_1, ..., L_n$ respectively, such that each Ω_i is in some Riemannian disk $B(p, r)$, then $L \geq (L_1 + ... + L_n)/2$ since any component of $\partial\Omega$ is in at most two $\partial\Omega_i$'s. So by equation (3.7)

$$CL^2 \geq \frac{1}{4}\sum_{i=1}^{n} CL_i^2 \geq \sum_{i=1}^{n} |\Omega_i| = |\Omega| \tag{3.8}$$

and we are done.

Now we assume $L < r$ and $\partial\Omega$ consists of Jordan curves $\gamma_1, ..., \gamma_n$ with lengths $L_1, ..., L_n$ respectively. Since $L_i \leq r$, γ_i is in some Riemannian disk $B(p, r)$. By the Jordan-Schoenflies theorem, γ_i separates M into a smaller domain $U_i \subset B(p, r)$ and a

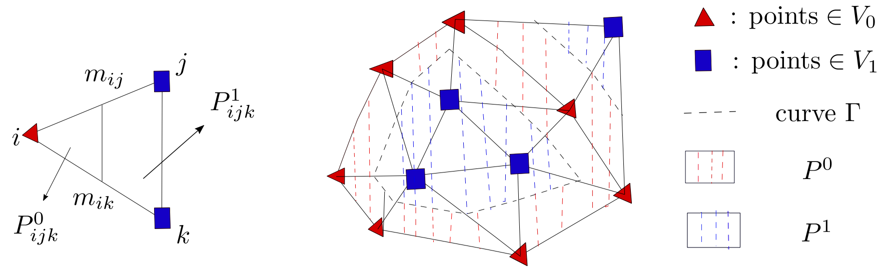

Figure 3.4: Decomposition of a triangulated surface

larger domain $V_i = M - \bar{U}_i$, and $\bar{U}_i$ is a topological closed disk. For any $i \neq j$, since γ_i and γ_j are disjoint, $\bar{U}_i \subset \bar{U}_j$ or $\bar{U}_j \subset \bar{U}_i$ or $\bar{U}_i \cap \bar{U}_j = \emptyset$. So $\cup_{i=1}^n \bar{U}_i$ is a finite disjoint union of topological disks, and thus $M - \cup_{i=1}^n \bar{U}_i$ is connected. If $\Omega \subset \cup_{i=1}^n U_i$, then by equation (3.8) we are done. Otherwise, $M - \cup_{i=1}^n \bar{U}_i \subset \Omega$ and $M - \bar{\Omega} \subset \cup_{i=1}^n U_i$, and again by equation (3.8) $CL^2 \geq |M - \bar{\Omega}|$ and we are done. □

Now we prove part (b) of Lemma 10 for the special cases that (M, g) has constant curvature 0 or ± 1.

Proof of Part (b) of theorem 10 for the cases of constant curvature 0 *or* ± 1. In this proof each triangle $\triangle ijk \in F(T)$ is identified as a geodesic triangle in (M, g). Assume $\delta < 0.1$, $V_0 \subset V$, and $V_1 = V - V_0$. Let

$$E_0 = \{ij \in E : i, j \in V_0\}, \quad E_1 = \{ij \in E : i, j \in V_1\}.$$

Notice that $\partial V_0 = \partial V_1$ and $E = E_0 \cup E_1 \cup \partial V_0$ is a disjoint union.

For any triangle $\triangle ijk \in F(T)$, 0 or 2 of its edges are in ∂V_0. So $F(T) = F_0 \cup F_2$ where

$$F_0 = \{\triangle ijk \in F(T) : \triangle ijk \text{ has 0 edges in } \partial V_0\}, \text{ and}$$
$$F_2 = \{\triangle ijk \in F(T) : \triangle ijk \text{ has 2 edges in } \partial V_0\}.$$

If $\triangle ijk \in F_2$ and $ij, ik \in \partial V_0$, let γ_{ijk} be the geodesic segment in $\triangle ijk$ connecting the middle points m_{ij} of ij, and m_{ik} of ik. Then by the triangle inequality $\frac{1}{2}(l_{ij} + l_{ik}) \geq s(\gamma_{ijk})$. γ_{ijk} cut $\triangle ijk$ into two relative open domains P^0_{ijk} and P^1_{ijk} such that $P^0_{ijk} \cap V_0 \neq \emptyset$ and $P^1_{ijk} \cap V_1 \neq \emptyset$. Given $\triangle ijk \in F_0$,

(1) if $i, j, k \in V_0$, denote $P^0_{ijk} = \triangle ijk$ and $P^1_{ijk} = \emptyset$, and

(2) if $i, j, k \in V_1$, denote $P^1_{ijk} = \triangle ijk$ and $P^0_{ijk} = \emptyset$.

The union

$$\Gamma = \bigcup_{\triangle ijk \in F_2} \gamma_{ijk}$$

is a finite disjoint union of piecewise smooth Jordan curves in (M, g), and

$$P^0 = \bigcup_{\triangle ijk \in F(T)} P^0_{ijk}, \quad \text{and} \quad P^1 = \bigcup_{\triangle ijk \in F(T)} P^1_{ijk}$$

are two open domains of (M, g) such that $\partial P^0 = \Gamma$ and $P^1 = M - \bar{P}^0$. The above notations are shown in Figure 3.4. By Lemma 3.7, it suffices to prove that if $\delta < 0.1$,

$$|P^1| \geq \frac{\epsilon}{60}(|V|_l - |V_0|_l), \tag{3.9}$$

and

$$|P^0| \geq \frac{\epsilon}{60}|V_0|_l, \tag{3.10}$$

and

$$s(\Gamma) \leq |\partial V_0|_l. \tag{3.11}$$

By part (b) of Lemma 3.2 and Remark 3.1, we have that

$$\begin{aligned} |V|_l - |V_0|_l = \sum_{ij \in E_1 \cup \partial V_0} l_{ij}^2 \leq \frac{4}{\epsilon} \sum_{ij \in E_1 \cup \partial V_0} (|\triangle ijk| + |\triangle ijk'|) \leq \frac{12}{\epsilon} \sum_{\triangle ijk \in F : \triangle ijk \cap P^1 \neq \emptyset} |\triangle ijk| \\ \leq \frac{60}{\epsilon} \sum_{\triangle ijk \in F} |P^1_{ijk}| = \frac{60}{\epsilon}|P^1|, \end{aligned}$$

and

$$|V_0|_l = \sum_{ij \in E_0} l_{ij}^2 \leq \sum_{ij \in E_0 \cup \partial V_0} l_{ij}^2 \leq \frac{60}{\epsilon}|P^0|,$$

and

$$|\partial V_0|_l = \sum_{ij \in \partial V_0} l_{ij} = \sum_{\triangle ijk \in F_2 : jk \notin \partial V_0} \frac{1}{2}(l_{ij} + l_{ik}) \geq \sum_{\triangle ijk \in F_2} s(\gamma_{ijk}) = s(\Gamma).$$

□

Now let us prove part (b) of Theorem 10 for general smooth surfaces.

Proof of Part (b) of Theorem 10. By the Uniformization theorem, there exists $u = u_{M,g} \in C^\infty(M)$ such that $e^{2u}g$ has constant curvature ± 1 or 0. By part (a) of Theorem 10, if δ is sufficiently small, we can find a geodesic triangulation T' in $(M, e^{2u}g)$

such that $V(T) = V(T')$, and T, T' are homotopic relative to V. Furthermore by the inequalities in (a) and (d) in the proof of Theorem 10 (a), if δ is sufficiently small, any inner angle of T' in $(M, e^{2u}g)$ is at least $\epsilon/2$. Let $\bar{l} \in \mathbb{R}^{E(T)} \cong \mathbb{R}^{E(T')}$ denote the geodesic edge lengths in $(M, e^{2u}g)$. Then by our result on surfaces of constant curvature ± 1 or 0, if $\delta = \delta(M, e^{2u}g)$ is sufficiently small, $(T, \bar{l})$ is C-isoperimetric for some constant $C = C(M, e^{2u}g) > 0$. Since $e^{-\|u\|_\infty} \leq \bar{l}_{ij}/l_{ij} \leq e^{\|u\|_\infty}$, (T, l) is $(e^{4\|u\|_\infty}C)$-isoperimetric. □

3.4 Discrete Elliptic Estimate

Recall that

Theorem 11. *Assume (G, l) is C_1-isoperimetric, and $x \in \mathbb{R}^E_A, \eta \in \mathbb{R}^E_{>0}, C_2 > 0, C_3 > 0$ are such that*

(1) $|x_{ij}| \leq C_2 l_{ij}^2$ for any $ij \in E$, and

(2) $\eta_{ij} \geq C_3$ for any $ij \in E$.

Then

$$||\Delta_\eta^{-1} \circ div(x))|| \leq \frac{4C_2\sqrt{C_1 + 1}}{C_3} ||l|| \cdot |V|_l^{1/2}.$$

Furthermore if $y \in \mathbb{R}^V$ and $C_4 > 0$ and $D \in \mathbb{R}^{V \times V}$ is a diagonal matrix such that

$$|y_i| < C_4 D_{ii} ||l|| \cdot |V|_l^{1/2}$$

for any $i \in V$, then

$$||(D - \Delta_\eta)^{-1}(div(x) + y)|| \leq \left(C_4 + \frac{8C_2\sqrt{C_1 + 1}}{C_3} \right) ||l|| \cdot |V|_l^{1/2}.$$

We will first prove Lemma 9 assuming Lemma 3.8, and then prove Lemma 3.8.

Proof. Assume

(1) $z = \Delta^{-1}(div(x))$, and

(2) $a, b \in V$ are such that $z_a = \max_i z_i \geq 0$ and $z_b = \min_i z_i \leq 0$, and $a \neq b$, and

(3) $u \in \mathbb{R}^V$ is such that

$$(\Delta u)_a = 1, \quad \text{and} \quad (\Delta u)_b = -1, \quad \text{and} \quad (\Delta u)_i = 0 \quad \forall i \neq a, b.$$

By the Green's identity Lemma 3 and Lemma 3.8,

$$||z|| \leq z_a - z_b = \sum_i z_i(\Delta u)_i = \sum_i u_i(\Delta z)_i = \sum_i u_i \cdot div(x)_i$$
$$= \sum_i u_i \sum_{j:j\sim i} x_{ij} = \sum_{ij\in E}(u_i - u_j) \cdot x_{ij} \leq C_2 \sum_{ij\in E} |u_i - u_j| \cdot l_{ij}^2 \leq \frac{4C_2\sqrt{C_1+1}}{C_3}||l|| \cdot |V|_l^{1/2}.$$

Let

$$w = (D - \Delta)^{-1}(div(x) + y) + z,$$

and then

$$(D - \Delta)w = div(x) + y + (D - \Delta)z = y + Dz. \tag{3.12}$$

Assume $w_i = \max_k w_k$, and then by comparing the i-th component in (3.12) we have

$$D_{ii}w_i \leq ((D - \Delta)w)_i = y_i + D_{ii}z_i \leq y_i + D_{ii}||z||.$$

So

$$\max_k w_k = w_i \leq ||z|| + y_i/D_{ii} \leq ||z|| + \max_k(y_k/D_{kk})$$

and similarly we also have that

$$\min_k w_k \geq -||z|| + \min_k(y_k/D_{kk}).$$

So

$$||(D - \Delta)^{-1}(div(x) + y)|| \leq ||w|| + ||z|| \leq 2||z|| + \max_k(|y_k|/D_{kk})$$

and we are done.

□

Lemma 3.8. *Assume (G, l) is C_1-isoperimetric, and the weight $\eta \in \mathbb{R}^E_{>0}$ satisfies that $\eta_{ij} \geq C_2$ for some constant $C_2 > 0$. Assume $u \in \mathbb{R}^V$ satisfies that*

$$(\Delta u)_a = 1, \quad \textit{and} \quad (\Delta u)_b = -1, \quad \textit{and} \quad (\Delta u)_i = 0 \quad \forall i \neq a, b.$$

Then

$$\sum_{ij\in E} l_{ij}^2|u_i-u_j| \le \frac{4\sqrt{C_1+1}}{C_2}||l||\cdot|V|_l^{1/2}.$$

Proof. We consider the 1-skeleton X of the graph G with edge length l. More specifically X can be constructed as follows. Let $\tilde{X}$ be a disjoint union of line segments $\{e_{ij} : ij \in E\}$ where each e_{ij} has length l_{ij} and two endpoints v_{ij}^i, v_{ij}^j. Then we obtain a connected quotient space X by identifying the points in $v_i := \{v_{ij}^i : ij \in E\}$ for any $i \in V$.

Assume μ is the standard 1-dimensional Lebesgue measure on X such that $\mu(e_{ij}) = l_{ij}$. Let ν be another measure on X such that $d\nu/d\mu \equiv l_{ij}$ on edge e_{ij}. Then we have that $\nu(e_{ij}) = l_{ij}^2$ and $\nu(X) = |V|_l$.

Assume $u : V \to \mathbb{R}$ is linearly extended to the 1-skeleton X, and then by maximum principle, $u_a = \min(u)$ and $u_b = \max(u)$. Let $\bar{u} \in (u_a, u_b)$ be such that

$$\nu(x \in X : u(x) < \bar{u}) \le |V|_l/2,$$

$$\nu(x \in X : u(x) > \bar{u}) \le |V|_l/2.$$

Let $f(x) = l_{ij}|u_i - u_j|$ for $x \in e_{ij}$, and then f is well-defined almost everywhere on X, and

$$\sum_{ij\in E} l_{ij}^2|u_i-u_j| = \int_X f(x)d\mu \le \int_{u_a\le u(x)\le \bar{u}} f(x)d\mu + \int_{\bar{u}\le u(x)\le u_b} f(x)d\mu.$$

We will prove

$$\int_{u_a\le u(x)\le\bar{u}} f(x)d\mu \le \frac{2\sqrt{C_1+1}}{C_2}||l||\cdot\sqrt{\nu(u(x)<\bar{u})} \le \frac{2\sqrt{C_1+1}}{C_2}||l||\cdot|V|_l^{1/2}$$

and then by the symmetry $\int_{\bar{u}\le u(x)\le u_b} f(x)d\mu$ has the same upper bound and we are done.

