MULTI-CHANNEL SCATTERING THEORY: LARGE TIME ASYMPTOTICS OF SCHRÖDINGER TYPE EQUATIONS WITH GENERAL DATA

by

XIAOXU WU

A dissertation submitted to the

School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Avy Soffer

and approved by

New Brunswick, New Jersey

May, 2023

ABSTRACT OF THE DISSERTATION

Multi-channel Scattering Theory: Large time asymptotics of Schrödinger type equations with general data

By Xiaoxu Wu

Dissertation Director:

Avy Soffer

This thesis appears to focus on scattering theory for both Schrödinger-type equations and Klein-Gordon equations. Chapter 1 provides an introduction to the background of the thesis, while Chapter 2 investigates the long-time behavior of solutions to the Schrödinger equation. Chapter 3 focuses on the construction of solutions to Schrödinger equations with non-trivial weakly localized parts, asymptotic self-similar solutions as time goes to infinity. Chapter 4 studies the long-time behavior of solutions to Klein-Gordon equations. In Chapter 5, the author presents a proof of Local Decay Estimates for Schrödinger-type equations. Finally, Chapter 6 establishes the L^p boundedness of wave operators for linear Schrödinger equations with time-dependent potentials and provides applications of this result to nonlinear dispersive equations and Hartree nonlinear Schrödinger equations.

Acknowledgements

The pursuit of a PhD degree was an incredible, exciting, and fulfilling journey. I am deeply thankful to have had the privilege of working with my advisor, Professor Avy Soffer, who not only guided me through my research, but also became a cherished friend. He taught me the importance of determination and consistently encouraged me, even during moments of self-doubt. I am eternally grateful for his support and mentorship.

I would also like to extend my gratitude to the Mathematics Department at Rutgers University and specifically to my thesis defense committee members, Professor Fioralba Cakoni, Professor Ian Jauslin, and Professor Michael Kiessling. Your invaluable insights and contributions have helped me to grow both personally and professionally. I appreciate your support and look forward to future collaborations. I also want to express my thanks to Professor Benjamin Dodson for serving as an external member of my thesis defense and to Professor Ting Zhang for showing interest in my research.

The members of the Mathematics Department have made this journey an unforgettable experience, and I am grateful for all of the interactions and relationships I have formed. I would also like to extend my gratitude to my friends Jiao He (Université Paris-Saclay), Qiuye Jia (Stanford University), Xiao Ma (Princeton University), Chengyang Shao (University of Chicago), Chenmin Sun (Université Paris-Est Créteil), Jian Wang (Stony Brook University), Yuchen Wei (Rutgers University), Jiayan Wu (Zhejiang University), Mengxuan Yang (University of California, Berkeley), Huiming Zhang (Peking University), Yiyue Zhang (University of California Irvine), and Xiaoping Zhu (Rutgers University). Thank you for your support through presentations, helping me find postdoctoral positions, reviewing my slides, and discussing various topics with me.

I am also grateful to my roommates Ruonan Xu and Jue Jin, my friends Lingxi Gao, Huiyi Xia, Yizhuo Hu, Qiqi Hu, Meiyicong Lin, Xiangting Ji, Yiyan Wang, Yang Yang, Yuanyuan Huang and Peipei Hong, as well as my family. I especially want to thank my roommates for teaching me how to live a meaningful life. Finally, I would like to acknowledge my family for their unwavering support throughout this journey.

Dedication

To those who inspired it

Table of Contents

Chapter 1

Introduction

Let $H_0 := -\Delta_x$. The linear Schrödinger equation

$$i\partial_t \psi(x,t) = (H_0 + V)\psi(x,t), \quad \psi(x,0) = \psi_0 \in L^2_x(\mathbb{R}^n) \tag{1.1}$$

is known as the analogue of Newton's second law for a quantum system. Its solutions do not describe trajectory of a particle as in the classical mechanics, but rather describe the wave function of the system, which carries information about the probability density for finding the particle at a position at time t. In fact, the square of the absolute value of the solutions gives this probability density and it holds the L^2 conservation

$$\|\psi(x,t)\|_{L^2_x(\mathbb{R}^n)} = \|\psi_0(x)\|_{L^2_x(\mathbb{R}^n)}. \tag{1.2}$$

In the free case($V = 0$), the solution to the Schrödinger equation can be represented using the unitary operator e^{-itH_0} acting on the initial wave function $\psi_0(x)$. This evolution can be described in terms of a Fourier transform, as follows:

$$e^{-itH_0}\psi_0 = \mathcal{F}_\xi^{-1}[e^{-it\xi^2}\hat{\psi}_0(\xi)](x) = \frac{c_n}{t^{n/2}} \int e^{i\frac{k^2}{4t}} \psi_0(x-k) d^n k. \tag{1.3}$$

This solution satisfies the global estimate:

$$\|e^{-itH_0}\psi_0\|_{L^\infty_x(\mathbb{R}^n)} \lesssim |t|^{-n/2} \|\psi_0\|_{L^1_x(\mathbb{R}^n)} \tag{1.4}$$

if $\psi_0 \in L^1_x(\mathbb{R}^n)$. From (1.4), it can be observed that the wave function $\psi(x,t)$ exhibits scattering phenomena, meaning that the mass contained in any finite region decays over time. This is demonstrated by:

$$\|\chi(|x| \le M) e^{-itH_0}\psi_0\|_{L^2_x(\mathbb{R}^n)} \lesssim_M |t|^{-n/2} \|\psi_0\|_{L^1_x(\mathbb{R}^n)} \tag{1.5}$$

for any $M \ge 1$. Scattering theory for Schrödinger equations deals with the study of these scattering phenomena when the potential energy $V(x,t)$ is of a more general type.

Unfortunately, when $V \neq 0$, in principle, it is not possible to solve for $\psi(x,t)$. As a result, it is difficult to determine whether the wave function will scatter under the evolution. In the case where V is time-independent, the Hamiltonian $H := H_0 + V(x)$ may have an eigenfunction, $f(x)$. If the initial wave function is equal to the eigenfunction, i.e. $\psi(x,0) = f(x)$, then the solution is given by:

$$\psi(x,t) = e^{-it\lambda} f \tag{1.6}$$

where λ stands for the eigenvalue of f. In this scenario, the distribution of the wave (i.e. the square of the absolute value of the solution) remains unchanged over time, and such an initial state is referred to as a bound state. Solitons are generalized bound states in nonlinear settings. For a free system, there are no bound states.

An initial state ψ_0 is referred to as a scattering state if:

$$\|\psi(t) - e^{-itH_0}\psi_+\|_{L^2_x(\mathbb{R}^n)} \to 0 \tag{1.7}$$

as $t \to \infty$ for some $\psi_+ \in L^2_x(\mathbb{R}^n)$. Scattering theory is the study of the long-time behavior of the solutions of scattering states. When V is time-independent and localized in space, if the initial wave function ψ_0 is not a bound state, it belongs to the continuous spectrum of H. If $|V(x)| \lesssim \langle x\rangle^{-\sigma}$, for some $\sigma > 3$, then the space of all scattering states is equal to the continuous spectrum of H, see e.g. [73]. The situation becomes more complicated in the case of time-dependent or nonlinear potential. This thesis focuses on the study of general Schrödinger type equations and, as an application, similar results are proven for Klein-Gordon equations.

In general, the scattering theory is established through the proof of asymptotic completeness (AC) and dispersive estimates. Throughout this thesis, AC refers to the fact that the solution $\psi(x,t)$ satisfies that, as t approaches either positive or negative infinity, it approaches the linear combination of several asymptotic states, $\psi_{a,\pm}(x,t)$

$$\|\psi(x,t) - \sum_a \psi_{a,\pm}(x,t)\|_{L^2_x(\mathbb{R}^n)} \to 0. \tag{1.8}$$

Here $\psi_{a,\pm}$ is the solution to a simpler system for which AC is easier to establish, such as a free wave. To establish AC, one must find all possible asymptotic states. In this thesis, we focus on the study of scattering theory for general types of nonlinear Schrödinger equations with time-dependent or nonlinear potentials. The scattering theory is established through the proof of asymptotic completeness (AC) and dispersive estimates. Formally, AC refers to the fact that the solution $\psi(x,t)$ approaches the linear combination of several asymptotic states, such as free waves and solitons, as t approaches either positive or negative infinity. To establish AC, one must find all possible asymptotic states. We demonstrate that nonlinear Schrödinger equations can have several different asymptotic states, and the soliton resolution conjecture of Schrödinger equations is a form of AC.

Dispersive estimates are also crucial in establishing AC. Known methods such as Mourre's method and resolvent estimates are applicable when the potential is either time-independent or periodic in time. However, for more general types of potentials, the lack of techniques makes it difficult. Some progress has been made for specific types of potentials, but the problem remains challenging. In this thesis, we develop new tools for general types of nonlinear Schrödinger equations. We first characterize the space of all free scattering states and then develop a new method of proving local decay estimates without using Mourre's method or resolvent estimates. This method is applicable to the case when the interaction $V(x,t)$ is quasi-periodic in t, which is the first method that is workable when the interaction is not periodic in time.

The organization of this thesis is as follows. In Chapter 2, we introduce a method for capturing all free waves, which is considered the first step towards proving the soliton resolution conjecture. Additionally, this method does not rely on dispersive estimates, as opposed to previous approaches to proving AC, which were based on such estimates. In Chapter 3, we discuss the existence of the non-trivial weakly localized part, which will spread in space but is not free. We show the existence of non-trivial weakly localized parts in some nonlinear Schrödinger equations. In Chapter 4, we prove similar results to Klein-Gordon equations. In Chapter 5, we introduce a new method that does not rely on resolvent estimates or Mourre estimates. Based on the knowledge of AC, we prove local decay estimates of the form

$$\|\langle x\rangle^{-\eta}U(t,0)P_c\psi\|_{L^2_{x,t}(\mathbb{R}^{5+1})} \lesssim_\eta \|\psi\|_{L^2_x(\mathbb{R}^5)} \tag{1.9}$$

for any $\eta > 5/2$ when V is localized in space and quasi-periodic in time. (1.9) results in an end-point Strichartz estimate

$$\|U(t,0)P_c\psi\|_{L^2_t L^{10/3}_x(\mathbb{R}^{5+1})} \lesssim \|\psi\|_{L^2_x(\mathbb{R}^5)}. \tag{1.10}$$

Wave operators play a crucial role in the proof of dispersive operators. In Chapter 6, we initiate the study of L^p bondedness of wave operators of time-dependent problems.

We believe that the methods introduced in this thesis will pave the way for solving numerous open problems.

Chapter 2

Long-time behavior of Schrödinger type equations

2.1 Introduction

The analysis of dispersive wave equations and systems is of critical importance in the study of evolution equations in Physics and Geometry.

It is well known that the asymptotic solutions of such equations, if they exist, show a dizzying zoo of possible solutions. Besides the "free wave", which corresponds to a solution of the equation without interaction terms, a multitude or other solutions may appear.

Such solutions are localized around possibly moving center of mass. They include nonlinear bound states, solitons, breathers, hedgehogs, vortices etc... The analysis of such equations is usually done on a case by case basis, due to this complexity. [85]

A natural question then follows: is it true that in general, solutions of dispersive equations converge in appropriate norm (L^2 or H^1) to a free wave and independently moving localized parts?

In fact this is precisely the statement of Asymptotic Completeness in the case of N-body Scattering. In this case the possible outgoing clusters are clearly identified, as bound states of subsystems.

But when the interaction term includes time dependent potentials (even localized in space) and more general nonlinear terms, we do not have an a-priory knowledge of the possible asymptotic states.

In the case of time independent interaction terms, one can use spectral theory. The scattering states evolve from the continuous spectrum, and the localized part is formed by the point spectrum. Once the interaction is time dependent/nonlinear that is not possible.

In fact, there are no general scattering results for localized time dependent potentials. The exceptions are charge transfer hamiltonians [104, 33, 101, 63, 72], decaying in time potentials and small potentials [39, 71], time periodic potentials [103, 39] and random (in time) potentials [7]. See also[6, 5]. For potentials with asymptotic energy distribution more could be done [79].

A very recent progress for more general localized potentials without smallness assumptions is obtained in [87]. Some tools from this work will be used in this paper.

Turning to the nonlinear case, Tao [94, 96, 97] has shown that the asymptotic decomposition holds for NLS with inter-critical nonlinearities, in 3 or higher dimensions, in the case of radial initial data.

In particular, in a very high dimension, and with an interaction that is a sum of smooth compactly supported potential and repulsive nonlinearity, Tao was able to show that the localized part is smooth and localized.