Let $u_a = p_0 < p_1 < \cdots < p_s = \bar{u}$ such that $\{p_0 \cdots p_{s-1}\} = \{u_i : i \in V, u_i < \bar{u}\}$. Noticing that $\int_{u(x)=p} f(x)d\mu = 0$ for any $p \in \mathbb{R}$, it suffices to prove that for any $k \in \{1, ..., s\}$

$$\int_{p_{k-1}<u(x)<p_k} f(x)d\mu \le \frac{2\sqrt{C_1+1}}{C_2}||l||\cdot\left(\sqrt{\nu(u(x)<p_k)} - \sqrt{\nu(u(x)<p_{k-1})}\right).$$

In the remaining of the proof, we fix a $k \in \{1, ..., s\}$ and let $V_k = \{i \in V : u(i) \leq p_{k-1}\}$. Then for any $i \in V_k$ and $ij \in \partial V_k$, $u_j \geq u_i$ and then

$$\begin{aligned}
0 &= \sum_{i \in V_k - \{a\}} (\Delta u)_i = \sum_{i \in V_k - \{a\}} \sum_{j \sim i} \eta_{ij}(u_j - u_i) \\
&= \sum_{i: i \sim a} \eta_{ia}(u_a - u_i) + \sum_{ij \in \partial V_k : i \in V_k} \eta_{ij}(u_j - u_i) \\
&\geq -(\Delta u)_a + C_2 \sum_{ij \in \partial V_k} |u_j - u_i|.
\end{aligned}$$

Therefore,

$$C_2 \sum_{ij \in \partial V_k} |u_j - u_i| \leq (\Delta u)_a = 1. \tag{3.13}$$

Let $e'_{ij} = \{x : p_{k-1} < u(x) < p_k\} \cap e_{ij}$, and $l'_{ij} = \mu(e'_{ij})$. Then $l'_{ij} = 0$ if $ij \notin \partial V_k$. If $ij \in \partial V_k$, let i' and j' be the two endpoints of e'_{ij}. Then $\{u(i'), u(j')\} = \{p_{k-1}, p_k\}$, and

$$\frac{l'_{ij}}{l_{ij}} = \frac{|u(j') - u(i')|}{|u_j - u_i|} = \frac{p_k - p_{k-1}}{|u_j - u_i|}.$$

So

$$\int_{p_{k-1} < u(x) < p_k} f(x) d\mu = \sum_{ij \in \partial V_k} l'_{ij} l_{ij} |u_i - u_j| = (p_k - p_{k-1}) \sum_{ij \in \partial V_k} l_{ij}^2 \leq (p_k - p_{k-1}) ||l|| \cdot |\partial V_k|_l. \tag{3.14}$$

On the other hand, by inequality (3.13) and Cauchy's inequality,

$$\begin{aligned}
\nu(p_{k-1} < u(x) < p_k) &= \sum_{ij \in \partial V_k} l'_{ij} l_{ij} = (p_k - p_{k-1}) \sum_{ij \in \partial V_k} \frac{l_{ij}^2}{|u_j - u_i|} \\
\geq &(p_k - p_{k-1}) \left(\sum_{ij \in \partial V_k} \frac{l_{ij}^2}{|u_j - u_i|} \right) \cdot \left(\sum_{ij \in \partial V_k} |u_j - u_i| \right) \cdot C_2 \\
\geq &C_2 (p_k - p_{k-1}) (\sum_{ij \in \partial V_k} l_{ij})^2 = C_2 (p_k - p_{k-1}) |\partial V_k|_l^2.
\end{aligned} \tag{3.15}$$

Since (G, l) is C_1-isoperimetric, we have that

$$\nu(u(x) < p_k) \leq |V_k|_l + \sum_{ij \in \partial V_k} l_{ij}^2 \leq C_1 |\partial V_k|_l^2 + |\partial V_k|_l^2 = (C_1 + 1) |\partial V_k|_l^2. \tag{3.16}$$

Divide(3.15) by $\sqrt{(3.16)}$ and then we have

$$\frac{\nu(p_{k-1} < u(x) < p_k)}{\sqrt{\nu(u(x) < p_k)}} \geq \frac{C_2}{\sqrt{C_1 + 1}} (p_k - p_{k-1}) |\partial V_k|_l. \tag{3.17}$$

Combining equations (3.14) and (3.17) and then

$$\begin{aligned}
&\int_{p_{k-1}<u(x)<p_k} f(x)d\mu \leq \frac{\sqrt{C_1+1}}{C_2}\cdot ||l||\cdot \frac{\nu(p_{k-1}<u(x)<p_k)}{\sqrt{\nu(u(x)<p_k)}} \\
\leq&\frac{\sqrt{C_1+1}}{C_2}\cdot ||l||\cdot \frac{\nu(u(x)<p_k)-\nu(u(x)<p_{k-1})}{\sqrt{\nu(u(x)<p_k)}} \\
\leq&\frac{2\sqrt{C_1+1}}{C_2}\cdot ||l||\cdot \frac{\nu(u(x)<p_k)-\nu(u(x)<p_{k-1})}{\sqrt{\nu(u(x)<p_k)}+\sqrt{\nu(u(x)<p_{k-1})}} \\
=&\frac{2\sqrt{C_1+1}}{C_2}\cdot ||l||\cdot \left(\sqrt{\nu(u(x)<p_k)}-\sqrt{\nu(u(x)<p_{k-1})}\right)
\end{aligned}$$

and we are done. $\square$

Chapter 4

Proof for the case of genus $g = 1$

4.1 Differential of discrete curvatures for discrete flat metric

First, we will derive the differential for discrete curvature of PL flat metric under a discrete conformal change.

Lemma 4.1. *Given a Euclidean triangle $\triangle ABC$, if we view A, B, C as functions of the edge lengths a, b, c, then*

$$\frac{\partial A}{\partial b} = -\frac{\cot C}{b}, \qquad \frac{\partial A}{\partial a} = \frac{\cot B + \cot C}{a} = \frac{1}{b \sin C}.$$

Furthermore if $(u_A, u_B, u_C) \in \mathbb{R}^3$ is a discrete conformal factor, and

$$a = e^{\frac{1}{2}(u_B+u_C)} a_0, \quad b = e^{\frac{1}{2}(u_A+u_C)} b_0, \quad c = e^{\frac{1}{2}(u_A+u_B)} c_0$$

for some constants $a_0, b_0, c_0 \in \mathbb{R}_{>0}$, then

$$\frac{\partial A}{\partial u_B} = \frac{1}{2}\cot C, \quad \frac{\partial A}{\partial u_A} = -\frac{1}{2}(\cot B + \cot C). \tag{4.1}$$

Proof. Take the partial derivative on

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

and we have

$$-\sin A \frac{\partial A}{\partial b} = \frac{2b}{2bc} - \frac{b^2 + c^2 - a^2}{2b^2 c} = \frac{b^2 + a^2 - c^2}{2b^2 c} = \frac{a \cos C}{bc},$$

and

$$\frac{\partial A}{\partial b} = -\frac{a \cos C}{bc \sin A} = -\frac{\cos C}{b \sin C} = -\frac{\cot C}{b}.$$

Similarly

$$-\sin A \frac{\partial A}{\partial a} = -\frac{a}{bc}$$

and

$$\frac{\partial A}{\partial a}=\frac{a}{bc\sin A}=\frac{1}{b\sin C}=\frac{\sin A}{a\sin B\sin C}=\frac{\sin B\cos C+\sin C\cos B}{a\sin B\sin C}=\frac{\cot B+\cot C}{a}.$$

Then equation (4.1) can be computed easily. $\square$

Now we are in the position to get the following proposition, which was first discovered by Luo [1].

Proposition 4. *Given $(T,l)_E$ and $u\in\mathbb{R}^{V(T)}$ such that $u*l$ satisfies the triangle inequalities, define the cotangent weight $\eta\in\mathbb{R}^E$ as*

$$\eta_{ij}(u)=\frac{1}{2}\cot\theta_{ij}^k(u)+\frac{1}{2}\cot\theta_{ij}^{k'}(u)$$

where $\triangle ijk$ and $\triangle ijk'$ are adjacent triangles in $F(T)$. Then

$$\frac{\partial K}{\partial u}(u)=-\Delta_{\eta(u)}.$$

Before giving the sketch the proof, we need an estimate for the angle difference between two Euclidean triangles.

Lemma 4.2. *Given a Euclidean triangle $\triangle ABC$ with all the angles are at least $\epsilon>0$. If*

$$|\tilde{a}-a|\le\delta a,\quad|\tilde{b}-b|\le\delta a,\quad|\tilde{c}-c|\le\delta c,$$

where $\delta<\epsilon^2/48$, then $\tilde{a},\tilde{b},\tilde{c}$ can be the edges lengths of a Euclidean triangle with opposite inner angles $\tilde{A},\tilde{B},\tilde{C}$ respectively. Furthermore, we have

$$|\tilde{A}-A|\le\frac{24}{\epsilon}\delta,$$

and

$$\Big||\triangle\tilde{A}\tilde{B}\tilde{C}|-|\triangle ABC|\Big|\le\frac{576}{\epsilon^2}\delta\cdot|\triangle ABC|.$$

Proof. Let

$$u_A(t)=t\cdot(\log\frac{\tilde{b}}{b}+\log\frac{\tilde{b}}{c}-\log\frac{\tilde{a}}{a})$$

and $u_B(t),u_C(t)$ be defined similarly. Then $|u_i'|\le-3\log(1-\delta)\le 6\delta$ for $i\in\{A,B,C\}$, since

$$\delta\le\frac{\epsilon^2}{48}\le\frac{(\pi/3)^2}{48}\le 0.1.$$

Assume

$$a(t) = e^{\frac{1}{2}(u_B(t)+u_C(t))}a, \quad b(t) = e^{\frac{1}{2}(u_A(t)+u_C(t))}b, \quad c(t) = e^{\frac{1}{2}(u_A(t)+u_B(t))}c,$$

and then $a(1) = \tilde{a}, b(1) = \tilde{b}, c(1) = \tilde{c}$. Let $A(t), B(t), C(t)$ be the inner angles of the triangle with edge lengths $a(t), b(t), c(t)$, if well-defined.

Let $T_0 \in [0, \infty]$ be the maximum real number such that for any $t \in [0, T_0)$, all $A(t), B(t), C(t) > \epsilon/2$. Then $T_0 > 0$ and for any $t \in [0, T_0)$, by Lemma 4.1

$$|A'(t)| = \left|\frac{\partial A}{\partial u_A}u'_A + \frac{\partial A}{\partial u_B}u'_B + \frac{\partial A}{\partial u_C}u'_C\right| \leq 2\cot\frac{\epsilon}{2}\cdot\max(|u'_A|, |u'_B|, |u'_C|) \leq 12\delta\cot\frac{\epsilon}{2} \leq \frac{24}{\epsilon}\delta,$$

and similarly $|B'(t)|, |C'(t)| \leq 24\delta/\epsilon$. So $T_0 \geq (\epsilon/2)/(24\delta/\epsilon) = \epsilon^2/(48\delta) > 1$, and $|\tilde{A} - A| \leq 24\delta/\epsilon$.

By Lemma 4.1 for $t \in (0, 1)$

$$\frac{\partial|\triangle ABC|}{\partial a} = \frac{\partial(\frac{1}{2}bc\sin A)}{\partial A}\cdot\frac{\partial A}{\partial a} = \frac{1}{2}bc\cos A\cdot\frac{a}{bc\sin A} = \frac{a(t)}{2\tan A(t)}$$

and then by the chain rule

$$\left|\frac{d|\triangle ABC(t)|}{dt}\right| \leq |u'|\cdot\left(\left|\frac{a^2}{2\tan A}\right| + \left|\frac{b^2}{2\tan B}\right| + \left|\frac{c^2}{2\tan C}\right|\right) \leq 6\delta\cdot\frac{a(t)^2 + b(t)^2 + c(t)^2}{\epsilon},$$

where $a(t) \leq e^{|u(t)|}a \leq e^{6\delta t}a \leq 2a$ and $b(t) \leq 2b$ and $c(t) \leq 2c$.

Then by Lemma 3.2,

$$\big||\triangle\tilde{A}\tilde{B}\tilde{C}| - |\triangle ABC|\big| \leq \frac{24\delta}{\epsilon}(a^2 + b^2 + c^2) \leq \frac{24\delta}{\epsilon}\cdot 3\cdot\frac{8}{\epsilon}\cdot|\triangle ABC| = \frac{576}{\epsilon^2}\delta|\triangle ABC|.$$

□

4.2 Sketch of the proof for the case of genus g = 1

By Theorem 10, if δ is sufficiently small then there exists a geodesic triangulation T' of $(M, e^{2\bar{u}}g)$ homotopic to T relative to $V(T) = V(T')$. Let $\bar{l} \in \mathbb{R}^{E(T)} \cong \mathbb{R}^{E(T')}$ denote the geodesic lengths of the edges of T' in $(M, e^{2\bar{u}}g)$, and then $(T, \bar{l})_E$ is isometric to $(M, e^{2\bar{u}}g)$ and globally flat.

For simplicity, we will frequently use the notion $a = O(b)$ to denote that if $\delta = \delta(M, g, \epsilon)$ is sufficiently small, then $|a| \leq C\cdot b$ for some constant $C = C(M, g, \epsilon)$. For example, $l_{ij} = O(l_{jk})$ for any $\triangle ijk \in F(T)$, and $(\bar{u} * l)_{ij} = O(l_{ij})$, and $\bar{l}_{ij} = O(l_{ij})$. The remaining of the proof is divided into three steps.

(1) Firstly we show that $(T, \bar{u} * l)_E$ is close to the globally flat polyhedral metric $(T, \bar{l})_E$, in the sense that

$$(\bar{u} * l)_{ij} - \bar{l}_{ij} = O(l_{ij}^3)$$

and

$$K(\bar{u}) = div(x)$$

for some flow $x \in \mathbb{R}_A^E$ satisfying $x_{ij} = O(l_{ij}^2)$.

(2) Secondly, we construct a smooth path $u(t) : [0, 1] \to \mathbb{R}^V$ with $u(0) = \bar{u}$ such that the identity holds

$$K(u(t)) = (1 - t)K(\bar{u}).$$

Furthermore we show that $||u'(t)|| = O(||l||)$, and then $(T, u(1) * l)_E$ is globally flat and $||u(1) - \bar{u}|_{V(T)}|| = O(||l||)$.

(3) Lastly we show that $Area((T, u(1) * l)_E) - 1 = O(||l||)$, so the area normalization condition can be satisfied by slightly scaling $(T, u(1) * l)_E$.

The uniqueness of the discrete uniformization conformal factor is proved by Bobenko-Pinkall-Springborn (see Theorem 3.1.4 in [2]).

4.3 Proof of the torus case

4.3.1 Step 1: Deviation of angle and discrete curvature of PL Euclidean angle

By Theorem 10, $(T, \bar{l})_E$ is $\frac{1}{2}\epsilon$-regular if δ is sufficiently small. For simplicity we denote $\bar{u}|_{V(T)}$ as $\bar{u}$. By lemma 3.3,

$$|\bar{l}_{ij} - (\bar{u} * l)_{ij}| = O(l_{ij}^3),$$

and then by Lemma 4.2

$$\alpha_{jk}^i := \bar{\theta}_{jk}^i - \theta_{jk}^i(\bar{u}) = O(l_{ij}^2)$$

where $\bar{\theta}_{jk}^i$ denotes the inner angle in $(T, \bar{l})_E$. So $(T, \bar{u} * l)_E$ is $\frac{1}{3}\epsilon$-regular if δ is sufficiently small. Let $x \in \mathbb{R}_A^E$ be such that

$$x_{ij} = \frac{\alpha_{jk}^i - \alpha_{ik}^j}{3} + \frac{\alpha_{jk'}^i - \alpha_{ik'}^j}{3}$$

where $\triangle ijk$ and $\triangle ijk'$ are adjacent triangles sharing edge ij. From $\alpha^i_{jk}+\alpha^j_{ik}+\alpha^k_{ij}=0$ and $\sum_{jk:\triangle_{ijk}\in F(T)}\bar{\theta}^i_{jk}=2\pi$, we have

$$div(x)_i=\sum_{j:j\sim i}x_{ij}=\sum_{jk:\triangle ijk\in F(T)}\left(\frac{\alpha^i_{jk}-\alpha^j_{ik}}{3}+\frac{\alpha^i_{jk}-\alpha^k_{ij}}{3}\right)=\sum_{jk:\triangle ijk\in F(T)}\alpha^i_{jk}=K_i(\bar{u}),$$

which implies

$$x_{ij}=O(l_{ij}^2). \tag{4.2}$$

4.3.2 Step 2: Construction of the path

Let

$$\tilde{\Omega}=\{u\in\mathbf{1}^\perp : u*l \text{ satisfies the triangle inequalities and } (T,u*l)_E \text{ is } \frac{\epsilon}{5}\text{-regular}\}$$

and

$$\Omega=\{u\in\tilde{\Omega} : ||u-\bar{u}||\leq 1,\ (T,u*l)_E \text{ is } \frac{\epsilon}{4}\text{-regular}\}.$$

Since $(T,\bar{u}*l)_E$ is $\frac{1}{3}\epsilon$-regular, $\bar{u}$ is in the interior of Ω. Now consider the following ODE on $int(\tilde{\Omega})$,

$$\begin{cases} u'(t)=\Delta^{-1}_{\eta(u)}K(\bar{u})=\Delta^{-1}_{\eta(u)}\circ div(x) \\ u(0)=\bar{u}\end{cases}, \tag{4.3}$$

where

$$\eta_{ij}(u)=\frac{\cot\theta^k_{ij}(u)+\cot\theta^{k'}_{ij}(u)}{2}=\frac{\sin(\theta^k_{ij}(u)+\theta^{k'}_{ij}(u))}{2\sin\theta^k_{ij}(u)\sin\theta^{k'}_{ij}(u)}\geq\frac{1}{2}\sin(\theta^k_{ij}(u)+\theta^{k'}_{ij}(u))\geq\frac{1}{2}\sin\frac{\epsilon}{5}, \tag{4.4}$$

where $\triangle ijk,\triangle ijk'$ are adjacent triangles sharing edge ij. According to lemma 3.1, the right-hand side of (4.3) is a smooth function of u. By Proposition 4, this ODE (4.3) has a unique solution $u(t)$ satisfying

$$\frac{dK(u(t))}{dt}=\frac{\partial K}{\partial u}u'(t)=K(\bar{u})$$

. Therefore

$$K(u(t))=(1-t)K(\bar{u})$$

Assume the maximum existing open interval of $u(t)$ in $int(\Omega)$ is $[0,T_0)$ where $T_0\in(0,\infty]$. By Theorem 10, (T,l) is C-isoperimetric for some constant $C=C(M,g,\epsilon)$.

Then for any $u \in \Omega$, $(T, u * l)$ is $(e^{4(|\bar{u}|+1)}C)$-isoperimetric by the fact that $|u| \leq |\bar{u}| + 1$ at any vertex. Then by Lemma 9 and equations (6.7),(4.3),(4.4), for any $t \in [0, T_0)$

$$||u'(t)|| = O(||l|| \cdot |V|_l^{1/2}). \tag{4.5}$$

By Lemma 3.2 and the fact that $(T, \bar{l})_E$ is $\frac{1}{2}\epsilon$-regular,

$$\begin{aligned} |V|_l = \sum_{ij \in E} l_{ij}^2 = O(\sum_{ij \in E} \bar{l}_{ij}^2) = O(\sum_{\triangle ijk \in F} (\bar{l}_{ij}^2 + \bar{l}_{jk}^2 + \bar{l}_{ik}^2)) \\ = O(\sum_{\triangle ijk \in F} |(\triangle ijk, \bar{l})_E|) = O(|(T, \bar{l})_E|) = O(1). \end{aligned} \tag{4.6}$$

Here recall that $|(\triangle ijk, \bar{l})_E|$ denotes the area of the Euclidean triangle, and $|(T, \bar{l})_E|$ denotes the area of the piecewise flat surface.

Combining the estimates (4.5) and (4.6), we have that for any $t \in [0, T_0)$

$$||u'(t)|| = O(||l||).$$

If $T_0 < 1$, by Lemma 4.2

$$||u(T_0) - \bar{u}|| = O(||l||) \quad \text{and} \quad |\theta^i_{jk}(u(T_0)) - \theta^i_{jk}(\bar{u})| = O(||l||), \forall \triangle ijk \in F(T).$$

and thus $u(T_0) \in int(\Omega)$ if δ is sufficiently small. But this contradicts with the maximality of T_0. So $T_0 \geq 1$ and $(T, u(1))_E$ is globally flat and $||u(1) - \bar{u}||| = O(||l||)$.

4.3.3 Step 3: Normalization of area

To prove part (a) of the theorem, we only need to scale the polyhedral metric $(T, u(1) * l)_E$ to make its area equal to 1. To get the estimate in part (b), it remains to show

$$\log |(T, u(1) * l)_E| = O(||l||).$$

Since

$$\begin{aligned} &|(u(1) * l)_{ij} - \bar{l}_{ij}| = |(u(1) * l)_{ij} - (\bar{u} * l)_{ij}| + |(\bar{u} * l)_{ij} - \bar{l}_{ij}| \\ \leq &(e^{|u(1) - \bar{u}|} - 1)(\bar{u} * l)_{ij} + O(l_{ij}^3) = O(||l|| \cdot \bar{l}_{ij}). \end{aligned}$$

Since $(T, \bar{l})_E$ is $\frac{1}{2}\epsilon$-regular, by Lemma 4.2 if δ is sufficiently small then for any $\triangle ijk \in F$

$$\log \frac{|(\triangle ijk, u(1) * l)_E|}{|(\triangle ijk, \bar{l})_E|} = O(||l||)$$

and
$$\log|(T, u(1) * l)_E| = \log \frac{\sum_{\triangle ijk \in F} |(\triangle ijk, u(1) * l)_E|}{\sum_{\triangle ijk \in F} |(\triangle ijk, \bar{l})_E|} = O(||l||).$$

Chapter 5

Proof for the case of genus $g > 1$

5.1 Differential for PL hyperbolic metric

Parallel to the case for PL flat metric, we have the differential formulae for the PL hyperbolic metric under the discrete conformal change.

Lemma 5.1. *Given a hyperbolic triangle $\triangle ABC$, if we view A, B, C as functions of the edge lengths a, b, c, then*

$$\frac{\partial A}{\partial b} = -\frac{\cot C}{\sinh b}, \qquad \frac{\partial A}{\partial a} = \frac{1}{\sinh b \sin C}.$$

Furthermore if $(u_A, u_B, u_C) \in \mathbb{R}^3$ is a discrete conformal factor, and

$$\sinh\frac{a}{2} = e^{\frac{1}{2}(u_B+u_C)}\sinh\frac{a_0}{2}, \quad \sinh\frac{b}{2} = e^{\frac{1}{2}(u_A+u_C)}\sinh\frac{b_0}{2}, \quad \sinh\frac{c}{2} = e^{\frac{1}{2}(u_A+u_B)}\sinh\frac{c_0}{2}$$

for some constants $a_0, b_0, c_0 \in \mathbb{R}_{>0}$, then

$$\frac{\partial A}{\partial u_B} = \frac{1}{2}\cot\tilde{C}(1 - \tanh^2\frac{c}{2}), \tag{5.1}$$

and

$$\frac{\partial A}{\partial u_A} = -\frac{1}{2}\cot\tilde{B}(1 + \tanh^2\frac{b}{2}) - \frac{1}{2}\cot\tilde{C}(1 + \tanh^2\frac{c}{2}), \tag{5.2}$$

where $\tilde{B} = \frac{1}{2}(\pi + B - A - C)$ and $\tilde{C} = \frac{1}{2}(\pi + C - A - B)$.

Proof. Take the partial derivative on

$$\cos A = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}$$

and we have

$$\begin{aligned} -\sin A\frac{\partial A}{\partial b} =& \frac{\sinh b\cosh c}{\sinh b \sinh c} - \frac{\cosh^2 b\cosh c - \cosh a\cosh b}{\sinh^2 b\sinh c} \\ =& \frac{\cosh a\cosh b - \cosh c}{\sinh^2 b \sinh c} = \frac{\sinh a}{\sinh b\sinh c}\cos C, \end{aligned}$$

and then by the hyperbolic law of sines,

$$\frac{\partial A}{\partial b} = -\frac{\cos C}{\sinh b \sin C} = -\frac{\cot C}{\sinh b}.$$

Similarly

$$-\sin A \frac{\partial A}{\partial a} = -\frac{\sinh a}{\sinh b \sinh c}$$

and then again by the hyperbolic law of sines

$$\frac{\partial A}{\partial a} = \frac{1}{\sinh b \sinh C}.$$

To prove (5.1) and (5.2) we need to compute

$$\frac{\partial c}{\partial u_A} = \frac{\partial \sinh(c/2)}{\partial u_A} \Big/ \frac{\partial \sinh(c/2)}{\partial c} = \frac{1}{2} \sinh \frac{c}{2} \Big/ \left(\frac{1}{2} \cosh \frac{c}{2} \right) = \tanh \frac{c}{2},$$

and other similar formulae hold.

Since

$$\tanh \frac{x}{2} = \frac{\sinh \frac{x}{2}}{\cosh \frac{x}{2}} = \frac{\sinh \frac{x}{2} \cosh \frac{x}{2}}{\cosh^2 \frac{x}{2}} = \frac{\sinh x}{\cosh x + 1}$$

and

$$\cosh b + 1 = \frac{\cos A \cos C + \cos B}{\sin A \sin C} + 1 = \frac{\cos(A - C) + \cos B}{\sin A \sin C}, \tag{5.3}$$

we have

$$\begin{aligned} & \frac{\tanh \frac{b}{2}}{\tanh \frac{c}{2}} = \frac{\sinh b}{\sinh c} \cdot \frac{\cosh c + 1}{\cosh b + 1} \\ = & \frac{\sin B}{\sin C} \cdot \frac{\cos(A - B) + \cos C}{\sin A \sin B} \cdot \frac{\sin A \sin C}{\cos(A - C) + \cos B} \\ = & \frac{\cos(A - B) + \cos C}{\cos(A - C) + \cos B}, \end{aligned}$$

and then

$$
\begin{aligned}
-\cos A+\frac{\tanh\frac{b}{2}}{\tanh\frac{c}{2}}=&\frac{\cos(A-B)+\cos C-\cos(A-C)\cos A-\cos B\cos A}{\cos(A-C)+\cos B}\\
=&\frac{(\cos(A-B)-\cos B\cos A)+(\cos C-\cos(A-C)\cos A)}{\cos(A-C)+\cos B}\\
=&\frac{\sin A\sin B+\sin A\sin(A-C)}{\cos(A-C)+\cos B}\\
=&\sin A\cdot\frac{\sin B+\sin(A-C)}{\cos(A-C)+\cos B}\\
=&\sin A\cdot\frac{\sin\frac{B+A-C}{2}\cos\frac{B-A+C}{2}}{\cos\frac{A+B-C}{2}\cos\frac{A-B-C}{2}}\\
=&\sin A\cdot\tan\frac{A+B-C}{2}\\
=&\sin A\cot\tilde{C},
\end{aligned}
$$

and then

$$
\begin{aligned}
\frac{\partial B}{\partial u_A}=&\frac{\partial B}{\partial c}\frac{\partial c}{\partial u_A}+\frac{\partial B}{\partial b}\frac{\partial b}{\partial u_A}\\
=&-\frac{\cot A}{\sinh c}\tanh\frac{c}{2}+\frac{1}{\sin A\sinh c}\tanh\frac{b}{2}\\
=&\frac{1}{2\cosh^2\frac{c}{2}}(-\frac{\cos A}{\sin A}+\frac{1}{\sin A}\frac{\tanh\frac{b}{2}}{\tanh\frac{c}{2}})\\
=&\frac{1}{2}(1-\tanh^2\frac{c}{2})\frac{1}{\sin A}(-\cos A+\frac{\tanh\frac{b}{2}}{\tanh\frac{c}{2}})\\
=&\frac{1}{2}(1-\tanh^2\frac{c}{2})\cot\tilde{C}.
\end{aligned}
$$

By the symmetry equation (5.1) is true, and for the equation (5.2), we have that

$$
\frac{\partial A}{\partial u_A}=\frac{\partial A}{\partial c}\frac{\partial c}{\partial u_A}+\frac{\partial A}{\partial b}\frac{\partial b}{\partial u_A}=-\frac{\cot B}{\sinh c}\tanh\frac{c}{2}-\frac{\cot C}{\sinh b}\tanh\frac{b}{2}.
$$

So we need to show

$$
-\frac{\cot B}{\sinh c}\tanh\frac{c}{2}-\frac{\cot C}{\sinh b}\tanh\frac{b}{2}=-\frac{1}{2}\cot\tilde{C}(1+\tanh^2\frac{c}{2})-\frac{1}{2}\cot\tilde{B}(1+\tanh^2\frac{b}{2}).
$$

Since

$$
\frac{\tanh\frac{x}{2}}{\sinh x}=\frac{\sinh\frac{x}{2}}{2\sinh\frac{x}{2}\cosh^2\frac{x}{2}}=\frac{1}{\cosh^2\frac{x}{2}}=\frac{2}{\cosh x+1}
$$

and

$$
1+\tanh^2\frac{x}{2}=\frac{\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}}{\cosh^2\frac{x}{2}}=\frac{2\cosh x}{\cosh x+1},
$$

we only need to show

$$\frac{\cot B}{\cosh c+1}+\frac{\cot C}{\cosh b+1}=\cot\tilde{C}\frac{\cosh c}{\cosh c+1}+\cot\tilde{B}\frac{\cosh b}{\cosh b+1}.$$

We will show that

$$\cot\tilde{B}\frac{\cosh b}{\cosh b+1}-\frac{\cot B}{\cosh c+1}$$

is anti-symmetric with respect to B and C. Recall equation (5.3) and we have that

$$\cosh b+1=\frac{\cos(A-C)+\cos B}{\sin A\sin C}=\frac{2\cos\frac{A+B-C}{2}\cos\frac{B+C-A}{2}}{\sin A\sin C},$$

and

$$\frac{\cosh b}{\cosh b+1}=\frac{\cos A\cos C+\cos B}{\cos(A-C)+\cos B}=\frac{\cos A\cos C+\cos B}{2\cos\frac{A+B-C}{2}\cos\frac{B+C-A}{2}}$$

and

$$\cot\tilde{B}=\tan(\frac{\pi}{2}-\tilde{B})=\tan\frac{A+C-B}{2}.$$

So

$$\begin{aligned}
&\cot\tilde{B}\frac{\cosh b}{\cosh b+1}-\frac{\cot B}{\cosh c+1}\\
=&\tan\frac{A+C-B}{2}\cdot\frac{\cos A\cos C+\cos B}{2\cos\frac{A+B-C}{2}\cos\frac{B+C-A}{2}}-\cot B\frac{\sin A\sin B}{2\cos\frac{A+C-B}{2}\cos\frac{B+C-A}{2}}\\
=&\frac{\sin\frac{A+C-B}{2}(\cos A\cos C+\cos B)-\sin A\cos B\cos\frac{A+B-C}{2}}{2\cos\frac{A+C-B}{2}\cos\frac{B+C-A}{2}\cos\frac{A+B-C}{2}}.
\end{aligned}$$

The denominator in the above fraction is symmetric, so we only need to show the numerator is anti-symmetric with respect to B, C.

$$\begin{aligned}
&\sin\frac{A+C-B}{2}(\cos A\cos C+\cos B)-\sin A\cos B\cos\frac{A+B-C}{2}\\
=&(\sin\frac{A}{2}\cos\frac{C-B}{2}+\sin\frac{C-B}{2}\cos\frac{A}{2})(\cos A\cos C+\cos B)\\
&-\sin A\cos B(\cos\frac{A}{2}\cos\frac{C-B}{2}+\sin\frac{A}{2}\sin\frac{C-B}{2})\\
=&\sin\frac{C-B}{2}(\cos\frac{A}{2}\cos A\cos C+\cos\frac{A}{2}\cos B-\sin A\cos B\sin\frac{A}{2})\\
&+\cos\frac{C-B}{2}(\sin\frac{A}{2}\cos A\cos C+\sin\frac{A}{2}\cos B-\sin A\cos B\cos\frac{A}{2})\\
=&\sin\frac{C-B}{2}(\cos\frac{A}{2}\cos A\cos C+\cos A\cos B\cos\frac{A}{2})\\
&+\cos\frac{C-B}{2}(\sin\frac{A}{2}\cos A\cos C-\cos A\cos B\sin\frac{A}{2})\\
=&\sin\frac{C-B}{2}\cos A\cos\frac{A}{2}(\cos C+\cos B)+\cos\frac{C-B}{2}\sin\frac{A}{2}\cos A(\cos C-\cos B)
\end{aligned}$$

is indeed anti-symmetric with respect to B, C. $\square$

Now we are in the position to get the following proposition, which was first discovered by Bobenko et.al [2].