In other cases, Tao showed the localized part is only weakly localized and smooth.

Tao's work uses direct estimates of the incoming and outgoing parts of the solution to control the nonlinear part, via Duhamel representation. In a ceratin sense, it is in the spirit of Enss' work. See also [70].

In contrast, the new approach of Liu-Soffer [58, 57] is based on proving a-priory estimates on the full dynamics, which hold in a suitably localized regions of the extended phase-space. In this way it was possible to show the asymptotic decomposition for general localized interactions, including time and space dependent ones, which are localized. Radial initial data is assumed.

More detailed information is obtained on the localized part of the solution. Besides being smooth, its expanding part (if it exists) can grow at most like $|x| \quad \sqrt{t}$, and furthermore, is concentrated in a thin set of the extended phase-space.

The free part of the solution concentrates on the *propagation set* where $x = vt, v = 2P$, and P being the dual to the space variable, the momentum, is given by the operator $i\nabla_x$.

The weakly localized part is found to be localized in the regions where

$$|x|/t^{\alpha} \sim 1 \quad \textbf{and} \quad |P| \sim t^{-\alpha}, \quad \forall\, 0<\alpha \leq 1/2.$$

It therefore shows that the spreading part follows a self similar pattern.

The method of proof is based on three main parts: first, construct the Free Channel Wave Operator. Then prove localization of the remainder localized part, and use it to prove the smoothness of the localized part. Finally, by using further propagation estimates which are adapted to localized solutions, prove the concentration on thin sets of the phase-space corresponding to self similar solutions.

It should be emphasized that the spreading localized solutions, if they exist, were shown to have a non-small nuclei part around the origin. This is true for both the results of Tao and Liu-Soffer.

Therefore, these are not pure self-similar solutions, as appear in the special cases of critical nonlinearities. See e.g. [92, 20].

We will follow here this point of view.

The key tool from scattering theory that is used to study multichannel scattering is the notion of ***channel wave operator***, which we denote by

$$\Omega_a\psi(0) \equiv s-\lim_{t\to\infty} e^{iH_a t}U(t,0)\psi(0). \tag{2.1}$$

Here the limit is taken in the strong sense in L^2. Note that since $U(t,0)$ is nonlinear in general, then so is the wave operator Ω_a.

$U(t,0)\psi(0)$ is the solution of the dispersive equation with initial data $\psi(0)$ and dynamics (linear or nonlinear) $U(t)=U(t,0)$ generated by a hamiltonian $H(t)$.

The asymptotic dynamics is generated by a Hamiltonian H_a for a given channel denoted by a. In this work we will only construct the ***free channel***, where $H_a = -\Delta$.

A crucial observation is that one can modify the definition of the Channel wave operators to

$$\Omega_a\psi(0) \equiv s-\lim_{t\to\infty} e^{iH_a t}J_aU(t)\psi(0). \tag{2.2}$$

See [75].

Here J_a is any bounded operator satisfying the following:

$$s-\lim_{t\to\infty}(I_d - J_a)e^{-iH_at}P_c(H_a)\phi = 0. \tag{2.3}$$

P_c denotes the projection on the continuous spectral part of H_a.

This construction can be easily generalized to the case when the *asymptotic* dynamics is nonlinear.

In practice, we should choose J_a to be a member of a partition of unity which is equal to 1 on the extended phase space where the asymptotic solution converges to; to be useful, it should also be decaying (in some vague sense) on the support of the interaction that couples the channel a to the rest of the solution.

Now, to prove that the limit exists we use the Cook's method.

For this, we need to show the integrability of the derivative w.r.t. time of the quantity $e^{iH_at}J_aU(t)\psi(0)$.

Taking the derivative (w.r.t. time) gives two types of terms:

$$e^{iH_at}\{i[H_a,J_a]+\frac{\partial J_a}{\partial t}\}U(t)\psi(0) - e^{iH_at}iJ_a(H(t) - H_a)U(t)\psi(0) = \tag{2.4}$$

$$e^{iH_at}D_{H_a}(J_a)U(t)\psi(0)(-i)e^{iH_at}J_aN_0U(t)\psi(0). \tag{2.5}$$

$$H(t) = -\Delta + N_0, \tag{2.6}$$

$$H_a \equiv -\Delta. \tag{2.7}$$

Here

$$D_HB \equiv i[H,B]+\frac{\partial B}{\partial t}. \tag{2.8}$$

$$B = B^*, \tag{2.9}$$

and N_0 represents the interaction term.

By choosing

$$J_a = F(\frac{|x|}{t^\alpha} - 1),$$

it is easy to see that such J_a satisfies our requirement, as on its support the interaction term vanishes like $t^{-m\alpha}$ for a localized interaction vanishing like $|x|^{-m}$ at infinity.

Furthermore, it is not hard to prove that on the support of $I_d - F = \bar{F}(\frac{|x|}{t^\alpha} - 1)$ any solution of the free Schrödinger equation vanishes strongly in L^2.

However, the Heisenberg Derivative part coming from D_H is not necessarily integrable in time, under the full dynamics.

The solution can have a part that stays on the boundary of the support of F, or revisit it infinitely many times.

To resolve this problem, as was done in the N-body case [75] and in the general nonlinear case [58, 57], we further microlocalize the partition of unity, such that on the boundary, the solution can be shown to decay (by propagation estimates).

In [75] these boundaries are cones in the configuration space, and then one needs to microlocalize the momentum to point either out or into the cone.

In [58, 57] one microlocalizes the partition F by localizing on the incoming/outgoing parts of the solution.

This microlocalization needs to be done in a way that allows proving ***propagation estimates*** there [75, 18].

It should be clear by now, that this method is tied to a distinguished point in space, and requires the interaction term to be localized around it. The function F can only annihilate a localized term, and the notion of incoming and outgoing is tied to the choice of origin.

Therefore, to go to the general initial data case, we need a more general type of constructions. This is the content of this work.

The key new construction is a free channel wave operator, with a different type of localization in the phase space.

This localization is constructed by projecting in the phase-space on a neighborhood of the thin propagation set in the extended phase space.

As the free wave concentrates where $x = 2Pt$, we use the projection $(t > 0)$

$$J_{\text{free}} = F_c\left(\frac{|x - 2Pt|}{(t+1)^\alpha} - 1\right); \qquad \alpha < 1.$$

It is a property of the free dynamics that the solution vanishes outside the support of F_c as time goes to infinity. The fundamental properties of this operator that we use are the following:

$$e^{iHt}F_c\left(\frac{|x|}{(t+1)^\alpha}-1\right)e^{-iHt}=F_c\left(\frac{|x-2Pt|}{(t+1)^\alpha}-1\right). \tag{2.10}$$

$$\|F_c\phi\|_p=\|e^{ix^2/4t}F_c\phi\|_p \lesssim \|Pe^{ix^2/4t}F_c\phi\|_2^a\|e^{ix^2/4t}F_c\phi\|_2^{1-a} \tag{2.11}$$

$$\lesssim \|(1/t)|2Pt-x|F_c\phi\|_2^a\|F_c\phi\|_2^{1-a} \lesssim t^{(-1+\alpha)a}\|F_c\phi\|_2=t^{(-1+\alpha)a}\|F\left(\frac{|x|}{t^\alpha}-1\right)e^{-iHt}\phi\|_2 \tag{2.12}$$

The constants $p>2, a$ depend on dimension.

For example, in three space dimensions, $p=6, a=1$.

Furthermore, the Heisenberg Derivative of this operator is positive:

$$D_H F_c\left(\frac{|x-2Pt|}{(t+1)^\alpha}-1\right)=-\alpha\frac{|x-2Pt|}{(t+1)^{1+\alpha}}F_c' \geq 0. \tag{2.13}$$

This is due to the fact that D_H of the operator $|x-2Pt|^2$ is zero.

This operator and its functions have a long history.

In fact this operator is the multiplier that gives the conformal identity for the Schrödinger equation.

It was used to prove sharp propagation estimates in [74, 78, 76, 77, 32, 17].

In a completely different way it was used in [55, 56].

Using propagation estimates similar to [75], the problem of showing the existence of the free channel wave operator, defined in terms of the above F_c is reduced to proving the propagation estimate that follows from using F_c as a propagation observable.

Since the Heisenberg derivative is positive, it remains to verify for what interaction terms the following is true:

$$\int_1^\infty \|J_aN_0U(t)\psi(0)\|_2dt<\infty.$$

We use cancellation lemmas [87], to verify the conditions on the interaction terms for the various cases. We also use the existence of free channel wave operators in L^p for $p>p_c(n)$, with $p_c(3)=6$. This is proved in [87].

2.2 Problem and Results

Let $H_0 := -\Delta_x$. We consider the general class of Nonlinear Schrödinger type equations of the form:

$$\begin{cases} i\partial_t \psi - H_0\psi = N(\psi) \\ \psi(x,0) = \psi_0 \in H^a_x(\mathbb{R}^n) \end{cases}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{2.14}$$

with space dimension $n \geq 1$.

Here $N(\psi) = V(x,t)\psi, N(|\psi|)\psi$ or $V(x,t)\psi + N(|\psi|)\psi$. $V(x,t)$ and $N(\psi)$ are NOT necessary to be real. Throughout the paper, we always assume that the solution to system (2.14) has a uniform L^2_x boundedness

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{L^2_x} \lesssim_{\|\psi_0\|_{H^a_x}} 1 \tag{2.15}$$

provided $\psi_0 \in H^a_x$. H^a_x denotes L^2 Sobolev space. Since we can treat the nonlinearity as a time-dependent perturbation, for simplicity, $N(\psi) = V(x,t)\psi$ when there is no ambiguity. The interaction terms $N(|\psi|)$ that we consider are a combination of the following cases:

1. (localized potentials) for $H^a_x = L^2_x$, $N(\psi) = V(x,t)\psi$ with $\langle x\rangle^\delta |V(x,t)| \in L^\infty_t L^\infty_x(\mathbb{R}^n \times \mathbb{R})$ for some $\delta > 1$. V can be nonlinear.

2. (L^p potentials) $V(x,t) \in L^\infty_t H^a_x(\mathbb{R}^3 \times \mathbb{R})$. Typical general examples are

   (a) (charge transfer potentials) for $a = 0$, general charge transfer potential $N(\psi) = \sum_{j=1}^{N} V_j(x - tv_j, t)\psi$, such that $V_j \in L^2_x$, $j = 1, \cdots, N$, and $v_j \neq v_l$ if $j \neq l$.

   (b) (nonlinear potentials) for some $a \in [0,1]$, $N(\psi) = N(|\psi|)\psi$, such that

   $$\|N(|\psi|)\|_{H^a_x} \leq C(\|\psi\|_{H^a_x}), \tag{2.16}$$

   and assume that $\psi_0 \in H^a_x$ will lead

   $$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^a_x} \lesssim_{\psi_0} 1. \tag{2.17}$$

Here $\langle \cdot \rangle : \mathbb{R}^n \to \mathbb{R}, x \mapsto \sqrt{|x|^2+1}$. Typical examples for nonlinear potentials are

$$N(\psi) = P(|\psi|)\psi, \quad |P(z)| \leq z^n, \quad P(z) \text{ smooth} \tag{2.18}$$

and

$$(n=3) \quad N(\psi) = -[\frac{1}{|x|^{3/2-\delta}} * |\psi|^2](x)\psi(x), \quad \delta \in (0, \frac{3}{2}). \tag{2.19}$$

$$(n=3) \quad N(\psi) = -|\psi|\psi \quad \text{Strauss Exponent.} \tag{2.20}$$

Let

$$L^p_{\delta,x} := \{f(x) : \langle x\rangle^\delta f(x) \in L^p_x\}, \quad \text{for } 1 \leq p \leq \infty. \tag{2.21}$$

Let $\bar{F}_c(\lambda), F_j(\lambda) (j=1,2)$ denote smooth characteristic functions of the interval $[1, +\infty)$ and

$$F_c(\lambda \leq a) := 1 - \bar{F}_c(\lambda/a), \quad F_j(\lambda > a) := F_j(\lambda/a), \quad j = 1, 2, \tag{2.22}$$

$$\bar{F}_c(\lambda \geq a) := \bar{F}_c(\lambda/a), \quad \bar{F}_j(\lambda \leq a) := 1 - F_j(\lambda/a). \tag{2.23}$$

Let $H_0 := -\Delta_x$. Here are our main results:

Theorem 2.2.1. *Let $\psi(t)$ be a global solution of equation (2.14) satisfying (2). For $\alpha \in (0, 1-2/n)$, $n \geq 3$, the channel wave operator*

$$\Omega_\alpha \psi(0) := s\text{-}\lim_{t\to\infty} e^{itH_0} F_c(\frac{|x-2tP|}{t^\alpha} \leq 1)\psi(t), \text{ exists in } H^a_x(\mathbb{R}^n). \tag{2.24}$$

Remark 1. *In order to control the non-free part, the weakly localized part, even in 3 or higher dimensions, we require that the interaction term is localized, see Theorem 2.2.2.*

Theorem 2.2.2. *If $V(x,t)$ satisfies (1) ($V(x,t)$ is localized in x), then for $n \geq 1$, some $\epsilon \in (0, 1/2)$, if $\alpha \in (0, \min(1/2+\epsilon, 1+\frac{\delta-2}{n}))$, $b \in (0, \frac{1}{2}-\epsilon)$,*

1. *the free channel wave operator*

$$\Omega_{\alpha,\epsilon}\psi(0) := s\text{-}\lim_{t\to\infty} e^{itH_0} F_c(\frac{|x-2tP|}{t^\alpha} \leq 1) F_1(t^b|P| > 1)\psi(t) \tag{2.25}$$

exists in $L^2_x(\mathbb{R}^n)$.

2. *furthermore, if $\delta > 2$ and $\alpha \in [1/2, 1/2+\epsilon)$, we have the following asymptotic decomposition*

$$\lim_{t\to\infty} \|\psi(t) - e^{-itH_0}\phi_+ - \psi_{w,b,\epsilon}(t)\|_{L^2_x(\mathbb{R}^n)} = 0 \tag{2.26}$$

where $\phi_+ \in L^2_x$ and $\psi_{w,b,\epsilon}$ is the weakly localized part of the solution, with the following property: It is localized in the region $|x| \lesssim t^{1/2+\epsilon}$ when $t \geq 1$, in the following sense

$$(\psi_{w,b,\epsilon}, |x|\psi_{w,b,\epsilon})_{L^2_x} \lesssim_\epsilon t^{1/2+\epsilon}. \tag{2.27}$$

Let $W^{s,p}_x(\mathbb{R}^n), H^s_x(\mathbb{R}^n)$ denote Sobolev spaces.

Theorem 2.2.3. *When $V(x,t) = \sum_{j=1}^{N} V_j(x - tv_j, t)$, if $\langle x\rangle^{1+\delta} V_j(x) \in W^{1,\infty}_x(\mathbb{R}^n) \cap H^1_x(\mathbb{R}^n)$ for some $\delta > 1$ and if*

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1_x} \lesssim \|\psi_0\|_{H^1_x}, \tag{2.28}$$

then for $n \geq 5, \epsilon \in (0, 1/2)$ we have the following asymptotic decomposition

$$\lim_{t\to\infty} \|\psi(t) - e^{-itH_0}\phi_+ - \sum_{j=1}^{N} \psi_{w,b,\epsilon,j}(t)\|_{L^2_x(\mathbb{R}^n)} = 0. \tag{2.29}$$

$\phi_+ \in L^2_x$ and $\psi_{w,b,\epsilon,j}$ are the weakly localized parts of the solution, with the following property: They are localized in the region $|x - tv_j| \lesssim t^{1/2+\epsilon}$ when $t \geq 1$, in the sense that

$$(e^{itP\cdot v_j}\psi_{w,b,\epsilon,j}, |x| e^{itP\cdot v_j}\psi_{w,b,\epsilon,j})_{L^2_x} \lesssim_\epsilon t^{1/2+\epsilon}. \tag{2.30}$$

Remark 2. *In fact, assumption (2.28) can be removed by using L^p theory for the charge transfer wave operators. It is known to hold [33].*

As an application, we prove Strichartz estimates for Nonlinear Schrödinger equations if $\psi_{w,b,\epsilon}(t)$ is small for $t \geq T$ with some sufficiently large T:

Proposition 2.2.1. *If N satisfies (2), $n=3$, and the additional assumptions (2.31)-(2.34)*

$$\left\|\frac{N(f) - N(g)}{|f| - |g|}\right\|_{L^{3/2}_x} \leq C_0 \| |f| + |g| \|^k_{H^a_x} \tag{2.31}$$

for some $k \geq 0$, $C_0 > 0$, *and if there exists* $T_0 > 0$ *such that when* $t \geq T_0$,

$$CC_0 \|\bar{F}_c(\frac{|x - 2tP|}{t^\alpha} > 1)\psi(t)\|^k_{H^\alpha_x} < 1, \tag{2.32}$$

where

$$C := \sup_{F \in L^2_t L^6_x} \frac{\|e^{-itH_0}\int_0^t ds e^{isH_0} F(x,s)\|_{L^2_t L^6_x}}{\|F(x,t)\|_{L^2_t L^{6/5}_x}}, \tag{2.33}$$

$\psi(t)$ *satisfies local Strichartz estimate, that is, for any* $t_0 > 0$,

$$\|\chi(|t| \leq t_0)\psi(t)\|_{L^2_t L^6_x} \lesssim_{t_0} \|\psi_0\|_{L^2_x}, \tag{2.34}$$

then in 3 *dimensions,* $\psi(t)$ *enjoys the Strichartz estimate, which implies*

$$\|\psi(t) - e^{-itH_0}\phi_+\|_{L^2_x} \to 0 \tag{2.35}$$

as $t \to \infty$ *for some* $\phi_+ \in L^2_x$.

Remark 3. *The same result can be extended to all higher dimensions.*

2.3 Propagation Estimate, Relative Propagation Estimate, tT potentials, estimates for interaction terms and commutator estimates

2.3.1 Propagation Estimate

Given an operator family $B(t)$, we denote

$$\langle B \rangle_t := (\psi(t), B(t)\psi(t))_{L^2_x(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \psi(t)^* B(t)\psi(t) d^n x \tag{2.36}$$

where $\psi(t)$ denotes the solution to (5.1). Suppose a family of self-adjoint operators $B(t)$ satisfy the following estimate:

$$\partial_t \langle B \rangle_t = (\psi(t), C^* C\psi(t))_{L^2_x(\mathbb{R}^n)} + g(t) \tag{2.37}$$

$$g(t) \in L^1(dt), \quad C^* C \geq 0. \tag{2.38}$$

We then call the family $B(t)$ a **Propagation Observable**(PROB)[[42], [79],[75]].

Upon integration over time, we obtain the bound:

$$\int_{t_0}^T \|C(t)\psi(t)\|^2_{L^2_x(\mathbb{R}^n)} dt = (\psi(T), B(T)\psi(T)) - (\psi(t_0), B(t_0)\psi(t_0)) - \int_{t_0}^T g(s)ds \tag{2.39}$$

$$\leq \sup_{t \in [t_0, T]} \left|(\psi(t), B(t)\psi(t))_{L^2_x(\mathbb{R}^n)}\right| + C_g, \quad C_g := \|g(t)\|_{L^1_t(\mathbb{R})}. \tag{2.40}$$

2.3.2 Relative Propagation Estimate

In this paper, we use a modified **PRES** , Relative **PRES**(RPRES). Given an operator $\tilde{B}$, we denote

$$\langle \tilde{B} : \phi(t) \rangle_t := (\phi(t), \tilde{B}(t)\phi(t))_{L^2_x(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \phi(t)^* \tilde{B}(t)\phi(t) d^n x. \tag{2.41}$$

Suppose a family of self-adjoint operators $\tilde{B}(t)$ satisfy the following estimate:

$$\partial_t \langle \tilde{B} : \phi(t) \rangle_t \geq \langle \phi(t), C^* C \phi(t) \rangle + g(t) \tag{2.42}$$

$$g(t) \in L^1(dt), \quad C^* C \geq 0. \tag{2.43}$$

We then call the family $\tilde{B}(t)$ a **Relative Propagation Observable**(RPROB).

Upon integration over time, we obtain the bound:

$$\int_{t_0}^{T} \|C(t)\phi(t)\|^2_{L^2_x(\mathbb{R}^n)} dt \leq \sup_{t \in [t_0, T]} \left| (\phi(t), \tilde{B}(t)\phi(t))_{L^2_x(\mathbb{R}^n)} \right| + C_g, \quad C_g := \|g(t)\|_{L^1_t(\mathbb{R})}. \tag{2.44}$$

In this paper, we take $\phi(t) = e^{itH_0}\psi(t)$. $C^* C(t)$ is a multiplication operator $\partial_t[F_c](\frac{|x|}{t^\alpha} - 1) \in L^1_t[1, \infty)$ or a multiplication operator in frequency space $\partial_t[F_1](t^b|q| - 1) \in L^1_t[1, \infty)$($q$ denotes the frequency variable). Then by using Hölder's inequality, (2.44) implies that for $T \geq t_0 \geq 1$,

$$\left\| \int_{t_0}^{T} dt C(t)\psi(t) \right\|_{L^2_x} \leq \left(\int_{t_0}^{T} dt |C(t)| \right)^{1/2} \left(\int_{t_0}^{T} \|C(t)\psi(t)\|^2_{L^2_x} dt \right)^{1/2} \to 0 \tag{2.45}$$

as $t_0 \to \infty$. We call this estimate **Relative Propagation Estimate**(RPRES).

2.3.3 Time translated (tT) Potential

Given a potential V, **time translated (tT) Potential**, the translation being the flow under the free hamiltonian, $H_0 := -\Delta_x$, is defined by

$$\mathcal{K}_t(V) := e^{itH_0} V e^{-itH_0}, \tag{2.46}$$

see [87]. $\mathcal{K}_t(V)$ has the following representation formulas

$$\mathcal{K}_t(V) = \frac{1}{(2\pi)^{n/2}} \int d^n\xi \hat{V}(\xi, t) e^{i(x+2tP)\cdot\xi}, \tag{2.47}$$

$$\mathcal{K}_t(V) = V(x + 2tP, t). \tag{2.48}$$

Here $P := -i\nabla_x$, $\hat{V}(\xi, t)$ denotes the Fourier transform of $V(x,t)$ in x variables

$$\hat{V}(\xi, t) := \frac{1}{(2\pi)^{n/2}} \int d^n x e^{-ix\cdot\xi} V(x,t). \tag{2.49}$$

In the following, we also use c_n to represent $1/(2\pi)^{n/2}$.

2.3.4 Estimates for interaction terms

Let

$$\Omega_\alpha(t)\psi(0) := \langle P\rangle^{-a} e^{itH_0} F_c(\frac{|x - 2tP|}{t^\alpha} \leq 1)\langle P\rangle^a \psi(t) \tag{2.50}$$

for $\psi(0) \in \mathcal{H}^a_x$. Use (2.48) to rewrite $\Omega_\alpha(T)\psi(0)$ and then **Cook's method** to expand it

$$\Omega_\alpha(t)\psi(0) = \langle P\rangle^{-a} F_c(\frac{|x|}{t^\alpha} \leq 1) e^{itH_0}\langle P\rangle^a \psi(t), \tag{2.51}$$

$$\begin{aligned}\Omega_\alpha(T)\psi(0) = \Omega_\alpha(1)\psi(0) + \int_1^T dt \langle P\rangle^{-a}\partial_t[F_c(\frac{|x|}{t^\alpha} \leq 1)] e^{itH_0}\langle P\rangle^a \psi(t) + \\ \int_1^T dt \langle P\rangle^{-a} F_c(\frac{|x|}{t^\alpha} \leq 1) e^{itH_0}\langle P\rangle^a V(x,t)\psi(t) \\ =: \Omega_\alpha(1)\psi(0) + \psi_p(T) + \int_1^T dt \psi_{in}(t).\end{aligned} \tag{2.52}$$

We refer $\psi_{in}(t)$ to as an **interaction term**. If $\psi_{in}(t) \in L^1_t[1,\infty)$, then by using **RPRES**, we obtain the existence of free channel wave operator on $\mathcal{H}^a_x$. So it is sufficient to estimate the interaction term.

We also use L^p decay estimates of the free flow

$$\| e^{-itH_0} f(x)\|_{L^p_x(\mathbb{R}^n)} \lesssim_n \frac{1}{|t|^{\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}} \|f(x)\|_{L^{p'}_x(\mathbb{R}^n)}, \quad f \in L^{p'}_x(\mathbb{R}^n) \tag{2.53}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad 2 \leq p \leq \infty. \tag{2.54}$$

Throughout the paper, we need following estimates for the interaction terms.