Proposition 5 (Proposition 6.1.7 in [2])**.** *Given* $(T,l)_H$ *and* $u \in \mathbb{R}^{V(T)}$ *such that* $u *_h l$ *satisfies the triangle inequalities, denote*

$$\tilde{\theta}^i_{jk}(u) = \frac{1}{2}(\pi + \theta^i_{jk}(u) - \theta^j_{ik}(u) - \theta^k_{ij}(u))$$

and

$$w_{ij}(u) = \frac{1}{2}\cot\tilde{\theta}^k_{ij}(u) + \frac{1}{2}\cot\tilde{\theta}^{k'}_{ij}(u)$$

where $\triangle ijk$ *and* $\triangle ijk'$ *are adjacent triangles in* $F(T)$*. Then*

$$\frac{\partial K}{\partial u}(u) = D(u) - \Delta_{\eta(u)}$$

where

$$\eta_{ij}(u) = w_{ij}(u)(1 - \tanh^2\frac{(u *_h l)_{ij}}{2}),$$

and $D = D(u)$ *is a diagonal matrix such that*

$$D_{ii}(u) = 2\sum_{j:ij\in E} w_{ij}(u)\tanh^2\frac{(u *_h l)_{ij}}{2}.$$

We also need an estimate for the angle difference between two hyperbolic triangles.

Lemma 5.2. *Given a hyperbolic triangle* $\triangle ABC$ *with all angles are at least* $\epsilon > 0$*. If positive numbers* $\tilde{a}, \tilde{b}, \tilde{c}$ *satisfy*

$$a \le 0.1, \quad b \le 0.1, \quad c \le 0.1,$$

and

$$|\tilde{a} - a| \le \delta a, \quad |\tilde{b} - b| \le \delta a, \quad |\tilde{c} - c| \le \delta c,$$

where $\delta < \epsilon^3/60$*, then* $\tilde{a}, \tilde{b}, \tilde{c}$ *can be edge lengths of a hyperbolic triangle* $\triangle\tilde{A}\tilde{B}\tilde{C}$ *with opposite inner angles* $\tilde{A}, \tilde{B}, \tilde{C}$ *respectively, such that*

$$|\tilde{A} - A| \le \frac{30}{\epsilon^2}\delta,$$

and

$$\left| |\triangle\tilde{A}\tilde{B}\tilde{C}| - |\triangle ABC| \right| \le \frac{120}{\epsilon^2}\delta \cdot |\triangle ABC|.$$

Proof. Let

$$a(t) = t\tilde{a} + (1-t)a, \quad b(t) = t\tilde{b} + (1-t)b, \quad c(t) = t\tilde{c} + (1-t)c,$$

and $A(t), B(t), C(t)$ be the inner angles of the triangle with edge lengths $a(t), b(t), c(t)$, if well-defined.

Let $T_0 \in [0, \infty]$ be the maximum real number such that for any $t \in [0, T_0)$, all $A(t), B(t), C(t) > \epsilon/2$. Notice that $\delta < \epsilon^3/60 < 0.1$ and then $\sinh a(t) \in [a, 2a]$ for any $t \in [0, T_0)$. Therefore by Lemma 5.1,

$$\begin{aligned}
|\dot{A}(t)| =& \left| \frac{\partial A}{\partial a}\dot{a} + \frac{\partial A}{\partial b}\dot{b} + \frac{\partial A}{\partial c}\dot{c} \right| \\
\leq& \frac{|\tilde{a} - a|}{\sinh b(t)\sin(\epsilon/2)} + \frac{\cot(\epsilon/2)|\tilde{b} - b|}{\sinh b(t)} + \frac{\cot(\epsilon/2)|\tilde{c} - c|}{\sinh c(t)} \\
\leq& \frac{|\tilde{a} - a|}{\sinh a(t)\sin^2(\epsilon/2)} + \frac{\cot(\epsilon/2)|\tilde{b} - b|}{\sinh b(t)} + \frac{\cot(\epsilon/2)|\tilde{c} - c|}{\sinh c(t)} \\
\leq& 2\delta \left(\frac{1}{\sin^2(\epsilon/2)} + 2\cot(\epsilon/2) \right) \\
\leq& 2(\frac{\pi^2}{\epsilon^2} + \frac{4}{\epsilon})\delta \leq \frac{30}{\epsilon^2}\delta.
\end{aligned}$$

Similary, we can get $|\dot{B}(t)|, |\dot{C}(t)| \leq 30\delta/\epsilon^2$. So $T_0 \geq (\epsilon/2)/(30\delta/\epsilon^2) = \epsilon^3/(60\delta) > 1$, and

$$|\tilde{A} - A| \leq 30\delta/\epsilon^2, \quad |\tilde{B} - B| \leq 30\delta/\epsilon^2, \quad |\tilde{C} - C| \leq 30\delta/\epsilon^2.$$

By Lemma 5.1, we have

$$\begin{aligned}
&\left| \frac{\partial(A+B+C)}{\partial a} \right| = \left| \frac{1}{\sinh b \sin C} - \frac{\cot C}{\sinh a} - \frac{\cot B}{\sinh a} \right| = \left| \frac{1}{\sinh a} \frac{\sin A - \sin(B+C)}{\sin B \sin C} \right| \\
\leq& \frac{1}{\sinh a} \frac{|\sin(\pi - A) - \sin(B+C)|}{\sin^2(\epsilon/2)} \leq \frac{\pi^2(\pi - A(t) - B(t) - C(t))}{\epsilon^2 \sinh a(t)} \leq \frac{2\pi^2}{\epsilon^2} \frac{|\triangle ABC(t)|}{a},
\end{aligned}$$

for $t \in [0, 1]$. Therefore

$$\begin{aligned}
&\left| \frac{d|\triangle ABC|}{dt} \right| = |\dot{A} + \dot{B} + \dot{C}| \leq \frac{2\pi^2}{\epsilon^2} |\triangle ABC(t)| \left(\frac{|\tilde{a} - a|}{a} + \frac{|\tilde{b} - b|}{b} + \frac{|\tilde{c} - c|}{c} \right) \\
\leq& \frac{6\pi^2\delta}{\epsilon^2} |\triangle ABC(t)|.
\end{aligned}$$

So

$$\frac{|\triangle \tilde{A}\tilde{B}\tilde{C}|}{|\triangle ABC|} \in [e^{-6\pi^2\delta/\epsilon^2}, e^{6\pi^2\delta/\epsilon^2}] \in [1 - 6\pi^2\delta/\epsilon^2, 1 + 120\delta/\epsilon^2]$$

and

$$\big||\triangle\tilde{A}\tilde{B}\tilde{C}| - |\triangle ABC|\big| \leq \frac{120}{\epsilon^2} \cdot \delta \cdot |\triangle ABC|.$$

□

5.2 Sketch of the proof for the case of genus g > 1

By Theorem 10, there exists a geodesic triangulation T' of $(M, e^{2\bar{u}}g)$ homotopic to T relative to $V(T) = V(T')$ if δ is sufficiently small. Let $\bar{l} \in \mathbb{R}^{E(T)} \cong R^{E(T')}$ be the geodesic edge lengths of T' in $(M, e^{2\bar{u}}g)$, then $(T, \bar{l})_H$ is isometric to $(M, e^{2\bar{u}}g)$ and globally hyperbolic.

For simplicity, we will frequently use the notion $a = O(b)$ to denote that if $\delta = \delta(M, g, \epsilon)$ is sufficiently small, then $|a| \leq C \cdot b$ for some constant $C = C(M, g, \epsilon)$. For example, we have that

(a) $l_{ij} = O(l_{jk})$ for any $\triangle ijk \in F(T)$, and

(b) $(\bar{u} *_h l)_{ij} = O(l_{ij})$, and

(c) $\bar{l}_{ij} = O(l_{ij})$, and

(d) $\sinh(l_{ij}/2) = O(l_{ij})$.

The remaining of the proof is divided into two steps.

(1) Firstly we show that $(T, \bar{u} *_h l)_H$ is very close to the globally hyperbolic PL metric $(T, \bar{l})_H$, in the sense that

$$(\bar{u} *_h l)_{ij} - \bar{l}_{ij} = O(l_{ij}^3)$$

and

$$K(\bar{u}) = div(x) + y$$

for some $x \in \mathbb{R}^E_A$ and $y \in \mathbb{R}^V$ such that $x_{ij} = O(l_{ij}^2)$ and $y_i = O(l_{ij}^4)$.

(2) Secondly, we construct a smooth path $u(t) : [0, 1] \to \mathbb{R}^V$ with $u(0) = \bar{u}$ such that the following identity

$$K(u(t)) = (1 - t)K(\bar{u})$$

holds. Furthermore we will show that $||u'(t)|| = O(||l||)$, and $(T, u(1) *_h l)_H$ is globally hyperbolic with $||u(1) - \bar{u}|| = O(||l||)$.

The uniqueness of the discrete uniformization conformal factor is also proved by Bobenko-Pinkall-Springborn (see Theorem 6.1.6 in [2]), so we omit its proof here.

5.3 Proof of the hyperbolic case

5.3.1 Step 1: Deviation of angles of curvatures of PL hyperbolic metric

By Theorem 10, $(T, \bar{l})_H$ is $\frac{1}{2}\epsilon$-regular if δ is sufficiently small. For simplicity we denote $\bar{u}|_{V(T)}$ as $\bar{u}$. By lemma 3.3, we get

$$\bar{l}_{ij} - (\bar{u} * l)_{ij} = O(l_{ij}^3).$$

Using the fact that $|2\sinh(\frac{x}{2}) - x| \leq |x|^3$ for $|x| \leq 1$, we have

$$\bar{l}_{ij} - (\bar{u} *_h l)_{ij} = O(l_{ij}^3).$$

Denote $\bar{\theta}^i_{jk}$ as the inner angle in $(T, \bar{l})_H$, and then by Lemmas 5.2, 3.2 and Remark 3.1, we have

$$\alpha^i_{jk} := \bar{\theta}^i_{jk} - \theta^i_{jk}(\bar{u}) = O(l_{ij}^2)$$

and

$$\alpha^i_{jk} + \alpha^j_{ik} + \alpha^k_{ij} = |(\triangle ijk, \bar{u} *_h l)_H| - |(\triangle ijk, \bar{l})_H| = O(l_{ij}^2) \cdot |(\triangle ijk, \bar{l})_H| = O(l_{ij}^4).$$

So $(T, \bar{u} *_h l)_H$ is $\frac{1}{3}\epsilon$-regular if δ is sufficiently small. Let $x \in \mathbb{R}^E_A$ and $y \in \mathbb{R}^V$ such that

$$x_{ij} = \frac{\alpha^i_{jk} - \alpha^j_{ik}}{3} + \frac{\alpha^i_{jk'} - \alpha^j_{ik'}}{3} \quad \text{and} \quad y_i = \frac{1}{3} \sum_{jk:\triangle ijk \in F(T)} (\alpha^i_{jk} + \alpha^j_{ik} + \alpha^k_{ij})$$

where $\triangle ijk$ and $\triangle ijk'$ are adjacent triangles sharing edge ij. Then simliar to the Euclidean case,

$$div(x)_i + y_i = K_i(\bar{u}),$$

and

$$x_{ij} = O(l_{ij}^2), \tag{5.4}$$

By the fact that any vertex $i \in V(T')$ has at most $\lfloor 2\pi/(\epsilon/2) \rfloor = O(1)$ neighboring vertices, we have

$$y_i = O(l_{ij}^4) \tag{5.5}$$

5.3.2 Step 2: Construction of the smooth path

Let

$$\tilde{\Omega}_H = \{u \in \mathbf{1}^\perp : u *_h l \text{ satisfies the triangle inequalities and } (T, u *_h l)_H \text{ is } \frac{\epsilon}{5}\text{-regular}\}$$

and

$$\Omega_H = \{u \in \tilde{\Omega} : ||u - \bar{u}|| \le 1,\ (T, u *_h l)_H \text{ is } \frac{\epsilon}{4}\text{-regular}\}.$$

Since $(T, \bar{u} *_h l)_H$ is $\frac{1}{3}\epsilon$-regular, $\bar{u}$ is in the interior of Ω_H. Now consider the following ODE on $int(\tilde{\Omega}_H)$,

$$\begin{cases} u'(t) = (D(u) - \Delta_{\eta(u)})^{-1} K(\bar{u}) = (D(u) - \Delta_{\eta(u)})^{-1}(div(x) + y) \\ u(0) = \bar{u} \end{cases}, \tag{5.6}$$

where $D(u)$ and $\eta(u)$ are defined as in Proposition 5. For any triangle $\triangle ijk$ and $u \in int(\tilde{\Omega}_H)$, by Lemma 3.2 and Remark 3.1 we have

$$|(\triangle ijk, u *_h l)_H| = O(l_{ij}^2)$$

and

$$\frac{1}{2}(\pi + \theta_{ij}^k(u) - \theta_{ik}^j(u) - \theta_{jk}^i(u)) = \theta_{ij}^k(u) + \frac{1}{2}(\pi - \theta_{ij}^k(u) - \theta_{ik}^j(u) - \theta_{jk}^i(u)) = \theta_{ij}^k(u) + O(l_{ij}^2).$$

Now let $w(u)$ be defined as in Proposition 5. Then by the formula

$$\cot A + \cot B = \frac{\sin(A+B)}{\sin A \sin B} \ge \sin(A+B) \quad \text{for any } A, B \in (0, \pi),$$

we can derive the following inequality if δ is sufficiently small and $u \in int(\tilde{\Omega}_H)$.

$$w_{ij}(u) \ge \frac{1}{2}\sin(\theta_{ij}^k + \theta_{ij}^{k'} + O(l_{ij}^2)) \ge \frac{1}{2}\sin\frac{\epsilon}{5} + O(l_{ij}^2) \ge \frac{1}{4}\sin\frac{\epsilon}{5},$$

and

$$D_{ii}(u) \ge 2w_{ij}\tanh^2\frac{(u *_h l)_{ij}}{2} \ge \epsilon' l_{ij}^2, \quad \text{and} \quad \eta_{ij}(u) \ge \frac{1}{8}\sin\frac{\epsilon}{5} \tag{5.7}$$

for some constant $\epsilon' = \epsilon'(M, g, \epsilon) > 0$.