Proposition 2.3.1. *For $V(x,t) \in L^\infty_t L^2_{\delta,x}(\mathbb{R}^n \times \mathbb{R})$ with some $\delta > 1$, when $\epsilon \in (0, 1/2), \alpha \in (0, \min(1/2+\epsilon, 1 + \frac{\delta-2}{n})), b \in (0, 1/2 - \epsilon)$, we have that for $t \geq 1$,*

$$\|F_c(\frac{|x|}{t^\alpha} \leq 1) F_1(t^b |P| > 1) e^{itH_0} V(x,t) U(t,0)\|_{L^2_x \to L^2_x} \lesssim \frac{1}{t^{n/2(1-\alpha)}} \|V(x,t)\|_{L^\infty_t L^2_x}, \tag{2.55}$$

$$|(V(x,t)\psi(t), F_1(t^b|P|-1)K_t(F_c(\frac{|x|}{t^\alpha}-1))F_1(t^b|P|-1)\psi(t))_{L^2_x(\mathbb{R}^n)}| \lesssim_{n,\epsilon,\alpha} \frac{1}{t^{1+\beta}}\|V(x,t)\|_{L^1_t L^2_{\delta,x}(\mathbb{R}^n\times\mathbb{R})}\|\psi(t)\|^2_{L^1_t L^2_x(\mathbb{R}^n\times\mathbb{R})}, \quad (2.56)$$

$$|(V(x,t)\psi(t), e^{-itH_0}F_c(\frac{|x|}{t^\alpha}-1)F_1(t^b|P|-1)F_c(\frac{|x|}{t^\alpha}-1)e^{itH_0}\psi(t))_{L^2_x(\mathbb{R}^n)}| \lesssim_{n,\epsilon,\alpha} \frac{1}{t^{1+\beta}}\|V(x,t)\|_{L^1_t L^2_{\delta,x}(\mathbb{R}^n\times\mathbb{R})}\|\psi(t)\|^2_{L^1_t L^2_x(\mathbb{R}^n\times\mathbb{R})}, \quad (2.57)$$

with $\beta := \frac{\delta+n(1-\alpha)}{2}-1>0.$

Proof. Write $F_1(t^b|P|-1)$ as F_1 for short. Let

$$a_\psi(t) := F_c(\frac{|x|}{t^\alpha}-1)F_1e^{itH_0}V(x,t)\psi(t). \quad (2.58)$$

It suffices to show

$$\|a_\psi(t)\|_{L^2_x} \lesssim_n \frac{1}{t^{\frac{\delta+n(1-\alpha)}{2}}}\|V(x,t)\|_{L^1_t L^2_{\delta,x}}\|\psi_0\|_{L^2_x}. \quad (2.59)$$

Break it into two pieces

$$a_\psi(t) = F_c(\frac{|x|}{t^\alpha}-1)F_1e^{itH_0}\chi(|x|>\sqrt{t})V(x,t)\psi(t)+ F_c(\frac{|x|}{t^\alpha}-1)F_1e^{itH_0}\chi(|x|\leq\sqrt{t})V(x,t)\psi(t) =: a_{\psi,1}(t)+a_{\psi,2}(t). \quad (2.60)$$

For $a_{\psi,1}(t)$, we use localization of V to get decay in t, that is,

$$\begin{aligned}\|a_{\psi,1}(t)\|_{L^2_x} &\lesssim \|\chi(|x|>\sqrt{t})V(x,t)\psi(t)\|_{L^1_x}\|e^{itH_0}\|_{L^1_x\to L^1_x}\|F_c(\frac{|x|}{t^\alpha}-1)\| \\ (L^p \text{ decay estimates of free flow}) &\lesssim_n \frac{1}{t^{\delta/2}}\|V(x,t)\|_{L^1_t L^2_{\delta,x}}\|\psi_0\|_{L^2_x}\cdot\frac{1}{t^{n/2}}\|F_c(\frac{|x|}{t^\alpha}-1)\|_{L^2_x} \\ &\lesssim_n \frac{1}{t^{\frac{\delta+n(1-\alpha)}{2}}}\|V(x,t)\|_{L^1_t L^2_{\delta,x}}\|\psi_0\|_{L^2_x}. \end{aligned} \quad (2.61)$$

For $a_{\psi,2}(t)$, we have

$$|a_{\psi,2}(t)| \lesssim_{n,N} \frac{1}{t^N}\|V(x,t)\|_{L^1_t L^2_{\delta,x}}\|\psi_0\|^2_{L^2_x} \quad (2.62)$$

by using the method of non-stationary phase since

$$\chi(|x|\leq\sqrt{t})e^{i(x-y)q}F_1(t^{1/2-\epsilon}|q|>1)e^{itq^2}F_c(\frac{|y|}{t^\alpha}-1) = \frac{1}{i[(x-y)\cdot\hat{q}+2t|q|]}\partial_{|q|}[\chi(|x|\leq\sqrt{t})e^{i(x-y)q}F_1(t^{1/2-\epsilon}|q|>1)e^{itq^2}F_c(\frac{|y|}{t^\alpha}-1)] \quad (2.63)$$

with

$$|(x-y)\cdot\hat{q}+2t|q|| \gtrsim t^{1/2+\epsilon}. \tag{2.64}$$

Based on (2.61) and (2.62), we get (2.59) and finish the proof. $\square$

Remark 4. *Here δ can be 0 if we are in 3 or higher dimensions, see Theorem 2.3.1. Here we mainly focus on the one dimensional case.*

Theorem 2.3.1. *For $V(x,t)\in L_t^1 H_x^a(\mathbb{R}^n\times\mathbb{R})$, $a\in[0,1]$, $\alpha\in(0,1-2/n)$, we have that for $t\geq 1, n\geq 3$,*

$$\begin{aligned}|(V(x,t)\psi(t),\langle P\rangle^a K_t F_c(\frac{|x|}{t^\alpha}\leq 1))^2\langle P\rangle^a\psi(t))_{L_x^2(\mathbb{R}^n)}| &\lesssim_{n,\alpha} \\ \frac{1}{t^{1+\beta}}\|V(x,t)\|_{L_t^1 H_x^a(\mathbb{R}^n\times\mathbb{R})}&\|\psi(t)\|^2_{L_t^1 H_x^a(\mathbb{R}^n\times\mathbb{R})}.\end{aligned} \tag{2.65}$$

for some β satisfying

$$\beta := \frac{n(1-\alpha)}{2}-1>0. \tag{2.66}$$

Proof. Let

$$a_\psi(t) := (V(x,t)\psi(t),\langle P\rangle^a K_t(F_c(\frac{|x|}{t^\alpha}\leq 1))\langle P\rangle^a\psi(t))_{L_x^2(\mathbb{R}^n)}. \tag{2.67}$$

Then by using Hölder's inequality, product rule for fractional derivatives and L^p decay of the free flow, we have

$$\begin{aligned}|a_\psi(t)| &\lesssim_n \|V(x,t)\psi(t)\|_{W_x^{a,1}}\frac{1}{t^{n/2}}\|F_c e^{itH_0}\psi(t)\|_{W_x^{a,1}} \\ &\lesssim_n \|V(x,t)\psi(t)\|_{W_x^{a,1}}\frac{1}{t^{n/2}}\|F_c(\frac{|x|}{t^\alpha}\leq 1)\|_{H_x^a}\|e^{itH_0}\psi(t)\|_{H_x^a} \\ &\lesssim_n \frac{1}{t^{n(1-\alpha)/2}}\|V(x,t)\|_{L_t^1 H_x^a}\|\psi(t)\|^2_{L_t^1 H_x^a}.\end{aligned} \tag{2.68}$$

$\square$

Remark 5. *Based on the proof of Theorem 2.3.1, L^1 decay estimates of free flow is not necessary in 3 or higher dimensions. For example, $L^{6+\epsilon}$ decay will be sufficient in 3 dimensions. See section 2.6.*

Next, we need to identify the positive term in the commutators, and get an explicit expression for the reminder and decay estimate for the reminder. So, consider an expression of the form $F(x)G(P)+G(P)F(x)$. In our applications both variables x, P are scales with a fractional factor of t. Suppose F, G are both positive and bounded and smooth. Then, the positive part can be constructed as follows:

$$F(x)G(P)+G(P)F(x) = 2\sqrt{F}G\sqrt{F} + [\sqrt{F},[\sqrt{F},G]]. \tag{2.69}$$

The double commutator can then be estimated using the *commutator expansion Lemma* [75, 1, 35]:

$$[F(A),B] = F'(A)[A,B] + O([A,[A,B]]). \tag{2.70}$$

Using $A \sim |x|/t^{\alpha};\quad B \sim t^{b}P$, (and vice versa) we can derive the estimates below. Since in this case the commutators can be explicitly computed using Fourier transform, we also give a direct estimate below.

Let

$$\begin{aligned}\tilde{r}_1(t) = \langle e^{-itH_0} F_1(t^b|P|>1)F_c(\frac{|x|}{t^\alpha}-1)\partial_t[F_1(t^b|P|>1)]e^{itH_0} \\ - e^{-itH_0}\sqrt{F_c(\frac{|x|}{t^\alpha}-1)}\partial_t[F_1(t^b|P|>1)]F_1(t^b|P|>1)\sqrt{F_c(\frac{|x|}{t^\alpha}-1)}e^{itH_0}\rangle_t,\end{aligned} \tag{2.71}$$

$$\tilde{r}_2(t) := (\tilde{r}_1(t)) , \tag{2.72}$$

$$\begin{aligned}\tilde{r}_3(t) = \langle e^{-itH_0} F_c(\frac{|x|}{t^\alpha}-1)F_1(t^b|P|>1)\partial_t[F_c(\frac{|x|}{t^\alpha}-1)]e^{itH_0} \\ - e^{-itH_0}\sqrt{F_1(t^b|P|>1)}F_c(\frac{|x|}{t^\alpha}-1)\partial_t[F_c(\frac{|x|}{t^\alpha}-1)]\sqrt{F_1(t^b|P|>1)}e^{itH_0}\rangle_t,\end{aligned} \tag{2.73}$$

and

$$\tilde{r}_4(t) = \tilde{r}_3(t) \tag{2.74}$$

Let

$$\begin{aligned}A_{r,1} := \langle F_1(t^b|P|>1)F_c(\frac{|x|}{t^\alpha}-1)^2\partial_t[F_1(t^b|P|>1)] - F_c(\frac{|x|}{t^\alpha}-1) \\ \partial_t[F_1(t^b|P|>1)]F_1(t^b|P|>1)F_c(\frac{|x|}{t^\alpha}-1) : e^{itH_0}\psi(t) - \psi_0\rangle\end{aligned} \tag{2.75}$$

and

$$A_{r,2} := A(r,1) \tag{2.76}$$

$$\tilde{A}_{r,3}(t) = \langle F_c(\frac{|x|}{t^\alpha} - 1)F_1(t^b|P| > 1)\partial_t[F_c(\frac{|x|}{t^\alpha} - 1)]$$
$$\sqrt{F_1(t^b|P| > 1)}F_c(\frac{|x|}{t^\alpha} - 1)\partial_t[F_c(\frac{|x|}{t^\alpha} - 1)]\sqrt{F_1(t^b|P| > 1)} : e^{itH_0}\psi(t) - \psi_0\rangle, \quad (2.77)$$

and

$$\tilde{A}_{r,4}(t) := \tilde{A}_{r,4}(t) \quad (2.78)$$

Here recall that

$$\langle B(t) : \phi(t)\rangle = \int d^n x \phi(t)^* B(t)\phi(t). \quad (2.79)$$

Commutator estimates:

Lemma 2.3.1. *For* $t \geq 1$, $\epsilon \in (0, 1/2), 0 < b < \alpha$, $j = 1, 2, 3, 4$,

$$|\tilde{r}_j(t)| \lesssim_{\alpha,b} \frac{1}{t^{1+\alpha-b}}\|\psi_0\|^2_{L^2_x}, \quad |A_{r,j}(t)| \lesssim_{\alpha,b} \frac{1}{t^{1+\alpha-b}}\|\psi_0\|^2_{L^2_x}. \quad (2.80)$$

Proof. It follows from that for $l = 0, 1$,

$$\|[F_c(\frac{|x|}{t^\alpha} - 1), F_1^{(l)}(t^b|P| > 1)]\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha-b}} \quad (2.81)$$

with

$$F_1^{(l)}(k) := \frac{d^l}{dk^l}[F_1]. \quad (2.82)$$

Let

$$\tilde{F} := F_1^{(l)}. \quad (2.83)$$

Write $[F_c(\frac{|x|}{t^\alpha} - 1), \tilde{F}]$ as

$$[F_c(\frac{|x|}{t^\alpha} - 1), \tilde{F}] =$$
$$c_n \int d^n\xi \hat{\tilde{F}}(\xi)e^{it^bP\cdot\xi}\left[e^{-it^bP\cdot\xi}F_c(\frac{|x|}{t^\alpha} - 1)e^{it^bP\cdot\xi} - F_c(\frac{|x|}{t^\alpha} - 1)\right]$$
$$= c_n \int d^n\xi \hat{\tilde{F}}(\xi)e^{it^bP\cdot\xi}(F_c(\frac{|x - t^b\xi|}{t^\alpha} - 1) - F_c(\frac{|x|}{t^\alpha} - 1)). \quad (2.84)$$

Since

$$\frac{\left|F_c(\frac{|x-t^b\xi|}{t^\alpha} - 1) - F_c(\frac{|x|}{t^\alpha} - 1)\right|}{t^{b-\alpha}|\xi|} \lesssim \sup_{x\in\mathbb{R}^n} |F_c'(|x| - 1)| \lesssim 1, \quad (2.85)$$

we have that for each $\psi \in L^2_x$,

$$\|[F_c(\frac{|x|}{t^\alpha} - 1)), F_1^{(l)}(t^b|P| > 1)]\psi\|_{L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha-b}}\int d^n\xi|\hat{\tilde{F}}(\xi)||\xi|\|\psi(x)\|_{L^2_x(\mathbb{R}^n)}$$
$$\lesssim_n \frac{1}{t^{\alpha-b}}\|\psi\|_{L^2_x(\mathbb{R}^n)}. \quad (2.86)$$

We finish the proof. □

2.4 Localized Time-Dependent Potentials

In this section we prove Theorem 2.2.2. Part (a) of Theorem 2.2.2 follows directly from the following result:

Theorem 2.4.1. *If* $V(x,t) \in L^1_t L^2_{\delta,x}(\mathbb{R}^n \times \mathbb{R})$ *for some* $\delta > 1$, *then* $\Omega_{\alpha,\epsilon}$ *defined in* (2.25) *exists for* $\alpha \in (0, \min(1/2+\epsilon, 1+\frac{\delta-2}{n}))$.

Remark 6. *Here we need* $\delta > 1$ *since we want to generalize the proof to the cases in one or two dimensions. When* $n \geq 3$, $\delta > 1$ *is not needed and* $F_1(t^b|P| > 1)$ *is not needed either. See Theorem 2.2.1.*

Proof of Theorem 2.4.1. According to (2.48), we have

$$\Omega_{\alpha,\epsilon}(t) := e^{itH_0} F_c(\frac{|x-2tP|}{t^\alpha} \leq 1) F_1 U(t,0) = F_c(\frac{|x|}{t^\alpha} \leq 1) F_1 e^{itH_0} U(t,0). \tag{2.87}$$

Choose $\psi(0) \in L^2_x$. Use Cook's method to expand $\Omega_{\alpha,\epsilon}(t)$

$$\begin{aligned}\Omega_{\alpha,\epsilon}(t)\psi(0) = \Omega_{\alpha,\epsilon}(1)\psi(0) + (-i)\int_1^t ds F_c(\frac{|x|}{s^\alpha} \leq 1) F_1(s^b|P| > 1) e^{isH_0} V(x,s)\psi(s) + \\ \int_1^t ds \partial_s [F_c(\frac{|x|}{s^\alpha} \leq 1)] F_1(s^b|P| > 1) e^{isH_0}\psi(s) + \\ \int_1^t ds F_c(\frac{|x|}{s^\alpha} \leq 1)\partial_s[F_1(s^b|P| > 1)] e^{isH_0}\psi(s) \\ =: \Omega_{\alpha,\epsilon}(1)\psi(0) + \psi_{in}(t) + \psi_{p,1}(t) + \psi_{p,2}(t).\end{aligned} \tag{2.88}$$

For $\psi_{in}(t)$, using Proposition 2.3.1, one has that $\psi_{in}(\infty)$ exists in L^2_x. For $\psi_{p,1}(t)$ and $\psi_{p,2}(t)$, use **RPRES** by choosing

$$\begin{cases} B(t) := F_1 F_c(\frac{|x|}{t^\alpha} \leq 1) F_1 \\ \phi(t) = e^{itH_0}\psi(t) \end{cases} \tag{2.89}$$

and

$$\begin{cases} B(t) := F_c(\frac{|x|}{t^\alpha} \leq 1) F_1(t^b|P| > 1) F_c(\frac{|x|}{t^\alpha} \leq 1) \\ \phi(t) = e^{itH_0}\psi(t) \end{cases} \tag{2.90}$$

respectively. Then by using Proposition 2.3.1 and Lemma 2.3.1, one has that $\psi_{p,1}(\infty)$ and $\psi_{p,2}(\infty)$ exist in L^2_x. We finish the proof.

□

For the part with $\bar{F}_c$, fortunately, we have

$$w\text{-}\lim_{t\to\infty} e^{itH_0}\bar{F}_c\left(\frac{|x-2tP|}{t^{\alpha}}>1\right)\psi(t)=0. \tag{2.91}$$

Lemma 2.4.1. (2.91) *is valid.*

Proof of Lemma 2.4.1. Choose $u\in L^2_x$. For any $\alpha>0$,

$$\begin{aligned}|(u,e^{itH_0}\bar{F}_c(\frac{|x-2tP|}{t^{\alpha}}>1)\bar{F}_1\psi(t))_{L^2_x(\mathbb{R}^n)}| &\leq \|\bar{F}_c(\frac{|x|}{t^{\alpha}}>1)u\|_{L^2_x}\|\psi(t)\|_{L^2_x(\mathbb{R}^n)}\\ &= \|\bar{F}_c(\frac{|x|}{t^{\alpha}}>1)u\|_{L^2_x(\mathbb{R}^n)}\|\psi_0\|_{L^2_x(\mathbb{R}^n)}\to 0\end{aligned} \tag{2.92}$$

as $t\to\infty$. We finish the proof. □

Now we prepare to prove the second part of Theorem 2.2.2. Let

$$\psi_{\alpha,d}(t):=F_c\left(\frac{|x-2tP|}{t^{\alpha}}\leq 1\right)F_1(\psi(t)-e^{-itH_0}\psi_0). \tag{2.93}$$

Corollary 2.4.1 (Corollary of Theorem 2.4.1)**.** *If* $V(x,t)\in L^{\infty}_t L^2_{\delta,x}(\mathbb{R}^n\times\mathbb{R})$ *for some* $\delta>1$*, then*

$$\psi_{+,\alpha,d}:=s\text{-}\lim_{t\to\infty} e^{itH_0}\psi_{\alpha,d}(t) \tag{2.94}$$

exists in L^2_x *for* $\alpha\in(0,\min(1/2+\epsilon,1+\frac{\delta-2}{n})),0<b<\alpha$.

Remark 7. *When* $\delta>2$*,* $\alpha\in(0,1/2+\epsilon)$. *And*

$$w\text{-}\lim_{t\to\infty} e^{itH_0}\bar{F}_c\left(\frac{|x|}{t^{\alpha}}>1\right)(\psi(t)-e^{-itH_0}\psi(0))=0. \tag{2.95}$$

Proof. We use Cook's method to expand $e^{itH_0}\psi_{\alpha,d}(t)$

$$\begin{aligned}e^{itH_0}\psi_{\alpha,d}(t)=e^{iH_0}\psi_{\alpha,d}(1)+(-i)\int_1^t dsF_c(\frac{|x|}{s^{\alpha}}\leq 1)F_1(s^b|P|>1)e^{isH_0}V(x,s)\psi(s)+&\\ \int_1^t ds\partial_s[F_c(\frac{|x|}{s^{\alpha}}\leq 1)]F_1(s^b|P|>1)e^{isH_0}(\psi(s)-e^{-isH_0}\psi_0)+&\\ \int_1^t dsF_c(\frac{|x|}{s^{\alpha}}\leq 1)\partial_s[F_1(s^b|P|>1)]e^{isH_0}(\psi(s)-e^{-isH_0}\psi_0)&\\ =:\Omega_{\alpha,\epsilon}(1)\psi(0)+\psi_{in}(t)+\psi_{p,1}(t)+\psi_{p,2}(t).&\end{aligned} \tag{2.96}$$

Following a similar process as we did in Theorem 2.4.1, we get $\lim_{t\to\infty} e^{itH_0}\psi_{\alpha,d}(t)$ exists in L^2_x by using Proposition 2.3.1, Lemma 2.3.1 and **RPRES** via choosing

$$\begin{cases}B(t):=F_1F_c(\frac{|x|}{t^{\alpha}}\leq 1)F_1\\ \phi(t)=e^{itH_0}\psi(t)-\psi(0)\end{cases} \tag{2.97}$$

and

$$\begin{cases} B(t) := F_c(\frac{|x|}{t^\alpha} - 1)F_1(t^b|P| > 1)F_c(\frac{|x|}{t^\alpha} - 1) \\ \phi(t) = e^{itH_0}\psi(t) - \psi(0) \end{cases} \tag{2.98}$$

for $\psi_{p,1}(t)$ and $\psi_{p,2}(t)$ respectively. □

Now let us show some decomposition before proving the second part of Theorem 2.2.2.

First, let us give a sketch of proof.

Let

$$\psi_d(t) := \psi(t) - e^{-itH_0}\psi(0). \tag{2.99}$$

Based on Corollary 2.4.1, we have the free part of $\psi_d(t)$.

Now we look at $\bar{F}_c(|x - 2tP|/t^\alpha > 1)\psi_d(t)$. For the part with $F_2(x_j > t^{1/2+\epsilon})$, when $P_j \le 1/(50t^{1/2-\epsilon})$, it is an incoming wave. (For the notion of incoming/outgoing wave, see [4] for example.)

The idea behind the estimate for this part of the phase-space is that when the position($x_j > t^{1/2+\epsilon}$)is large positive, and the velocity is small or has an opposite sign ($P_j \le 1/(50t^{1/2-\epsilon})$), there is no propagation: the norm of the operator

$$F_2(x_j > t^{1/2+\epsilon})\bar{F}_1(P_j \le 1/(50t^{1/2-\epsilon}))e^{-ibH_0}\langle x\rangle^{-\delta} \tag{2.100}$$

decays in t, and with decay rate that is absolutely integrable over t when $b \in (0,t], \delta > 2$. It corresponds to the expectation that as time goes to infinity, a particle starting from the origin with a small or negative velocity, has a very small probability of moving to a positive position. The so-called **Maximum and Minimum velocity bound**, see Lemma 2.4.2.

We get that this part vanishes as $t \to \infty$,

$$\|F_2(x_j > t^{1/2+\epsilon})\bar{F}_1(P_1 \le 1/(50t^{1/2-\epsilon}))\bar{F}_c(\frac{|x - 2tP|}{t^\alpha} > 1)\psi_d(t)\|_{L^2_x(\mathbb{R}^n)} \to 0 \tag{2.101}$$

as $t \to \infty$.

When $P_j > 1/(50t^{1/2-\epsilon})$, this part is an outgoing wave. We write it as

$$F_2(x_j > t^{1/2+\epsilon})F_1(P_1 > 1/(50t^{1/2-\epsilon}))\bar{F}_c(\frac{|x-2tP|}{t^\alpha} > 1)\psi_d(t) =$$

$$F_2(x_j > t^{1/2+\epsilon})F_1 P_1 > 1/(50t^{1/2-\epsilon}))e^{-itH_0}\tilde{\psi}_{+,d}$$

$$-(-i)\int_t^\infty ds F_2(x_j > t^{1/2+\epsilon})F_1(P_1 > 1/(50t^{1/2-\epsilon}))e^{-i(t-s)H_0}V(s)\psi(s)$$

$$-F_2(x_j > t^{1/2+\epsilon})F_1(P_1 > 1/(50t^{1/2-\epsilon}))\bar{F}_c(\frac{|x-2tP|}{t^\alpha} > 1)\psi_d(t)$$

$$=: \psi_{d,1}(t) + \psi_{d,2}(t) + \psi_{d,3}(t) \quad (2.102)$$

with

$$\tilde{\psi}_{+,d} := \lim_{s\to\infty} e^{isH_0}(\psi(s) - e^{-isH_0}\psi(0)), \quad \text{on } L^2_x(\mathbb{R}^n). \quad (2.103)$$

$$\|\psi_{d,2}(t)\|_{L^2_x(\mathbb{R}^n)} \to 0, \quad \text{as } t \to \infty \quad (2.104)$$

since $-(t-s) \geq 0$ when $s \geq t$ and the norm of the operator

$$F_2(x_j > t^{1/2+\epsilon})\bar{F}_1(P_j > 1/(50t^{1/2-\epsilon}))e^{i(s-t)H_0}\langle x\rangle^{-\delta} \quad (2.105)$$

decays in t, and with decay rate that is absolutely integrable over t when $s \geq t, \delta > 2$. It corresponds to the observation that as time goes to negative infinity, a particle starting from the origin with a positive velocity, has a very small probability of moving to a positive position.