The right-hand side of equation (5.6) is a smooth function of u, so the ODE (5.6) has a unique solution $u(t)$ satisfying

$$\frac{dK(u(t))}{dt} = \frac{\partial K}{\partial u} u'(t) = K(\bar{u}).$$

Therefore

$$K(u(t)) = (1-t)K(\bar{u}).$$

Assume the maximum existence open interval of $u(t) \in int(\Omega_H)$ is $(0, T_0)$ where $T_0 \in (0, \infty]$. By Theorem 10, when δ is sufficiently small, (T, l) is C-isoperimetric for some constant $C = C(M, g, \epsilon)$. Then for any $u \in \Omega_H$, $(T, u *_h l)$ is $(e^{4(|\bar{u}|+1)}C)$-isoperimetric by the fact that $|u| \leq |\bar{u}| + 1$ at any vertex and

$$\frac{\sinh a}{a} \geq \frac{\sinh b}{b}, \quad \forall a \geq b > 0.$$

By Lemma 3.2 and Remark 3.1, it is not difficult to see

$$|V|_l = O(|V|_{\bar{l}}) = O(|(T, \bar{l})_H|) = O(1) \quad \text{and} \quad 1 = O(|(T, \bar{l})_H|) = O(|V|_{\bar{l}}) = O(|V|_l).$$

Then by Lemma 9 and equation (5.4)(5.5)(5.7), for any $t \in [0, T_0)$

$$||u'(t)|| = O(||l|| \cdot |V|_l^{1/2}) = O(||l||). \tag{5.8}$$

By Lemma 5.2, we have

$$||u(T_0) - \bar{u}|| = O(||l||) \quad \text{and} \quad \theta^i_{jk}(u(T_0)) - \theta^i_{jk}(\bar{u}) = O(||l||), \forall \triangle ijk \in F(T) \tag{5.9}$$

if $T_0 < 1$. Therefore $u(T_0) \in int(\Omega_H)$ if δ is sufficiently small, which contradicts with the maximality of T_0. So $T_0 \geq 1$ and $(T, u(1))_H$ is hyperbolic and $||u(1) - \bar{u}|| = O(||l||)$.

Chapter 6

Proof for the case of genus $g = 0$

In this chapter, we will state the proof for the case of surfaces of genus zero.

6.1 An equivalent formulation for spherical uniformization

Springborn [21][5] and Bobenko et al. [2] proposed an another notion of discrete uniformization, which is for flat PL metric that are homeomorphic to a sphere. We adapt their definitions as follows.

Let $\mathcal{P}$ be the set of the compact convex polyhedral surfaces P, satisfying that

(a) P is the boundary of the convex hull of a finite subset of $\mathbb{S}^2$, and

(b) 0 is strictly inside P, and

(c) each face of P is a triangle.

Given $P \in \mathcal{P}$, denote $V(P)$ as the set of its vertices, and T_P as the natural triangulation of P where each triangle is a face of P, and $l_P \in \mathbb{R}^{E(T_P)}$ as the edge length of T_P on P. For a flat PL metric $(T, l)_E$, which is a topological sphere, we say that u is a *discrete uniformization factor* of $(T, l)_E$ if $(T, u * l)_E$ is isometric to some $P \in \mathcal{P}$, through a map φ such that $\varphi(T) = T_P$.

We call a geodesic triangulation of the unit sphere *strictly Delaunay* if the circumference circle of each triangle contains no other vertex. The spherical empty circle condition is equivalent to the condition that if for any edge ij in T, two adjacent triangles $\triangle ijk, \triangle ijk'$ containing ij satisfies

$$\theta_{ij}^{k} + \theta_{ij}^{k'} < \theta_{jk}^{i} + \theta_{jk'}^{i} + \theta_{ik}^{j} + \theta_{ik'}^{j}. \tag{6.1}$$

If we consider this geodesic triangulation as a sphercial PL metric $(T, l)_S$ where $l(e)$ is the geodesic arc length of edge e, then the above condition is just condition (1.4).

The central projection $p : (x, y, z) \mapsto (x, y, z)/\sqrt{x^2 + y^2 + z^2}$ naturally gives rise to a bijection between $\mathcal{P}$ and the set of strictly Delaunay triangulations of the unit sphere. Here we always assume that a triangulation of $\mathbb{S}^2$ is a geodesic triangulation and each triangle is a proper subset of a hemisphere. For a vector $x \in \mathbb{R}^I$, we denote $\sin x$ as the vector in $\mathbb{R}^I$ such that $(\sin x)_i = \sin(x_i)$.

Proposition 6. *Let $P \mapsto p(T_P)$ be a bijection from $\mathcal{P}$ to the set of strictly Delaunay triangulations of $\mathbb{S}^2$. Furthermore we have that $l_P = 2 \sin \frac{l}{2}$ where l denotes the geodesic edge lengths of $p(T_P)$ on $\mathbb{S}^2$.*

Proof. Given $P \in \mathcal{P}$, for any triangle $\triangle ijk$, other vertices are on one side of the plane $\triangle ijk$ lies in by the convexity of P. Therefore other vertices lies outside the circumference circle of spherical triangle $p(\triangle ijk)$. So $p(T_P)$ is a strictly Delaunay triangulation of $\mathbb{S}^2$. See Figure 6.1 for illustrations.

If T is a strictly Delaunay triangulation of $\mathbb{S}^2$, we construct a polyhedral surface P as the union of all flat triangles $\triangle ijk$ where i, j, k are the three vertices of a triangle in T. Then $p|_P$ is a homeomorphism from P to $\mathbb{S}^2$. Since T satisfies the empty circle property, we conclude that the dihedral angle on any edge $ij \in E(T)$ is less than π by the similar argument in the above paragraph. So P is a convex polyhedral surface (See Lemma 6.1 in [22] for example).

Since all the vertices of P is on the unit sphere, therefore P satisfies the condition (a) of set $\mathcal{P}$. It is not hard to see that P satisfies condition (b) and (c). The condition $l_P = 2 \sin \frac{l}{2}$ follows easily from the construction of P. $\square$

As a consequence, we obtain an equivalence between the two notions of discrete uniformizations.

Corollary. *Assume T is topologically a sphere, then u is a discrete uniformization factor of $(T, 2 \sin \frac{l}{2})_E$ if and only if u is a discrete uniformization factor of $(T, l)_S$ and $(T, u *_s l)_S$ is strictly Delaunay.*

By such an equivalence, we can reformulate our theorem 4 as follows.

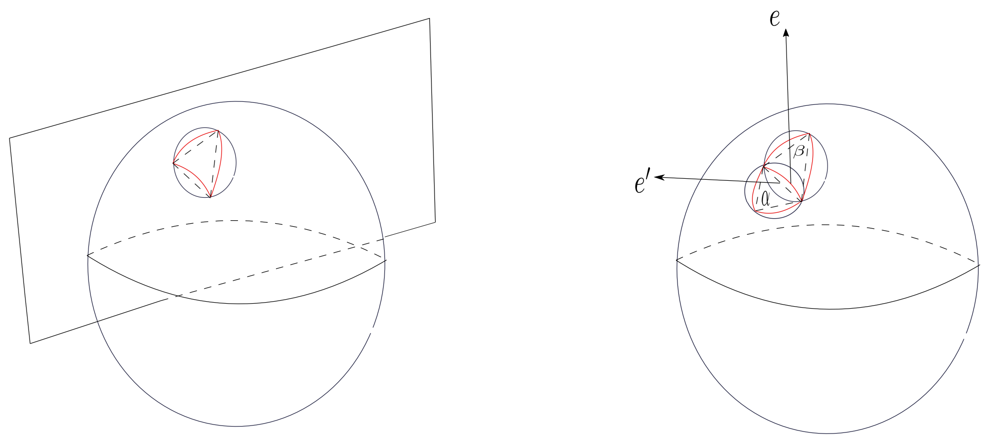

Figure 6.1: Equivalence between local Delaunay condition and local convexity.

Theorem 12. *Suppose (M, g) is a closed smooth Riemannian surface of genus zero with three marked points X, Y, Z, and $\bar{u} \in C^{\infty}(M)$ is the unique uniformization conformal factor such that $(M, e^{2\bar{u}}g)$ is isometric to the unit sphere $\mathbb{S}^2 \cong \hat{\mathbb{C}}$ through map ϕ, and $\phi(Z) = 0$, $\phi(Y) = 1$, $\phi(X) = \infty$. Assume T is a geodesic triangulation of (M, g) of geodesic edge length l such that its one-skeleton is a 4-vertex-connected graph. Then for any $\epsilon > 0$, there exist constants $\delta = \delta(M, g, X, Y, Z, \epsilon) > 0$ and $C = C(M, g, X, Y, Z, \epsilon) > 0$ such that if $(T, l)_S$ is ϵ-regular and $||l|| \leq \delta$, then*

(a) there exists a unique discrete conformal factor u on $V(T)$, such that $(T, u*(2\sin\frac{l}{2}))_E$ is isometric to some $P \in \mathcal{P}$ through a map ψ such that $\psi(Z) = 0$, $\psi(Y) = 1$, and $\psi(X) = \infty$, and

(b) $||u - \bar{u}|_{V(T)}|| \leq C||l||$.

We will prove this new version of our theorem. By the stereographic projection, we can consider triangulations of a flat polygon, instead of the polyhedrons inscribed in the unit sphere. To obtain a satisfactory flat PL metric, we construct a smooth path of conformal factor. The estimate in part (b) essentially follows from a discrete elliptic estimate on the this path. In Section 6.2, we will discuss the isoperimetric condition theorem for the PL metric with boudary and the corresponding version of discrete elliptic esimate. In Section 6.3, we will discuss the stereographic projection and the one-to-one correspondence between the convex polyhedral surfaces inscribed in the unit sphere and the the Delaunay triangulations of convex polygons. The proof of the theorem 12 will be given in Section 6.5.

6.2 Isoperimetric conditions and discrete elliptic estimates for spherical case

What we really need for the spherical case is the following modified version of Theorem 10.

Theorem 13. *Suppose (M, g) is a closed Riemannian surface, and T is a geodesic triangulation of (M, g) with geodesic length l such that $(T, l)_E$ is ϵ-regular. Assume $v \in V$ and $star(v) \subset V$ contains v and its neighbors in T, and let $\hat{V} = V - star(v)$ and $G = (\hat{V}, \hat{E})$ be the subgraph of $(V(T), E(T))$ generated by $\hat{V}$. Then there exists a constant $\delta = \delta(M, g, \epsilon)$ such that if $||l|| < \delta$, $(\hat{T}, l|_{\hat{E}})$ is C-isoperimetric for some constant $C = C(M, g, \epsilon) > 0$.*

Proof. By Theorem 10, we can find constants $\delta(M, g, \epsilon) > 0$ and $C(M, g, \epsilon) > 0$ such that (T, l) is C-isoperimetric if $||l|| < \delta$. Now assume $B = \{i \in \hat{V} : \exists j \in V - \hat{V} \text{ s.t. } ij \in E\}$ is the set of boundary vertices of G in T, and $V_0 \subset \hat{V}$, and $\hat{\partial} V_0$ (resp. ∂V) is the boundary of V_0 in G (resp. T). We consider the following three cases:

Case 1: $V_0 \cap B = \emptyset$. Then $|\hat{\partial} V_0|_l = |\partial V_0|_l$ and $|V|_l - |V_0|_l \geq |\hat{V}|_l - |V_0|_l$. Since (T, l) is C-isoperimetric, we have

$$C|\hat{\partial} V_0|_l^2 \geq \min\{|V_0|_l, |\hat{V}|_l - |V_0|_l\}.$$

Case 2: $B \subset V_0$. In this case, $\hat{\partial} V_0 = \partial(V_0 \cup star(v))$. Since (T, l) is C-isoperimetric, we have

$$C|\hat{\partial} V_0|_l^2 = C|\partial(V_0 \cup star(v))|_l^2 \geq \min\{|V_0 \cup star(v)|_l, |V|_l - |V_0 \cup star(v)|_l\}$$

Clearly, $|V_0 \cup star(v)|_l \geq |V_0|_l$, and $|V|_l - |V_0 \cup star(v)|_l = |\hat{V}|_l - |V_0|_l$. Then

$$C|\hat{\partial} V_0|_l^2 \geq \min\{|V_0|_l, |\hat{V}|_l - |V_0|_l\}.$$

Case 3: $V_0 \cap B \neq \emptyset$ and $B \not\subset V_0$. It is not difficult to show that B is connected in V since the 1-skeleton of T is 4-vertex-connected, so there is an edge $ij \in \hat{\partial} V_0$ such that $i \in B \cap V_0$ and $j \in B - V_0$. By the ϵ-regularity, the degree of each vertex in T is bounded by $\lfloor 2\pi/\epsilon \rfloor$ if $\delta(M, g, \epsilon)$ is sufficiently small, and the ratio of the two edge lengths in a triangle of T is at least $\sin\epsilon$. So there is a constant $C_1(M, g, \epsilon) > 0$ such that

$$C_1 l_{ij} \geq \sum_{xy \in E(T) - \hat{E}} l_{xy} \geq |\partial V_0|_l - |\hat{\partial} V_0|_l.$$

Then

$$C(1 + C_1)^2 |\hat{\partial} V_0|_l^2 \geq C|\partial V_0|_l^2 \geq \min\{|V_0|_l, |\hat{V}|_l - |V_0|_l\}.$$

□

The following discrete elliptic estimate is a key tool to prove the convergence theorem for the spherical case, which is reformulated from theorem 9.