Therefore by using Lemma 2.4.2 again, it vanishes as $t \to \infty$.

Due to Corollary 2.4.1, one has

$$\psi_{d,1}(t) = F_2(x_j > t^{1/2+\epsilon})F_1(t^{1/2-\epsilon}P_1 > 1/50)e^{-itH_0}\psi_{+,\alpha,d}. \quad (2.106)$$

Based on Corollary 2.4.1, one has

$$\|\psi_{d,1}(t) + \psi_{d,3}(t)\|_{L^2_x(\mathbb{R}^n)} \to 0 \quad \text{as } t \to \infty. \quad (2.107)$$

Based on (2.101), (2.107) and (2.104), one has

$$\|F_2(x_j > t^{1/2+\epsilon})\bar{F}_c(\frac{|x-2tP|}{t^\alpha} > 1)\psi_d(t)\|_{L^2_x(\mathbb{R}^n} \to 0 \quad (2.108)$$

as $t \to \infty$.

Similarly, one has

$$\|F_2(-x_j > t^{1/2+\epsilon})\bar{F}_c(\frac{|x-2tP|}{t^{\alpha}} > 1)\psi_d(t)\|_{L^2_x(\mathbb{R}^n)} \to 0 \tag{2.109}$$

as $t \to \infty$. Then the desired weakly localized part is defined by

$$\psi_{w,b,e}(t) := \left(\Pi_{j=1}^{j=n}\bar{F}_2(|x_j| \le t^{1/2+\epsilon})\right)\bar{F}_c(\frac{|x-2tP|}{t^{\alpha}} > 1)\psi_d(t). \tag{2.110}$$

Now let us introduce **Minimal and Maximal velocity bounds**.

Lemma 2.4.2 (Minimal and Maximal velocity bounds). *For* $a > 0, b \in (0,t]$, $\alpha \in (0, 1/2+\epsilon), c \ge 0$,

$$\|(F_2(\frac{x_1}{t^{1/2+\epsilon}} > 1))F_1(t^{1/2-\epsilon}P_1 > 1/50)e^{iaH_0}\langle x_1\rangle^{-\delta}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)} \lesssim_\epsilon \frac{1}{|t^{1/2+\epsilon}+\sqrt{a}|^{\delta}}, \tag{2.111}$$

$$\|(F_2(\frac{x_1}{t^{1/2+\epsilon}} > 1))F_1(t^{1/2-\epsilon}P_1 \le 1/50)e^{-ibH_0}\langle x_1\rangle^{-\delta}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)} \lesssim_\epsilon \frac{1}{|t^{1/2+\epsilon}+\sqrt{b}|^{\delta}}, \tag{2.112}$$

$$\|(F_2(\frac{x_1}{t^{1/2+\epsilon}} > 1))F_1(-t^{1/2-\epsilon}P_1 > 1/50)e^{iaH_0}\langle x_1\rangle^{-\delta}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)} \lesssim_\epsilon \frac{1}{|t^{1/2+\epsilon}+\sqrt{a}|^{\delta}}, \tag{2.113}$$

$$\|(F_2(\frac{x_1}{t^{1/2+\epsilon}} > 1))F_1(-t^{1/2-\epsilon}P_1 \le 1/50)e^{-ibH_0}\langle x_1\rangle^{-\delta}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)} \lesssim_\epsilon \frac{1}{|t^{1/2+\epsilon}+\sqrt{b}|^{\delta}}, \tag{2.114}$$

$$\|F_{2,t}(-x_j > t^{1/2+\epsilon})\bar{F}_1(-t^{1/2-\epsilon}P_j \le 1/50)\bar{F}_c(\frac{|x-2tP|}{t^{\alpha}} > 1)e^{-ibH_0}\langle x\rangle^{-\delta}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)} \lesssim_\epsilon \frac{1}{|t^{1/2+\epsilon}+\sqrt{b}|^{\delta}}. \tag{2.115}$$

Remark 8. *When we use Lemma 2.4.2, we need* $\delta > 2$ *in order to get integrability in* a *or* b *when* $|a|, |b| \ge 1$.

Proof of Lemma 2.4.2. These estimates are proved by using the method of non-stationary phase for constant coefficients. Break the LHS of (2.111) into two parts:

$$\begin{aligned}
&(F_2(\frac{x_1}{t^{1/2+\epsilon}} > 1))F_1(t^{1/2-\epsilon}P_1 > 1/50)e^{iaH_0}\langle x_1\rangle^{-\delta} = \\
&(F_2(\frac{x_1}{t^{1/2+\epsilon}} > 1))F_1(t^{1/2-\epsilon}P_1 > 1/50)e^{iaH_0}\langle x_1\rangle^{-\delta}\chi(|x_1| \ge (t^{1/2+\epsilon}+\sqrt{|a|})/1000)+ \\
&(F_2(\frac{x_1}{t^{1/2+\epsilon}} > 1))F_1(t^{1/2-\epsilon}P_1 > 1/50)e^{iaH_0}\langle x_1\rangle^{-\delta}\chi(|x_1| < (t^{1/2+\epsilon}+\sqrt{|a|})/1000) \\
&=: A_1 + A_2.
\end{aligned} \tag{2.116}$$

For A_1,

$$\begin{aligned}\|A_1\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)} \lesssim{}& \|(F_2(\frac{x_1}{t^{1/2+\epsilon}} > 1))F_1(t^{1/2-\epsilon}P_1 > 1/50)e^{iaH_0}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)} \frac{1}{|t^{1/2+\epsilon}+\sqrt{|a|}|^\delta}\\ &\lesssim \frac{1}{|t^{1/2+\epsilon}+\sqrt{|a|}|^\delta}. \end{aligned} \quad (2.117)$$

For A_2, going to Fourier space, by using the factor $(F_2(\frac{x_1}{t^{1/2+\epsilon}} > 1))F_1(t^{1/2-\epsilon}q_1 > 1/50)$ and factor $\chi(|y_1| < (t^{1/2+\epsilon}+\sqrt{|a|})/1000)$,

$$e^{ix_1q_1}e^{iaq_1^2}e^{-iq_1y_1} = \frac{1}{i(x_1+2aq_1-y_1)}\partial_{q_1}[e^{ix_1q_1}e^{iaq_1^2}e^{-iq_1y_1}] \quad (2.118)$$

with

$$|x_1+2aq_1-y_1| \gtrsim t^{1/2+\epsilon}\chi(|a|\leq t^{1+2\epsilon}) + \sqrt{|a|}\chi(|a|>t^{1+2\epsilon}) \gtrsim |t^{1/2+\epsilon}+\sqrt{|a|}|, \quad (2.119)$$

we have

$$\|A_2\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)} \lesssim_\epsilon \frac{1}{|t^{1/2+\epsilon}+\sqrt{|a|}|^\delta} \quad (2.120)$$

via taking integration by parts in q_1 for enough times. Thus, we get (2.111).

Similarly, we get (2.112), (2.113), (2.114) and (2.115).

□

For charge transfer potentials, we also need the following **Minimal and Maximal velocity bounds**.

Lemma 2.4.3 (Minimal and Maximal velocity bounds)**.** *For $a > 0, b \in (0,t]$, $\alpha \in (0, 1/2+\epsilon)$, $v_1 \in \mathbb{R}$, any $c \geq 0$,*

$$\begin{aligned}\|(F_2(\frac{x_1-tv_1}{t^{1/2+\epsilon}} > 1))F_1(t^{1/2-\epsilon}(P_1-\frac{v_1}{2}) > 1/50)e^{iaH_0}\langle x_1-tv_1-av_1\rangle^{-\delta}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)}\\ \lesssim_\epsilon \frac{1}{|t^{1/2+\epsilon}+\sqrt{a}|^\delta}, \end{aligned} \quad (2.121)$$

$$\begin{aligned}\|(F_2(\frac{x_1-tv_1}{t^{1/2+\epsilon}} > 1))F_1(t^{1/2-\epsilon}(P_1-\frac{v_1}{2}) \leq 1/50)e^{-ibH_0}\langle x_1-tv_1+bv_1\rangle^{-\delta}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)} \lesssim_\epsilon\\ \frac{1}{|t^{1/2+\epsilon}+\sqrt{b}|^\delta}, \end{aligned} \quad (2.122)$$

$$\|(F_2(\frac{x_1+tv_1}{t^{1/2+\epsilon}}>1))F_1(-t^{1/2-\epsilon}(P_1-\frac{v_1}{2})>1/50)e^{iaH_0}\langle x_1-tv_1-av_1\rangle^{-\delta}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)}\lesssim_\epsilon \frac{1}{|t^{1/2+\epsilon}+\sqrt{a}|^\delta}, \quad (2.123)$$

$$\|(F_2(\frac{x_1+tv_1}{t^{1/2+\epsilon}}>1))F_1(-t^{1/2-\epsilon}(P_1-\frac{v_1}{2})\leq 1/50)e^{-ibH_0}\langle x_1-tv_1+bv_1\rangle^{-\delta}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)}\lesssim_\epsilon \frac{1}{|t^{1/2+\epsilon}+\sqrt{b}|^\delta}, \quad (2.124)$$

$$\|F_2(-x_1-tv_1>t^{1/2+\epsilon})\bar{F}_1(-t^{1/2-\epsilon}(P_1-\frac{v_1}{2})\leq 1/50)\bar{F}_c(\frac{|x-2tP|}{t^\alpha}>1)e^{-ibH_0}\langle x_1-tv_1+bv_1\rangle^{-\delta}\|_{H^c_x(\mathbb{R}^n)\to H^c_x(\mathbb{R}^n)}\lesssim_\epsilon \frac{1}{|t^{1/2+\epsilon}+\sqrt{b}|^\delta}. \quad (2.125)$$

Proof. (2.121)-(2.125) follow by using Galilean transformation and Lemma 2.4.2.

□

Now we can prove Theorem 2.2.2 for time-dependent interactions.

Proof of Theorem 2.2.2(time-dependent linear). Based on Lemma 2.4.2, Corollary 2.4.1, (2.108) and (2.109), we have

$$\|\psi(t)-e^{-itH_0}(\psi_0+\psi_{+,\alpha,d})-\psi_{w,b,\epsilon}(t)\|_{L^2_x(\mathbb{R}^n)}=$$
$$\|(1-\Pi_{j=1}^{j=n}\bar{F}_2(|x_j|\leq t^{1/2+\epsilon}))\bar{F}_c(\frac{|x-2tP|}{t^\alpha}>1)\psi_d(t)\|_{L^2_x(\mathbb{R}^n)}\to 0 \quad (2.126)$$

as $t\to\infty$. And by definition of $\psi_{w,b,\epsilon}(t)$, see (2.110). (2.27) follows and we finish the proof.

□

Corollary 2.4.2. *If*

$$\|\langle P\rangle^a\langle x\rangle^\delta V(x,t)\|_{L^1_t L^2_x}, \text{ for some } \delta>2, a\in[0,1], \quad (2.127)$$

and if

$$s\text{-}\lim_{t\to\infty}\langle P\rangle^{-a}e^{itH_0}F_c(\frac{|x-2tP|}{t^\alpha}\leq 1)\langle P\rangle^a\psi_d(t), \text{ exists in } H^a_x, \quad (2.128)$$

then

$$\|\psi(t)-e^{-itH_0}(\psi_0+\psi_{+,\alpha,d})-\psi_{w,b,\epsilon}(t)\|_{H^a_x(\mathbb{R}^n)}\to 0 \quad (2.129)$$

where $\phi_+ \in H^a_x$ and $\psi_{w,b,\epsilon}$ is the weakly localized part of the solution. It shares the following property: It is localized in the region $|x| \sim t^{1/2+\epsilon}$ when $t \geq 1$, in the sense that

$$(\psi_{w,b,\epsilon}, |x|\psi_{w,b,\epsilon})_{H^a_x} \lesssim_\epsilon t^{1/2+\epsilon}. \tag{2.130}$$

Proof. The proof follows by using a similar argument of the second part of Theorem 2.2.2 and that $\psi(t), \psi_0 \in H^a_x$ implies $\psi_d(t) \in H^a_x$; then $\psi_{w,b,\epsilon} \in H^a_x(\psi_{w,b,\epsilon}$ is defined in (2.110)) for all $t \geq 1$.

Here $\|\bar{F}_c(\frac{|x-2tP|}{t^\alpha} > 1)\psi_d(t)\|_{H^a_x}$ is uniformly bounded in $t \in [1, \infty)$ since

$$\sup_{t \in \mathbb{R}} \|\psi(t)\|_{H^a_x} \lesssim_{N, \psi(0)} 1. \tag{2.131}$$

□

Proposition 2.4.1. *Let*

$$\phi(t) := \int_0^t ds e^{isH_0} V(x - sv_j, s)\tilde{\phi}(s) \tag{2.132}$$

for some $\tilde{\phi}(t)$ satisfying

$$\|\tilde{\phi}(t)\|_{L^2_x(\mathbb{R}^n)} \lesssim \|\tilde{\phi}(0)\|_{L^2_x(\mathbb{R}^n)}. \tag{2.133}$$

If in n dimensions,

$$s\text{-}\lim_{t \to \infty} F_c(\frac{|x|}{t^\alpha} - 1)\phi(t) \text{ exists in } L^2_x(\mathbb{R}^n) \text{ for some } \alpha \in (0, 1/2), \tag{2.134}$$

and if $V(x,t) \in L^\infty_t L^2_{\delta,x}(\mathbb{R}^n \times \mathbb{R})$ for some $\delta > 2$, then for $\epsilon > \max(0, \alpha - \frac{1}{2})$,

$$\|F_2(|x - tv| - t^{1/2+\epsilon})e^{-itH_0}\bar{F}_c(\frac{|x|}{t^\alpha} > 1)\phi(t)\|_{L^2_x(\mathbb{R}^n)} \to 0 \tag{2.135}$$

as $t \to \infty$.

Proof. Let $e_1 := \hat{v}$,

$$\phi_{\epsilon,v,+}(t) := F_2(x_1 - t|v| - t^{1/2+\epsilon}/100)e^{-itH_0}\bar{F}_c(\frac{|x|}{t^\alpha} > 1)\phi(t) \tag{2.136}$$

and

$$\phi_{\epsilon,v,-}(t) := F_2(-x_1 + t|v| - t^{1/2+\epsilon}/100)e^{-itH_0}\bar{F}_c(\frac{|x|}{t^\alpha} > 1)\phi(t). \tag{2.137}$$

Then

$$e^{-itH_0}\bar{F}_c(\frac{|x|}{t^\alpha}>1)\phi(t)=\bar{F}_2(|x_1-t|v||<t^{1/2+\epsilon}/100)e^{-itH_0}\bar{F}_c(\frac{|x|}{t^\alpha}>1)\phi(t)+$$
$$(\phi_{\epsilon,v,+}(t)+\phi_{\epsilon,v,-}(t))=:\phi_{\epsilon,v,0}(t)+(\phi_{\epsilon,v,+}(t)+\phi_{\epsilon,v,-}(t)). \quad (2.138)$$

For $\phi_{\epsilon,v,0}(t)$,

$$\|F_2(|x-tv|\geq t^{1/2+\epsilon})\phi_{\epsilon,v,0}(t)\|_{L^2_x}\to 0 \quad (2.139)$$

as $t\to\infty$, since with F_2, $|x_j|\geq t^{1/2+\epsilon}/100$ for some $j\in\{2,\cdots,n\}$, $V(x-tv)$ is localized in $x_2,\cdots,x_n$ and we can use the same argument as we did for localized potential to get its decay in t.

Based on the following Lemma, we will get

$$\|\phi_{\epsilon,v,-}(t)\|_{L^2_x}\to 0 \quad (2.140)$$

as $t\to\infty$.

Lemma 2.4.4. *If $V(x,t)\in L^\infty_t L^2_{\delta,x}(\mathbb{R}^n\times\mathbb{R})$ for some $\delta>2$, then for any $a\in[0,1]$,*

$$\|\phi_{\epsilon,v,-}(t)\|_{H^a_x(\mathbb{R}^n)}\to 0, \text{ as } t\to\infty. \quad (2.141)$$

Proof of Lemma 2.4.4. Break $\phi_{\epsilon,v,-}(t)$ into three pieces

$$\phi_{\epsilon,v,-}(t)=F_2(x_1-t|v|\leq t^{1/2+\epsilon}/100)F_1(t^{1/2-\epsilon}(P_1-|v|/2)>1/100)e^{-itH_0}\phi(t)$$
$$-F_2(x_1-t|v|\leq t^{1/2+\epsilon}/100)F_1(t^{1/2-\epsilon}(P_1-|v|/2)>1/100)e^{-itH_0}F_c(\frac{|x|}{t^\alpha}\leq 1)\phi(t)+$$
$$F_2(x_1-t|v|\leq t^{1/2+\epsilon}/100)\bar{F}_1(t^{1/2-\epsilon}(P_1-|v|/2)\leq 1/100)e^{-itH_0}\phi(t)$$
$$=:\phi_{\epsilon,v,-,1}(t)+\phi_{\epsilon,v,-,2}(t)+\phi_{\epsilon,v,-,3}(t). \quad (2.142)$$

According Lemma 2.4.3, we have

$$\|\phi_{\epsilon,v,-,3}\|_{H^a_x(\mathbb{R}^n)}\to 0, \text{ as } t\to\infty. \quad (2.143)$$

For $\phi_{\epsilon,v,\ ,1}(t)+\phi_{\epsilon,v,\ ,2}(t)$, write it as

$$\begin{aligned}\phi_{\epsilon,v,\ ,1}(t)+\phi_{\epsilon,v,\ ,2}(t) = \\ \Big[F_2(x_1-t|v|-t^{1/2+\epsilon}/100)F_1(t^{1/2-\epsilon}(P_1-|v|/2)>1/100)e^{-itH_0}\phi(\infty) \\ -F_2(x_1-t|v|-t^{1/2+\epsilon}/100)F_1(t^{1/2-\epsilon}(P_1-|v|/2)>1/100)e^{-itH_0}F_c(\frac{|x|}{t^\alpha}-1)\phi(t)\Big] \\ -F_2(x_1-t|v|-t^{1/2+\epsilon}/100)F_1(t^{1/2-\epsilon}(P_1-|v|/2)>1/100)e^{-itH_0}(\phi(\infty)-\phi(t)) \\ =: \psi_{\epsilon,j,\ ,m,1,1}(t)+\psi_{\epsilon,j,\ ,m,1,2}(t).\end{aligned} \tag{2.144}$$

Here

$$\phi(\infty) := \lim_{t\to\infty}\phi(t). \tag{2.145}$$

If $\phi(\infty)$ exists, then

$$\phi(\infty) = s\text{-}\lim_{t\to\infty}F_c(\frac{|x|}{t^\alpha}-1)\phi(t) \tag{2.146}$$

since

$$w\text{-}\lim_{t\to\infty}\bar{F}_c(\frac{|x|}{t^\alpha}>1)\phi(t)=0 \tag{2.147}$$

which follows from the same argument as what we did for (2.91), see Lemma 2.4.1. By using a similar argument as what we did for $\psi_{\epsilon,j,\ ,m,1,1}$ for (2.107), we get

$$\|\psi_{\epsilon,j,\ ,m,1,1}(t)\|_{H^a_x}\to 0 \tag{2.148}$$

as $t\to\infty$. By using Lemma 2.4.3,

$$\|\psi_{\epsilon,j,\ ,m,1,2}(t)\|_{H^a_x}\to 0 \tag{2.149}$$

as $t\to\infty$. We finish the proof. □

Based on Lemma 2.4.4, we get (2.140) and finish the proof. □

2.5 Charge Transfer Potentials and Nonlinear Potentials

In this section, we prove Theorem 2.2.1, Theorem 2.2.3 and Lemma 2.2.1.

Proof of Theorem 2.2.1. According to (2.48), we have

$$\Omega_\alpha = s\text{-}\lim_{t\to\infty}\langle P\rangle^{-a}F_c(\frac{|x|}{t^\alpha}-1)\langle P\rangle^a e^{itH_0}U(t,0). \tag{2.150}$$

Now we use propagation estimates to prove (2.25). Choose

$$B(t) := \langle P\rangle^{a} e^{-itH_0} F_c(\frac{|x|}{t^{\alpha}} - 1)^2 e^{itH_0} \langle P\rangle^{a}. \tag{2.151}$$

Then

$$|\langle B(t)\rangle| \lesssim \|\psi(t)\|^2_{H^a_x}. \tag{2.152}$$

Let

$$\mathbf{V}(t) := i[V(x,t), B(t)], \tag{2.153}$$

$$B_1(t) := \langle P\rangle^{a} e^{-itH_0} \partial_t[F_c(\frac{|x|}{t^{\alpha}} - 1)^2] e^{itH_0} \langle P\rangle^{a}. \tag{2.154}$$

Compute $\partial_t \langle B(t)\rangle$

$$\partial_t \langle B(t)\rangle = \langle \mathbf{V}(t)\rangle + \langle B_1(t)\rangle. \tag{2.155}$$

Since

$$\langle B_1(t)\rangle \geq 0 \text{ for all } t > 0, \tag{2.156}$$

and since due to Theorem 2.3.1, for $\alpha \in (0, 1 - 2/n)$,

$$\int_0^{\infty} dt |\langle \mathbf{V}(t)\rangle| < \infty, \tag{2.157}$$

we have that for $\alpha \in (0, 1 - 2/n)$,

$$\int_0^{\infty} dt \left| \partial_t [\| F_c(\frac{|x|}{t^{\alpha}} - 1) F_1 e^{itH_0} \langle P\rangle^{a} \psi(t)\|^2_{L^2_x(\mathbb{R}^n)}] \right| < \infty \tag{2.158}$$

which implies

$$\Omega_\alpha \psi_0 \text{ exists in } H^a_x(\mathbb{R}^n) \tag{2.159}$$

for all $\psi_0 \in H^a_x$. □

Corollary 2.5.1. *Let* $\psi(t)$ *be a global solution of equation* (2.14)*. For* $\alpha \in (0, 1 - 2/n)$*,* $n \geq 3$*, the channel wave operator*

$$\Omega_\alpha \psi(0) := s\text{-}\lim_{t\to\infty} \langle P\rangle^{-a} e^{itH_0} F_c(\frac{|x - 2tP|}{t^{\alpha}} - 1) \langle P\rangle^{a} \psi_d(t), \text{ exists in } H^a_x(\mathbb{R}^n). \tag{2.160}$$

Proof. By using propagation estimate with a modified choice of the flow(replace $\psi(t)$ with $\psi(t) - e^{-itH_0}\psi_0$), we get (2.160) via a similar argument of Theorem 2.2.1. □

In the following context, $\Omega_\alpha \psi_0$ captures all the free part of the solution:

Lemma 2.5.1. *If $V(x,t) \in L_t^\infty L_x^2(\mathbb{R}^n \times \mathbb{R})$, then for $n \geq 3$, for any $M > 0$,*

$$\|\bar{F}_1(|P| - M)(\Omega_+ - \Omega_\alpha)\|_{L_x^2(\mathbb{R}^n)\cap L_x^p(\mathbb{R}^n) \to L_x^p(\mathbb{R}^n)} = 0, \text{ for } p > 6. \tag{2.161}$$

Remark 9. *In the proof for Lemma 2.5.1, we will reprove the free channel exists in L_x^p for $p > 6$ in 3 dimensions. We have shown it in [87]. We give the proof for the convenience of the readers.*

Proof. Choose $\psi \in L_x^2(\mathbb{R}^n) \cap L_x^p(\mathbb{R}^n)$ and set $n = 3$. For $t \in [1, \infty)$,

$$\begin{aligned}\|\bar{F}_1(|P| - 100M)e^{itH_0}\psi(t)\|_{L_x^\infty(\mathbb{R}^n)} \\ \leq \|\bar{F}_1(|q| - 100M)\|_{L_q^2(\mathbb{R}^n)}\|e^{itH_0}\psi(t)\|_{L_x^2(\mathbb{R}^n)} \lesssim_M \|\psi_0\|_{L_x^2(\mathbb{R}^n)}\end{aligned} \tag{2.162}$$

and by interpolation,

$$\|\bar{F}_1(|P| - 100M)e^{itH_0}\psi(t)\|_{L_x^p(\mathbb{R}^n)} \lesssim_M \|\psi_0\|_{L_x^2(\mathbb{R}^n)}. \tag{2.163}$$

On the other hand, for $p > 6$, by using Duhamel's formula, we have

$$\begin{aligned}\|\Omega\psi_0 - e^{itH_0}\psi(t)\|_{L_t^p} &= \|\int_t^\infty ds e^{isH_0}V(x,s)\psi(s)\|_{L_t^p} \\ &\leq \int_t^\infty ds \frac{1}{s^{3(1/2-1/p)}}\|V(x,s)\psi(s)\|_{L_x^{p'}} \\ &\lesssim_p \frac{1}{t^{1/2-3/p}}\|V(x,t)\|_{L_t^\infty L_x^r(\mathbb{R}^3\times\mathbb{R})}\|\psi_0\|_{L_x^2} \to 0\end{aligned} \tag{2.164}$$

as $t \to \infty$. Here

$$\frac{1}{r} + \frac{1}{2} = \frac{1}{p'}. \tag{2.165}$$

Thus, $\Omega\psi_0$ exists in L_x^p for $p > 6$ and so does $\bar{F}_1(|P| - 100M)\Omega\psi_0$.