Lemma 6.1. *Given a constant $C > 0$ and a C-isoperimetric pair (G, l), consider the equation*

$$(D - \Delta_\eta)u = div(x) + y \tag{6.2}$$

on G where

(i) $\eta \in \mathbb{R}^E$ *is an edge weight such that for any $ij \in E$*

$$\eta_{ij} \geq \frac{1}{C},$$

and

(ii) $x \in \mathbb{R}^E_A$ *is a flow such that for any $ij \in E$*

$$|x_{ij}| \leq C l_{ij}^2,$$

and

(iii) $D \in \mathbb{R}^{V \times V}$ *is a nonzero nonnegative diagonal matrix, and*

(iv) $y \in \mathbb{R}^V$ *satisfies that for any $i \in V$*

$$|y_i| \leq C \cdot D_{ii} ||l|| \cdot |V|_l^{\frac{1}{2}}.$$

Then the solution $u \in \mathbb{R}^V$ *of equation (6.2) satisfies that*

$$||u|| \leq C'||l|| \cdot |V|_l^{\frac{1}{2}}.$$

for some constant $C' = C'(C) > 0$.

6.3 Stereographic Projections of polyhedral surfaces

We will use the stereographic projection to connect convex polyhedral surfaces inscribed in the unit spheres with triangulations of planar convex polygons. Denote N as the north pole $(0, 0, 1)$ of the unit sphere $\mathbb{S}^2$. The stereographic projection p_N is a map from $\mathbb{R}^3 \backslash \{z = 1\}$ to the xy-plane, which is identified as $\mathbb{R}^2$ or $\mathbb{C}$. The map p_N is defined as

$$p_N(x, y, z) = \frac{x}{1-z} + i\frac{y}{1-z}.$$

It is well-known that the restriction of p_N on $\mathbb{S}^2 \backslash \{N\}$ is a conformal diffeomorphism to $\mathbb{R}^2$, and maps any circle to a circle or a straight line. Given a convex polyhedral surface P in $\mathcal{P}$ such that N is a vertex of P, denote $\mathring{T}_P$ as the subtriangulation of T_P with the open 1-star neighborhood of N in T_P being removed. In this case, $\mathring{P}$ denotes the carrier of $\mathring{T}_P$ and is a topological closed disk. We use $|\cdot|_2$ to denote the standard l^2-norm.

Lemma 6.2. *(a) Assume* $P \in \mathcal{P}$ *and* P *contains* N *as a vertex, then* p_N *is injective on* $\mathring{P}$, *and* $Q = p_N(\mathring{P})$ *is a convex polygon, and* $T_Q = p_N(\mathring{T}_P)$ *is a geodesic triangulation of* Q. *Furthermore, if we naturally identify* $\mathring{T}_P$ *and* T_Q, *and denote* l_P *(resp.* l_Q*) as its edge length on* P *(resp.* Q*), then*

$$l_Q = w * l_P \quad \text{where} \quad w_i = \log \frac{2}{|i - N|_2^2} = \log \frac{|p_N(i)|_2^2 + 1}{2}, \quad \forall i \in V(\mathring{T}_P). \tag{6.3}$$

(b) Assume Q *is a convex polygon in* $\mathbb{R}^2$, *and* T_Q *is a strictly Delaunay triangulation of* Q *such that* 0 *is an interior vertex and* $K_i > 0$ *for any boundary vertex* i *in* $V(T_Q)$. *Then there exists a convex polyhedral surface* $P \in \mathcal{P}$ *such that* $N \in P$ *and* $p_N(\mathring{P}) = Q$ *and* $p_N(\mathring{T}_P) = T_Q$.

Proof. (a) Let us first prove the injectivity by contradiction. Suppose x, y are two different points on $\mathring{P}$ such that $p_N(x) = p_N(y)$. Then N, x, y are co-linear and pairwise different. Without loss of generality, assume y is between x and N. Then it is not difficult to show that the line segment $\overline{xN} \subset P$. Suppose $\triangle ijk$ is a face of P containing the line segment $\overline{xN}$. Then N has to be one of the vertex of $\triangle ijk$, and x, y are contained in the edge in $\triangle ijk$ opposite to N. But this implies that N, x, y are not co-linear, which leads to a contradiction.

So Q is a polygon, and T_Q is a geodesic triangulation of Q. Any inner angle of the polygon Q is less than π, since the dihedral angle on any edge $Ni \in E(T_P)$ is less than π. Equation (6.3) can be proved by a standard computation.

(b) We will first construct a polyhedron P and then show that it is satisfactory. The set of vertices of P is given by $V_P = (p_N|_{\mathbb{S}^2})^{-1}(V(T_Q)) \cup \{N\}$, and the set of faces of P is given by a set of flat triangles in $\mathbb{R}^3$ with vertices in V_P. The $\triangle ijk$ is a triangle in P if and only if $p_N(\triangle ijk)$ is a triangle in T_Q, or $\{i, j, k\} = \{N, x, y\}$ where $p_N(xy)$ is a boundary edge of T_Q.

It is ordinary to verify that the union P of such triangles is a topological sphere, and these flat triangles naturally give a triangulation T_P of P. It remains to show that $P \in \mathcal{P}$, or indeed that any dihedral angle in T_P is less than π.

Assume ij is an edge in T_P. If $i = N$, the dihedral angle at ij is less than π because the discrete curvature at $p_N(j)$ in T_Q is greater than 0. If $p_N(ij)$ is a boundary edge in T_Q, assume $\triangle ijk \in F(T_P)$ and $k \neq N$, and then the dihedral angle at ij is less than π because $p_N(k)$ and 0 are in the same half plane divided by $p_N(ij)$. Now we can assume that $p_N(ij)$ is an inner edge in T_Q, and $p_N(\triangle ijk), p_N(\triangle ijk')$ are two triangles in T_Q. Since T_Q is strictly Delaunay, $p_N(k')$ is strictly outside of the circumcircle of $p_N(\triangle ijk)$. Since $p_N|_{\mathbb{S}^2}$ preserves circles, k' is strictly outside of the spherical circumcircle of $\{i, j, k\}$ on $\mathbb{S}^2$. So the dihedral angle at ij is less than π. $\square$

In the following lemma we prove that the stereographic projection preserves the ϵ-regularity.

Lemma 6.3. *Assume $P \in \mathcal{P}$, and $N \in P$, and $T = p(T_P)$ is a geodesic triangulation* *of* $\mathbb{S}^2$*, and* $Q = p_N(\mathring{P})$*, and* $T_Q = p_N(\mathring{T}_P)$*, and* l *(resp.* l_Q*) denotes the edge length of* T *(resp.* T_Q*) on* $\mathbb{S}^2$ *(resp.* Q*). Then for any* $\epsilon > 0$*, there exists constants* $\epsilon' = \epsilon'(\epsilon) > 0$ *and* $\delta = \delta(\epsilon) > 0$ *such that if* $(T, l)_S$ *is* ϵ*-regular and* $||l|| < \delta$*, then* $(T_Q, l_Q)_E$ *is* ϵ'*-regular, and* $K_i \geq \epsilon'$ *for any boundary vertex* i *in* T_Q*.*

Proof. Let θ_{ij}^k denote the inner angles in $(T, l)_S$, and ϕ_{ij}^k denote the inner angles in $(T_Q, l_Q)_E$. We need to prove following three statements: (a) ϕ_{ij}^k are bounded below by $\epsilon' > 0$, and (b) T_Q is strictly Delaunay with angle sums $\phi_{ij}^k + \phi_{ij}^{k'}$ bounded above by $\pi - \epsilon'$, and (c) $K_i \geq \epsilon'$ for any boundary vertex i in T_Q.

Consider a pair of triangles $\triangle ijk$ and $\triangle ijk'$ in $(T, l)_S$, and by assumption $\theta_{ij}^k + \theta_{ij}^{k'} \leq \pi - \epsilon$. Let Θ_{ij} be the intersecting angle of the circumcircles of two triangles $\triangle ijk$ and $\triangle ijk'$ on $\mathbb{S}^2$. It is elementary to show that

$$\Theta_{ij} = \theta_{jk}^i + \theta_{ik}^j + \theta_{jk'}^i + \theta_{ik'}^j - (\theta_{ij}^k + \theta_{ij}^{k'}) > 2\pi - 2(\theta_{ij}^k + \theta_{ij}^{k'}) \geq 2\epsilon.$$

The stereographic projection preserves angles and circles, so the intersecting angle of the circumcircles of $p_N(\triangle ijk)$ and $p_N(\triangle ijk')$ in T_Q is also Θ_{ij}, if N is not contained in $\triangle ijk \cup \triangle ijk'$. Then it is also ordinary to show that this intersecting angle is

$$\Theta_{ij} = \phi_{jk}^i + \phi_{ik}^j + \phi_{jk'}^i + \phi_{ik'}^j - (\phi_{ij}^k + \phi_{ij}^{k'}) = 2\pi - 2(\phi_{ij}^k + \phi_{ij}^{k'}).$$

Therefore part (b) is true by

$$\phi_{ij}^k + \phi_{ij}^{k'} = \pi - \frac{\Theta_{ij}}{2} \leq \pi - \epsilon.$$

If $i = N$, then the circumcircles of $\triangle ijk$ and $\triangle ijk'$ are mapped to straight lines $p_N(jk)$ and $p_N(jk')$. So the angle between $p_N(jk)$ and $p_N(jk')$, or the inner angle of the polygon Q at $p_N(j)$, is equal to $\pi - \Theta_{ij} \leq \pi - 2\epsilon$. So part (c) is true.

Now we prove part (a). Assume $\triangle ijk$ is a triangle in T not containing N, and C is the circumcircle of $\triangle ijk$ on $\mathbb{S}^2$, and C' is the circle on $\mathbb{S}^2$ containing $\{j, k, N\}$. Then $p_N(C)$ is the circumcircle of $p_N(\triangle ijk)$ and $p_N(C')$ is the straight line $p_N(jk)$, and the intersecting angle of them is equal to the intersecting angle of C and C'. It is elementary to show that ϕ_{jk}^i is equal to an intersecting angle of $p_N(C)$ and $p_N(C')$, i.e., an intersecting angle of C and C'. See Figure 6.2 for illustration. We only need to

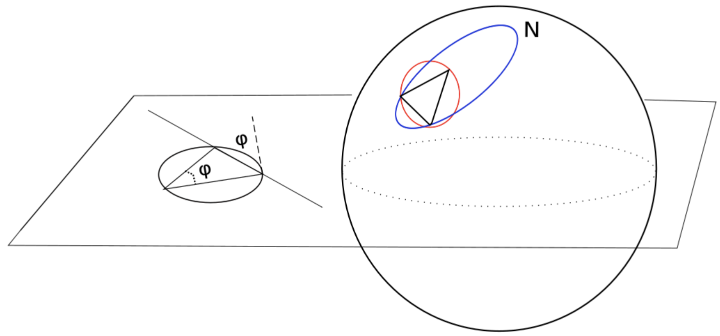

Figure 6.2: Projection of angles under stereographic projection.

show that the intersecting angle of C and C' are at least ϵ' for some constant $\epsilon'(\epsilon) > 0$, when $||l|| < \delta$ for some constant $\delta(\epsilon) > 0$.

Denote R as the spherical radius of the cirlce C. Since $(T, l)_S$ is ϵ-regular and $||l|| < \delta$, the degree (valence) of any vertex in T is at most $\lfloor 2\pi/\epsilon \rfloor$, and l_{ij}, l_{ik}, l_{jk} are at least $r_1 R$ for some constant $r_1 = r_1(\epsilon)$. Furthermore it is not difficult to show that there exists a constant $r_2(\epsilon) > 0$ such that the 1-star neighborhood of i in T contains the open spherical disk U_i centered at i with radius $r_2 R$. So $N \notin U_i$. We define U_j and U_k similarly.

Assume C_k is the circumcircle of the triangle in T that is adjacent to $\triangle ijk$ along the edge ij. Then the intersecting angle between C and C_k is $\Theta_{ij} \geq 2\epsilon$. Assume C'_k is the circle on $\mathbb{S}^2$ such that $i, j \in C'_k$ and the intersecting angle between C and C'_k is equal to 2ϵ. Denote D_k (resp. D'_k, D) as the open spherical disk bounded by C_k (resp. C'_k, C). Then $D'_k \subset D \cup D_k$ and has a diameter less than $r_3 R$ for some constant $r_3(\epsilon) > 0$. Then $N \notin D_k$ and $N \notin D$ by the convexity of P, and so $N \notin D'_k$. Define D'_i and D'_j similarly.

Without loss of generality, assume j' is the opposite point of j on $\mathbb{S}^2$, and $j' \neq N$. Denote X as the tangent plane of $\mathbb{S}^2$ at j in $\mathbb{R}^3$, and p' as the projection map centered at j' and mapping $\mathbb{S}^2 \backslash \{j'\}$ to X. Since p' preserves the angles and disks, $p'(D)$, $p'(D'_i)$, $p'(D'_j)$, $p'(D'_k)$, $p'(U_i)$, $p'(U_j)$, $p'(U_k)$ are all disks on X, and the intersecting angle between $p'(D)$ and $p'(D_i)$ (resp. $p(D_j)$, $p'(D_k)$) is 2ϵ, and we only need to show that the intersecting angle between $p'(C)$ and $p'(C')$ is at least ϵ' for some constant $\epsilon'(\epsilon) > 0$.

The projection p' is very close to an isometry near the point j, so if $\delta = \delta(\epsilon) > 0$ is sufficiently small,

$$\frac{1}{2}d(x, y) \leq d(p'(x), p'(y)) \leq 2d(x, y)$$

for any $x, y \in D \cup D_i' \cup D_j' \cup D_k' \cup U_i \cup U_j \cup U_k$. Assume R' is the radius of $p'(D)$ and it is not difficult to show that the radii of $p'(U_i)$ and $p'(U_j)$ and $p'(U_k)$ are at least $r_2R'/4$, and $d(p'(a), p'(b)) \geq r_1R'/4$ for any two different vertices a, b in $\{i, j, k\}$.

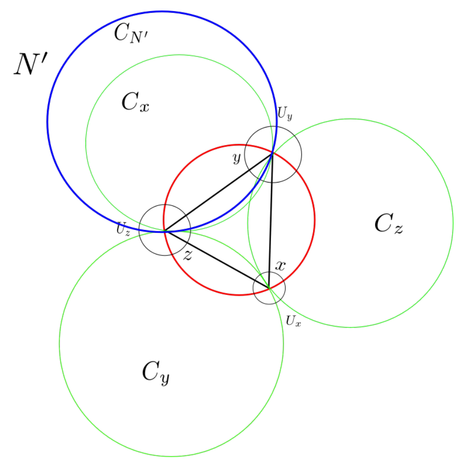

Figure 6.3: Seven circles.

By a scaling, it suffices to prove the following claim. Assume $\triangle xyz$ is a triangle inscribed in the unit circle $\mathbb{S}$ in the plane, and all the edge lengths are at least $r_1/4$, and C_x is the circle such that $y, z \in C_x$ and x is not inside C_x and the intersecting angle between C and C_x is 2ϵ, and C_y and C_z are defined similarly, and U_x is the open disk centered at x with radius $r_2/4$, and U_y and U_z are defined similarly, and N' is a point that is not strictly inside of the unit circle or C_x or C_y or C_z or U_x or U_y or U_z, and $C_{N'}$ is the circle (or the straight line) passing through y, z, N', then the intersecting angle θ between C and $C_{N'}$ is at least ϵ' for some constant $\epsilon'(\epsilon) > 0$. See Figure 6.3 for illustration.

If the above claim is not true, for some $\epsilon > 0$, one can pick a sequence $(x_n, y_n, z_n, N_n') \in (\mathbb{R}^2)^4$ such that the resulted intersecting angle θ_n goes to 0. By picking a subsequence, we may assume $x_n \to x \in \mathbb{S}$ and $y_n \to y \in \mathbb{S}$ and $z_n \to z \in \mathbb{S}$ and $N_n' \to N' \in \mathbb{R}^2 \cup \{\infty\}$, and then by the continuity (x, y, z, N') satisfies the conditions in the claim, and the resulted intersecting angle is $\theta = 0$. This means that N' is on the unit circle. However this is impossible because by the continuity

$$N' \notin D_x \cup D_y \cup D_z \cup U_x \cup U_y \cup U_z \supset \mathbb{S}$$

where D_x (resp. D_y, D_z) is the open disk bounded by C_x (resp. C_y, C_z). $\square$

6.4 Estimate of linear map between Euclidean triangles

Before proceeding to the proof of Theorem 12, we need the following lemma showing that a linear map between two Euclidean triangles is close to isometry if their corresponding edge lengths are close.

Lemma 6.4. *Assume $\triangle ABC$ ($\triangle \tilde{A}\tilde{B}\tilde{C}$) is a Euclidean triangle with edge lengths a, b, c (resp. $\tilde{a}, \tilde{b}, \tilde{c}$). We require that all the angles in $\triangle ABC$ are at least $\epsilon > 0$, $\delta < \epsilon^2/576$, and*

$$|\tilde{a} - a| \leq \delta a, \quad |\tilde{b} - b| \leq \delta a, \quad |\tilde{c} - c| \leq \delta c,$$

If λ_1, λ_2 are the two singular values of the unique linear map sending $\triangle ABC$ to $\triangle \tilde{A}\tilde{B}\tilde{C}$ preserving the correspondence of the vertices, then

$$1 - \frac{10^4}{\epsilon^4}\delta \leq \lambda_i \leq 1 + \frac{10^4}{\epsilon^4}\delta, \quad i = 1, 2.$$

Proof. By Lemma 3.5, $|A - \tilde{A}|, |B - B'|, |C - C'|$ are all less or equal to $24\delta/\epsilon < \epsilon/2$, and thus $\tilde{A}, B', C'$ are all at least $\epsilon/2$. Then again by Lemma 3.5 it is easy to show that

$$\tilde{A}^2 \leq \frac{2|\triangle \tilde{A}\tilde{B}\tilde{C}|}{\sin^2(\epsilon/2)} \leq \frac{64|\triangle ABC|}{\epsilon^2}.$$

It is well known that

$$\lambda_1\lambda_2 = \frac{|\triangle \tilde{A}\tilde{B}\tilde{C}|}{|\triangle ABC|}$$

and thus by Lemma 4.2

$$|\lambda_1\lambda_2 - 1| < \frac{576}{\epsilon^2}\delta. \tag{6.4}$$

In [23] we can find the formula

$$\lambda_1^2 + \lambda_2^2 = \frac{\tilde{a}^2 \cot A + \tilde{b}^2 \cot B + \tilde{c}^2 \cot C}{2|\triangle ABC|}.$$

Applying this formula to the special case $\triangle\tilde{A}\tilde{B}\tilde{C} = \triangle ABC$, we get

$$2 = \frac{\tilde{a}^2 \cot \tilde{A} + \tilde{b}^2 \cot \tilde{B} + \tilde{c}^2 \cot \tilde{C}}{2|\triangle\tilde{A}\tilde{B}\tilde{C}|},$$

, which implies

$$\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 = \frac{\tilde{a}^2(\cot A - \cot \tilde{A}) + \tilde{b}^2(\cot B - \cot \tilde{B}) + \tilde{c}^2(\cot C - \cot \tilde{C})}{2|\triangle ABC|}. \quad (6.5)$$

Denote $f(x) = \cot x$, then $f'(x) = -1/\sin^2 x$ and $f''(x) = 2\cos x/\sin^3 x$. By Taylor's expansion, there exists ξ_A between A and $\tilde{A}$ such that

$$\tilde{a}^2(\cot A - \cot \tilde{A}) = \tilde{a}^2[f'(\tilde{A})(A - \tilde{A}) + \frac{1}{2}f''(\xi_A)(A - \tilde{A})^2]$$

$$= -\frac{\tilde{a}^2}{\sin^2 \tilde{A}}(A - \tilde{A}) + \frac{\tilde{a}^2}{2}f''(\xi_A)(A - \tilde{A})^2 = -(2R)^2(A - \tilde{A}) + \frac{\tilde{a}^2}{2}f''(\xi_A)(A - \tilde{A})^2$$

where R is the radius of the cicumcircle of $\triangle\tilde{A}\tilde{B}\tilde{C}$, and

$$\left|\frac{\tilde{a}^2}{2}f''(\xi_A)(A - \tilde{A})^2\right| \leq \frac{64|\triangle ABC|}{\epsilon^2} \cdot \frac{2}{\sin^3(\epsilon/2)} \cdot \left(\frac{24\delta}{\epsilon}\right)^2 \leq |\triangle ABC| \cdot \frac{10^6 \cdot \delta^2}{\epsilon^7}.$$

Combining the similar computation for B and C, we get that the right hand side of equation (6.5) is less or equal to $3 \times 10^6\delta^2/\epsilon^7$, and thus $|\lambda_1 - \lambda_2| \leq \sqrt{3 \times 10^6\delta^2/\epsilon^7} \leq 10^4\delta/\epsilon^4$. Then by equation (6.4) and the fact that $10^4\delta/\epsilon^4 \geq 576\delta/\epsilon^2$, it is easy to prove that

$$1 - \frac{10^4}{\epsilon^4}\delta \leq \lambda_i \leq 1 + \frac{10^4}{\epsilon^4}\delta.$$

□

6.5 Sketch of the proof for the case of genus $g > 1$

Assume $\epsilon > 0$ is a fixed constant and $(T, l)_S$ is ϵ-regular and $||l|| < \delta$ where

$$\delta = \delta(M, g, X, Y, Z, \epsilon) > 0$$

is a sufficiently small constant to be determined. By Lemma 3.3 we may assume that $(T, l)_E$ is $(\epsilon/2)$-regular. By Theorem 10, we may assume that there exists a geodesic triangulation T' of $(M, e^{2\bar{u}}g)$ such that T' is homotopic to T relative to $V(T) = V(T')$. Denote $\bar{l} \in \mathbb{R}^{E(T)} \cong \mathbb{R}^{E(T')}$ as the geodesic edge length of T' in $(M, e^{2\bar{u}}g)$, and then

$(T,\bar{l})_S$ is isometric to the unit sphere $(M, e^{2\bar{u}}g)$ and has zero discrete curvatures. Again by Lemma 10 we may assume that $(T,\bar{l})_E$ is $(\epsilon/4)$-regular. Then by Lemma 3.3 we may assume $(T,\bar{l})_S$ is $(\epsilon/5)$-regular, and thus is strictly Delaunay, and then by Proposition 6 $p(T_P) = \phi(T')$ where $P \in \mathcal{P}$ is the boundary of the convex hull of $\phi(V(T))$. By Lemma 6.2, $Q = p_N(\mathring{P})$ is a convex polygon, and $T_Q = p_N(\mathring{T}_P)$ is a geodesic triangulation of Q. Denote $l_Q \in \mathbb{R}^{E(T_Q)}$ as the edge lengths in Q, and then by Lemma 6.3 there exists a constant $\epsilon'(M, g, X, Y, Z, \epsilon) > 0$ such that $(T_Q, l_Q)_E$ is ϵ'-regular and $K_i \geq \epsilon'$ for any boundary vertex in $(T_Q, l_Q)_E$. The combinatorial structures of T, T', T_P and $p(T_P) = \phi(T')$ are naturally identified. We also identify the combinatorial structures of $\mathring{T}_P$ and T_Q and just denote it as $\mathring{T}$. Denote $l_P = 2\sin(\bar{l}/2) \in \mathbb{R}^{E(T)}$ as the edge length of T_P on P, and then by equation (6.3) on $\mathring{T}$ we have

$$l_Q = w * l_P \quad \text{where} \quad w_i = \log \frac{2}{|\phi(i) - N|_2^2} = \log \frac{|p_N(\phi(i))|_2^2 + 1}{2}, \quad \forall i \in V(\mathring{T}).$$

Denote $l'_P = \bar{u} * 2\sin(l/2) \in \mathbb{R}^{E(T)}$ and $l'_Q = w * l'_P = (\bar{u} + w) * 2\sin(l/2) \in \mathbb{R}^{E(\mathring{T})}$, and $K(u) \in \mathbb{R}^{V(\mathring{T})}$ as the discrete curvature in $(\mathring{T}, u * 2\sin(l/2))_E$.

In the following proof, for simplicity we will use the notation $a = O(b)$ to represent that if $\delta(M, g, X, Y, Z, \epsilon)$ is sufficiently small, then $|a| \leq Cb$ for some constant $C(M, g, X, Y, Z, \epsilon) > 0$. We summarize the remaining part of the proof in three steps:

(a) Estimate the curvature $K(\bar{u} + w)$ of $(\mathring{T}, l'_Q)_E$ for interior vertices.

(b) Construct a smooth path $u(t) : [0,1] \to \mathbb{R}^{V(\mathring{T})}$ with $u(0) = \bar{u} + w$ such that the equality

$$K_i(u(t)) = (1-t)K_i(\bar{u} + w)$$

holds for any interior vertex i of $\mathring{T}$. Furthermore, we will also show that $|u'(t)| = O(||l||)$, and $(\mathring{T}, u(1) * 2\sin(l/2))_E$ is isometric to a convex polygon in the plane.

(c) After a proper normalization, which is a small perturbation, we use the inverse of the stereographic projection to construct the desired polyhedral surface $P \in \mathcal{P}$.

6.6 Proof of the spherical case

6.6.1 Step 1: Estimate of curvatures

By Lemma 3.3,

$$|\bar{l}_{ij} - (\bar{u} * l)_{ij}| = O(l_{ij}^3).$$

Notice the fact that $|x - 2\sin(x/2)| \leq 10x^3$ if $|x| < 0.01$, so

$$|(l_P)_{ij} - (l'_P)_{ij}| = O(l_{ij}^3),$$

and

$$\left|\frac{(l'_Q)_{ij} - (l_Q)_{ij}}{(l_Q)_{ij}}\right| = \left|\frac{(l'_P)_{ij} - (l_P)_{ij}}{(l_P)_{ij}}\right| = O(l_{ij}^2). \tag{6.6}$$

Given a triangle $\triangle ijk \in F(\mathring{T})$, denote $\theta^i_{jk}(u)$ (*resp.* $\bar{\theta}^i_{jk}$) as the inner angle at i in $\triangle ijk$ in $(\mathring{T}, u * 2\sin(l/2))_E$ (*resp.* $(\mathring{T}, l_Q)_E$), and $K_i(u)$ as the discrete curvature at i in $\triangle ijk$ in $(\mathring{T}, u * 2\sin(l/2))_E$.

Since $(\mathring{T}, l_Q)_E$ is ϵ'-regular, by equation (6.6) and Lemma 4.2,

$$\alpha^i_{jk} := \bar{\theta}^i_{jk} - \theta^i_{jk}(\bar{u} + w) = O(l_{ij}^2).$$

So for sufficiently small $\delta(M, g, \epsilon)$, we have

$$|\alpha^i_{jk}| \leq \frac{\epsilon'}{4}.$$

Then $(\mathring{T}, l'_Q)_E$ is $(\epsilon'/2)$-regular. Since $(\mathring{T}, l_Q)_E$ is globally flat, for any $i \in \mathrm{int}(\mathring{T})$,

$$\sum_{ijk \in F} \bar{\theta}^i_{jk} = 2\pi.$$

So

$$K_i(\bar{u} + w) = 2\pi - \sum_{ijk \in F} \theta^i_{jk}(\bar{u} + w) = \sum_{ijk \in F} (\bar{\theta}^i_{jk} - \theta^i_{jk}(\bar{u} + w)) = \sum_{ijk \in F} \alpha^i_{jk} \quad \text{if } i \in \mathrm{int}(\mathring{T}).$$

Set $x \in \mathbb{R}^{E(\mathring{T})}_A$ be such that if ij is an interior edge of $\mathring{T}$

$$x_{ij} = \frac{\alpha^i_{jk} - \alpha^j_{ik}}{3} + \frac{\alpha^i_{jk'} - \alpha^j_{ik'}}{3},$$

where $\triangle ijk$ and $\triangle ijk'$ are adjacent triangles in $\mathring{T}$, and if ij is a boundary edge of $\mathring{T}$

$$x_{ij} = \frac{\alpha^i_{jk} - \alpha^j_{ik}}{3},$$

where $\triangle ijk$ is a triangle in $\mathring{T}$. Then

$$x_{ij} = O(l_{ij}^2), \tag{6.7}$$

and it is straightforward to verify that for any $i \in \mathrm{int}(\mathring{T})$

$$div(x)_i = \sum_{j:j\sim i} x_{ij} = \sum_{ijk\in F} \alpha_{jk}^i = K_i(\bar{u}+w)$$

using $\alpha_{jk}^i + \alpha_{ik}^j + \alpha_{ij}^k = 0$.

6.6.2 Step 2: Construction of the path

Consider the sets defined by

$$\tilde{\Omega} = \{u \in \mathbb{R}^{V(\mathring{T})} : (\mathring{T}, u * 2\sin\frac{l}{2})_E \text{ satisfies the triangle inequality and is strictly Delaunay}\},$$

and

$$\Omega = \{u \in \tilde{\Omega} : \ (\mathring{T}, u * 2\sin\frac{l}{2})_E \text{ is } \frac{\epsilon'}{4}\text{-regular}, ||u - (\bar{u}+w)|| \leq 1\}.$$

Notice that $\tilde{\Omega}$ is an open domain in $\mathbb{R}^{V(\mathring{T})}$ and Ω is a compact subset of $\tilde{\Omega}$. By the construction, $(\bar{u}+w)$ is in the interior of Ω, since $(\mathring{T}, l'_Q)_E$ is $(\epsilon'/2)$-regular. Given $u \in \tilde{\Omega}$ and an interior edge ij in $\mathring{T}$, denote

$$\eta_{ij}(u) = \frac{1}{2}(\cot\theta_{ij}^k(u) + \cot\theta_{ij}^{k'}(u))$$

where $\triangle ijk$ and $\triangle ijk'$ are adjacent triangles in $\mathring{T}$. Then for $u \in \Omega$,

$$2\eta_{ij}(u) = \cot\theta_{ij}^k(u) + \cot\theta_{ij}^{k'}(u) = \frac{\sin(\theta_{ij}^k(u)+\theta_{ij}^{k'}(u))}{\sin\theta_{ij}^k(u)\sin\theta_{ij}^{k'}(u)} \geq \sin(\theta_{ij}^k(u)+\theta_{ij}^{k'}(u)) \geq \sin\frac{\epsilon'}{4} \tag{6.8}$$

for any interior edge ij in $\mathring{T}$.

Consider the following system of differential equations on $\tilde{\Omega}$,

$$\begin{aligned} \frac{\partial K_i}{\partial u}\frac{du}{dt} &= -K_i(\bar{u}+w) = -div(x)_i, & i \in \mathrm{int}(\mathring{T}), \quad (6.9)\\ \frac{du_i}{dt} &= (\log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X) - (\bar{u}_i + w_i), & i \in \mathrm{bdy}(\mathring{T}),\\ u(0) &= \bar{u}+w, \end{aligned}$$

where $\bar{u}_X$ is the value of $\bar{u}$ at the marked point X sent to the north pole, and l_{iX} is the length of the edge iX given by l. We want to show that the solution $u(t)$ exists on $[0,1]$, then it is easy to see that $K_i(u(1)) = 0$ for an interior vertex i of $\mathring{T}$, and

$$u_i(1) = \log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X$$

for a boundary vertex i of $\mathring{T}$.

For a boundary vertex i of $\mathring{T}$, $u_i(t)$ can be easily solved as

$$u_i(t) = t(\log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X) + (1-t)(\bar{u}_i + w_i),$$

and

$$\begin{aligned}\frac{du_i}{dt} =&(\log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X) - (\bar{u}_i + w_i)\\ =&\log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X - \bar{u}_i - \log\frac{2}{(l_P)^2_{iX}}\\ =&-2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X - \bar{u}_i + 2\log(l_P)_{iX}\\ =&-2\log(l'_P)_{iX} + 2\log(l_P)_{iX}\\ =&O(l^2_{iX}).\end{aligned}$$

Now let us focus on solving $u_i(t)$ for all the interior vertices of $\mathring{T}$. Let $\hat{V}$ be the set of interior vertices of $\mathring{T}$, and $G = (\hat{V}, \hat{E})$ be the subgraph of $(V(\mathring{T}), E(\mathring{T}))$ generated by $\hat{V}$. It is easy to show that G is nonempty and connected. Let $\hat{u} \in \mathbb{R}^{\hat{V}}$ and $\hat{x} \in \mathbb{R}^{\hat{E}}_A$ and $\hat{\eta} \in \mathbb{R}^{\hat{E}}$ be the restrictions of u and x and η respectively on $G = (\hat{V}, \hat{E})$, and $\hat{\Delta} = \Delta_{\hat{\eta}}$ be the associated discrete Laplacian on G. Then by Proposition 4, it is straightforward to verify that Equation (6.9) can be rewritten as

$$(D - \hat{\Delta})\frac{d\hat{u}}{dt} = -div(\hat{x}) + y, \tag{6.10}$$

where

(a) $D \in \mathbb{R}^{\hat{V}\times\hat{V}}$ is a nonzero diagonal matrix and

$$D_{ii} = \sum_{j\sim i: j\notin\hat{V}} \eta_{ij} \geq 0,$$

and

(b)

$$y_i = \sum_{j\sim i: j\notin\hat{V}} \eta_{ij}\frac{du_j}{dt} - \sum_{j\sim i: j\notin\hat{V}} x_{ij}.$$

For $u \in \tilde{\Omega}$, it is easy to show that $(D - \hat{\Delta})$ is positive definite, by the fact that G is connected, $\eta_{ij} > 0$ for any $ij \in \hat{E}$, and D is nonzero and non-negative. So equation (6.10) locally has a unique solution in $\tilde{\Omega}$.

Assume the maximum existence open interval for the solution $\hat{u}(t) \in \Omega$ is $(0, T_0)$ where $0 < T_0 \leq +\infty$. For $t \in [0, T_0)$, we have

$$\frac{d\hat{u}}{dt}(t) = O(||l|| \cdot |\hat{V}|_l^{1/2})$$

by Lemma 6.1, and Theorem 13, and equation (6.8) and (6.7), and the fact that

$$y_i = \sum_{j\sim i: j\notin\hat{V}} (\eta_{ij}\frac{du_j}{dt} - x_{ij}) = O\left(\sum_{j\sim i: j\notin\hat{V}} (\eta_{ij} l_{jX}^2 + l_{ij}^2)\right) = O(D_{ii}||l||^2) = O(D_{ii}||l|| \cdot |\hat{V}|_l^{1/2}).$$

Furthermore

$$\begin{aligned}|\hat{V}|_l \leq |V|_l = \sum_{ij\in E} l_{ij}^2 = O(\sum_{ij\in E} \bar{l}_{ij}^2) = O(\sum_{ijk\in F} (\bar{l}_{ij}^2 + \bar{l}_{jk}^2 + \bar{l}_{ik}^2))\\ = O(\sum_{ijk\in F} Area(\triangle ijk, \bar{l})_S) = O(Area((T, \bar{l})_S)) = O(Area(\mathbb{S}^2)) = O(1),\end{aligned}$$

and thus $(du/dt)(t) = O(||l||)$ for $t \in [0, T_0)$.

If $T_0 \leq 1$, combining Lemma 6.1, we have

$$||u(T_0) - (\bar{u} + w)|| = O(||l||), \quad \text{and} \quad |\theta_{jk}^i(u(T_0)) - \theta_{jk}^i(\bar{u} + w)| = O(||l||).$$

This implies that $u(T_0) \in \text{int}(\Omega)$ if δ is sufficiently small, which contradicts to the maximality of T_0. Thus, $T_0 > 1$ and $u(1)$ is well-defined. Furthermore we have that

(a) $K_i(u(1)) = 0$ for any interior vertex i of $\mathring{T}$, and

(b) $u_i(1) = \log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X$ for any boundary vertex i of $\mathring{T}$, and

(c) $u(1) - (\bar{u} + w) = O(||l||)$, and

(d) $(\mathring{T}, u(1) * 2\sin\frac{l}{2})_E$ is strictly Delaunay, and

(e) $K_i(u(1)) > 0$ for any boundary vertex i in $\mathring{T}$.

6.6.3 Step 3: Normalization and the inverse of the stereographic projection

We know that $(\mathring{T}, u(1) * 2\sin\frac{l}{2})_E$ is isometric to a closed convex polygon in $\mathbb{C}$. Let f be the piecewise linear map from $(\mathring{T}, u(1) * 2\sin\frac{l}{2})_E$ to $(\mathring{T}, l_Q)_E$ that preserves the triangulation and is linear on each triangle. From equation (6.6) and the fact that $u(1) - (\bar{u} + w) = O(||l||)$, we can deduce that

$$\Big|\frac{(u(1) * 2\sin\frac{l}{2})_{ij} - (l_Q)_{ij}}{(l_Q)_{ij}}\Big| = \Big|\frac{(u(1) * 2\sin\frac{l}{2})_{ij} - ((\bar{u} + w) * 2\sin\frac{l}{2})_{ij}}{(l_Q)_{ij}}\Big| + O(||l||^2) = O(||l||).$$

Then by Lemma 6.4, $\|Df\|_2$ and $\|Df^{-1}\|_2$ are both $(1 + C||l||)$-Lipschitz for some constant $C(M, g, X, Y, Z, \epsilon) > 0$. So the distance d_{YZ} between Y and Z in $(\mathring{T}, u(1) * 2\sin(l/2))_E$ lies in $[1 - C||l||, 1 + C||l||]$. So we can scale $(\mathring{T}, u(1) * 2\sin(l/2))_E$ by letting $\tilde{u} = u(1) - \log d_{YZ}$, and then $(\mathring{T}, \tilde{u} * 2\sin(l/2))_E$ is still isometric to a convex polygon and the distance between Y and Z is 1, and

$$\Big|\frac{\tilde{u} * 2\sin\frac{l}{2} - (l_Q)_{ij}}{(l_Q)_{ij}}\Big| = \Big|\frac{\tilde{u} * 2\sin\frac{l}{2} - u(1) * 2\sin\frac{l}{2}}{(l_Q)_{ij}}\Big| + O(||l||) = O(||l||).$$

Let g be the isometry from $(\mathring{T}, \tilde{u} * 2\sin\frac{l}{2})_E$ to a closed convex polygon Q_1 in $\mathbb{C}$ such that $g(Z) = 0$ and $g(Y) = 1$. Then for any $i \in \mathring{V}$, the above bi-Lipschitz property of f implies that

$$\left|\log\frac{|g(i)|_2}{|p_N(\phi(i))|_2}\right| = O(||l||).$$

Now we are ready to project the points in the plane back to the sphere. Let

$$V_1 = (p_N|_{\mathbb{S}^2})^{-1}(g(V(\mathring{T}))) \cup \{N\}$$

and P_1 be the convex hull of V_1. Then by part (b) of Lemma 6.2, $P_1 \in \mathcal{P}$ and $p_N(\mathring{P}_1) = Q_1$ and $p_N(\mathring{T}_{P_1}) = g(\mathring{T})$. Naturally identify the combinatorial structures of T and T_{P_1}, and denote $l_{P_1} \in \mathbb{R}^{E(T)}$ as the edge length on P_1. We will verify that

$$l_{P_1} = u * 2\sin\frac{l}{2}$$

where $u_X = \bar{u}_X + \log d_{YZ}$ and

$$u_i = \tilde{u}_i - w_i', \quad \text{where} \quad w_i' = \log\frac{|g(i)|^2 + 1}{2}$$

if $i \in V(\mathring{T})$. If $ij \in E(\mathring{T})$,

$$(l_{P_1})_{ij} = (u * 2\sin\frac{l}{2})_{ij}$$

by implementing Lemma 6.2 on l_{P_1} and l_{Q_1}. For edge $iX \in E(T)$, we have that

$$\begin{aligned}
&\log(u * 2\sin\frac{l}{2})_{iX} \\
=&\log(2\sin\frac{l_{iX}}{2}) + \frac{1}{2}(\bar{u}_X + \log d_{YZ} + \log 2 - 2\log(2\sin\frac{l_{iX}}{2}) - \bar{u}_X - \log d_{YZ} - w_i') \\
=&\frac{1}{2}(\log 2 - w_i') = \frac{1}{2}\log\frac{4}{|g(i)|^2+1} = \frac{1}{2}\log(l_{P_1})^2_{iX} = \log(l_{P_1})_{iX}.
\end{aligned}$$

So u is our desired discrete conformal factor. As we mentioned in Remark 1.4, such u is known to be unique. It remains to show $u_i - \bar{u}_i = O(||l||)$ for any $i \in V$. Notice that

$$|w_i' - w_i| = \left|\log\frac{|g(i)|_2^2+1}{2} - \log\frac{|p_N(\phi(i))|_2^2+1}{2}\right| = \left|\log\frac{|g(i)|_2^2+1}{|p_N(\phi(i)|_2^2+1}\right| = O(||l||).$$

So restricted on $V(\mathring{T})$, we have that

$$u-\bar{u} = (\tilde{u}-w')-\bar{u} = u(1)-\log d_{YZ}-w'-\bar{u} = (u(1)-\bar{u}-w)+(w-w')-\log d_{YZ} = O(||l||).$$

On vertex X we have that $u_X - \bar{u}_X = \log d_{YZ} = O(||l||)$.

Bibliography

[1] Feng Luo. Combinatorial yamabe flow on surfaces. *Communications in Contemporary Mathematics*, 6(05):765–780, 2004.

[2] Alexander I Bobenko, Ulrich Pinkall, and Boris A Springborn. Discrete conformal maps and ideal hyperbolic polyhedra. *Geometry & Topology*, 19(4):2155–2215, 2015.

[3] Gregory Leibon. Characterizing the Delaunay decompositions of compact hyperbolic surfaces. *Geometry & Topology*, 6(1):361 – 391, 2002.

[4] Y Colin De Verdiere, A Marin, et al. Triangulations presque équilatérales des surfaces. *Journal of Differential Geometry*, 32(1):199–207, 1990.

[5] Boris Springborn. Ideal hyperbolic polyhedra and discrete uniformization. *Discrete & Computational Geometry*, pages 1–46, 2019.

[6] Igor Rivin. Intrinsic geometry of convex ideal polyhedra in hyperbolic 3-space. *Lecture Notes in Pure and Applied Mathematics*, pages 275–275, 1994.

[7] A. I. Bobenko and B. A. Springborn. A discrete Laplace-Beltrami operator for simplicial surfaces. *Discrete and Computational Geometry*, 38:740–756, 2007.

[8] Xianfeng Gu, Ren Guo, Feng Luo, Jian Sun, and Tianqi Wu. A discrete uniformization theorem for polyhedral surfaces ii. *Journal of differential geometry*, 109(3):431–466, 2018.

[9] Xianfeng David Gu, Feng Luo, Jian Sun, and Tianqi Wu. A discrete uniformization theorem for polyhedral surfaces. *Journal of differential geometry*, 109(2):223–256, 2018.

[10] François Fillastre. Polyhedral hyperbolic metrics on surfaces. *Geometriae Dedicata*, 2009.

[11] R. C. Penner. The decorated Teichmüller space of punctured surfaces. *Communications in Mathematical Physics*, 113(2):299 – 339, 1987.

[12] A.D. Mednykh. Brahmagupta formula for cyclic quadrilaterals in the hyperbolic plane. *Sibirskie Elektronnye Matematicheskie Izvestiia*, 9, 2012.

[13] Daniel Zwillinger. *CRC standard mathematical tables and formulae.* CRC press, 2002.

[14] Elena Frenkel. *On area and volume in spherical and hyperbolic geometry.* PhD thesis, 2018.

[15] David Gu, Feng Luo, and Tianqi Wu. Convergence of discrete conformal geometry and computation of uniformization maps. *Asian Journal of Mathematics*, 23(1):21–34, 2019.

[16] Peter Petersen. *Riemannian geometry*, volume 171. Springer, 2006.

[17] Stephanie B Alexander, I David Berg, and Richard L Bishop. Geometric curvature bounds in riemannian manifolds with boundary. *Transactions of the American Mathematical Society*, 339(2):703–716, 1993.

[18] Martin R Bridson and André Haefliger. Metric spaces of non-positive curvature. 1999.

[19] Alfred Gray, Elsa Abbena, S Salamon, et al. Modern differential geometry of curves and surfaces with mathematica. 2006.

[20] Robert Osserman. The isoperimetric inequality. *Bulletin of the American Mathematical Society*, 84(6):1182–1238, 1978.

[21] Boris Springborn, Peter Schröder, and Ulrich Pinkall. Conformal equivalence of triangle meshes. In *ACM SIGGRAPH 2008 papers*, pages 1–11. 2008.

[22] Satyan L Devadoss and Joseph O'Rourke. *Discrete and computational geometry*. Princeton University Press, 2011.

[23] Ulrich Pinkall and Konrad Polthier. Computing discrete minimal surfaces and their conjugates. *Experimental mathematics*, 2(1):15–36, 1993.