Similarly,

$$\|F_c(\frac{|x|}{t^\alpha} - 1)\bar{F}_1(|P| - 100M)e^{itH_0}\psi(t)\|_{L_x^p(\mathbb{R}^n)} \lesssim_M \|\psi_0\|_{L_x^2(\mathbb{R}^n)}. \tag{2.166}$$

Set

$$\begin{aligned}&\|\bar{F}_1(|P| - 100M)\Omega_+\psi_0 - F_c(\frac{|x|}{t^\alpha} - 1)\bar{F}_1(|P| - 100M)e^{itH_0}\psi(t)\|_{L_x^p(\mathbb{R}^n)} \\ &\leq \|\bar{F}_1(|P| - 100M)\Omega_+\psi_0 - F_c(\frac{|x|}{t^\alpha} - 1)\bar{F}_1(|P| - 100M)\Omega_+\psi_0\|_{L_x^p(\mathbb{R}^n)} + \\ &\|F_c(\frac{|x|}{t^\alpha} - 1)\bar{F}_1(|P| - 100M)(\Omega_+ - e^{itH_0}U(t,0))\psi_0\|_{L_x^p(\mathbb{R}^n)} =: A_1(t) + A_2(t).\end{aligned} \tag{2.167}$$

By dominated convergent theorem,

$$A_1(t) \to 0, \text{ as } t \to \infty. \tag{2.168}$$

For $A_2(t)$, since

$$\|F_c(\frac{|x|}{t^\alpha} - 1)\bar{F}_1(|P| - 100M)(\Omega_+ - e^{itH_0}U(t,0))\psi_0\|_{L^2_x(\mathbb{R}^n)} \lesssim \|\psi_0\|_{L^2_x(\mathbb{R}^n)} \tag{2.169}$$

and since

$$\begin{aligned}
\|F_c(\frac{|x|}{t^\alpha} - 1)\bar{F}_1(|P| - 100M)(\Omega_+ - e^{itH_0}U(t,0))\psi_0\|_{L^\infty_x(\mathbb{R}^n)} = \\
\|F_c(\frac{|x|}{t^\alpha} - 1)\bar{F}_1(|P| - 100M)\int_t^\infty ds e^{isH_0}V(x,s)\psi(s)\|_{L^\infty_x(\mathbb{R}^n)} \\
\lesssim \int_t^\infty ds \frac{1}{s^{3/2}}\|V(x,t)\|_{L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})}\|\psi_0\|_{L^2_x(\mathbb{R}^n)} \\
\lesssim \frac{1}{t^{1/2}}\|V(x,t)\|_{L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})}\|\psi_0\|_{L^2_x(\mathbb{R}^n)},
\end{aligned} \tag{2.170}$$

by interpolation, for $p > 2$,

$$\|F_c(\frac{|x|}{t^\alpha} - 1)\bar{F}_1(|P| - 100M)(\Omega_+ - e^{itH_0}U(t,0))\psi_0\|_{L^p_x(\mathbb{R}^n)} \to 0, \text{ as } t \to \infty. \tag{2.171}$$

Thus, we have

$$\begin{aligned}
&\|F_1(|P| - M)\Omega_+\psi_0 - F_1(|P| - M)F_c(\frac{|x|}{t^\alpha} - 1)e^{itH_0}\psi(t)\|_{L^p_x(\mathbb{R}^n)} \\
&\|F_1(|P| - M)F_1(|P| - 100M)\Omega_+\psi_0 - F_1(|P| - M)F_c(\frac{|x|}{t^\alpha} - 1)F_1(|P| - 100M)e^{itH_0}\psi(t)\|_{L^p_x(\mathbb{R}^n)} \\
&+ \|F_1(|P| - M)F_c(\frac{|x|}{t^\alpha} - 1)\bar{F}_1(|P| > 100M)e^{itH_0}\psi(t)\|_{L^p_x(\mathbb{R}^n)} \\
&=: B_1(t) + B_2(t).
\end{aligned} \tag{2.172}$$

According (2.170),

$$B_1(t) \to 0, \text{ as } t \to \infty. \tag{2.173}$$

$$B_2(t) \to 0, \text{ as } t \to \infty \tag{2.174}$$

since

$$\|F_1(|P| - M)F_c(\frac{|x|}{t^\alpha} - 1)\bar{F}_1(|P| > 100M)\|_{L^2_x \to L^2_x} \lesssim_N \frac{1}{t^N} \to 0, \text{ as } t \to \infty. \tag{2.175}$$

We finish the proof. □

Charge transfer potentials and Nonlinear Interactions are examples:

Corollary 2.5.2. *If $V_j(x,t) \in L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})$ for all $j = 1, \ldots, N$, the assumption of Theorem 2.2.2 is satisfied and we arrive at the same conclusion in 3 or higher dimensions.*

Proof. In this case,

$$\Big\|\sum_{j=1}^{N} V_j(x - tv_j)\Big\|_{L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})} \le \sum_{j=1}^{N} \|V_j(x)\|_{L^2_x(\mathbb{R}^n)} < \infty. \tag{2.176}$$

□

Corollary 2.5.3. *If N satisfies (2), the assumption of Theorem 2.2.2 is satisfied and same conclusion holds in 3 or higher dimensions.*

Proof. In this case,

$$\|N(\psi(t))\|_{L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})} \lesssim C(\|\psi(t)\|_{L^\infty_t H^\alpha_x(\mathbb{R}^n \times \mathbb{R})}) < \infty. \tag{2.177}$$

□

For charge transfer potentials, we would like to show a similar localization result:

Proposition 2.5.1. *If $\langle x\rangle^{1+\delta} V_j(x,t) \in L^\infty_t W^{1,\infty}_x(\mathbb{R}^n \times \mathbb{R}) \cap L^\infty_t H^1_x(\mathbb{R}^n \times \mathbb{R})$ for some $\delta \in (0,1)$ and*

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1_x} \lesssim \|\psi_0\|_{H^1_x}, \tag{2.178}$$

then for $n \ge 5$, $\alpha \in (0, 1 - \frac{2}{n})$, $j = 1, 2,$

$$s\text{-}\lim_{t\to\infty} F_c\Big(\frac{|x|}{t^\alpha} \le 1\Big)\psi_j(t) \quad \text{exists in } L^2_x(\mathbb{R}^n). \tag{2.179}$$

Before we prove Proposition 2.5.1, we need some lemmas. Let

$$\psi_j(t) := -i\int_0^t ds\, e^{isH_0} V_j(x - sv_j, s)\psi(s), \quad j = 1, \ldots, N. \tag{2.180}$$

Then

$$\Big\|\sum_{j=1}^{N} \psi_j(t)\Big\|_{L^2_x(\mathbb{R}^n)} = \|e^{itH_0}\psi(t) - \psi_0(x)\|_{L^2_x(\mathbb{R}^n)} \le 2\|\psi_0\|_{L^2_x(\mathbb{R}^n)}. \tag{2.181}$$

We have following result for $\psi_j(t)(j = 1, \ldots, N, \epsilon \in (0, 1/2))$:

Lemma 2.5.2. *If* $\langle x\rangle^{1+\delta}V_j(x) \in W_x^{1,1}(\mathbb{R}^n) \cap H_x^1(\mathbb{R}^n)$ *for some* $\delta \in (0,1)$ *and*

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H_x^1(\mathbb{R}^n)} \lesssim C(\|\psi_0\|_{H_x^1(\mathbb{R}^n)}), \tag{2.182}$$

then for $n \geq 5$*,*

$$\|\psi_j(t)\|_{L_x^2(\mathbb{R}^n)} \lesssim \|\psi_0\|_{L_x^2(\mathbb{R}^n)}. \tag{2.183}$$

Proof. Based on (2.181), we have

$$\|\sum_{j=1}^{N} \psi_j(t)\|^2_{L_x^2(\mathbb{R}^n)} \leq 4\|\psi_0\|_{L_x^2(\mathbb{R}^n)} \tag{2.184}$$

which implies

$$\left(\sum_{j=1}^{N} \|\psi_j(t)\|^2_{L_x^2(\mathbb{R}^n)}\right) + \sum_{j\neq l} \Re(\psi_j(t), \psi_l(t))_{L_x^2(\mathbb{R}^n)} \leq 4\|\psi_0\|_{L_x^2(\mathbb{R}^n)}. \tag{2.185}$$

In order to prove (2.183), it is sufficient to show that

$$\left|(\psi_j(t), \psi_l(t))_{L_x^2(\mathbb{R}^n)}\right| \lesssim_{\|\psi_0\|_{L_x^2(\mathbb{R}^n)}} 1. \tag{2.186}$$

Now let us prove (2.186). Let

$$R_{jl}(t) := (\psi_j(t), \psi_l(t))_{L_x^2(\mathbb{R}^n)} \tag{2.187}$$

Write $R_{jl}(t)$ as

$$\begin{aligned} R_{jl}(t) &= \int_0^t ds_1 \int_0^t ds_2 (e^{is_1H_0}V_j(x-s_1v_j, s_1)\psi(s_1), e^{is_2H_0}V_l(x-s_2v_l, s_2)\psi(s_2))_{L_x^2(\mathbb{R}^n)} \\ &= \int_0^t ds_1 \int_0^t ds_2 (V_j(x-s_1v_j, s_1)\psi(s_1), e^{i(s_2-s_1)H_0}V_l(x-s_2v_l, s_2)\psi(s_2))_{L_x^2(\mathbb{R}^n)}. \end{aligned} \tag{2.188}$$

Break $R_{jl}(t)$ into two pieces

$$\begin{aligned} R_{jl}(t) = &\int_0^t ds_1 \int_0^t ds_2 \chi_1(s_1,s_2)(V_j(x-s_1v_j, s_1)\psi(s_1), e^{i(s_2-s_1)H_0}V_l(x-s_2v_l, s_2)\psi(s_2))_{L_x^2(\mathbb{R}^n)} + \\ &\int_0^t ds_1 \int_0^t ds_2 \bar{\chi}_1(s_1,s_2)(V_j(x-s_1v_j, s_1)\psi(s_1), e^{i(s_2-s_1)H_0}V_l(x-s_2v_l, s_2)\psi(s_2))_{L_x^2(\mathbb{R}^n)} \\ &=: R_1(t) + R_2(t) \end{aligned} \tag{2.189}$$

where

$$\chi_1(s_1,s_2) := \chi(|s_1-s_2| \leq \langle s_1\rangle/100), \quad \bar{\chi}_1(s_1,s_2) = 1-\chi_1(s_1,s_2). \tag{2.190}$$

For $R_1(t)$, in 5 or higher dimensions, by using L^p decay estimate for $e^{i(s_2-s_1)H_0}$ and unitarity of $e^{i(s_2-s_1)H_0}$, we have

$$
\begin{aligned}
|R_1(t)| \lesssim & \int_0^t ds_1 \int_0^t ds_2 \chi_1(s_1,s_2) \|V_j(x-tv_j,t)\psi(t)\|_{L_t^\infty L_x^1 \cap L_t^\infty L_x^2} \\
& \times \frac{1}{\langle s_1-s_2\rangle^{n/2}} \|V_l(x-tv_l,t)\psi(t)\|_{L_t^\infty L_x^2 \cap L_t^\infty L_x^1} \\
\lesssim & \int_0^t ds_1 \int_0^t ds_2 \chi_1(s_1,s_2) \times \frac{1}{\langle s_1-s_2\rangle^{n/2}} \|V_j(x,t)\|_{L_t^\infty L_x^\infty \cap L_x^2} \|V_l(x,t)\|_{L_t^\infty L_x^\infty \cap L_t^\infty L_x^2} \|\psi_0\|_{L_x^2}^2 \\
& \lesssim \|V_j(x,t)\|_{L_t^\infty L_x^\infty \cap L_t^\infty L_x^2} \|V_l(x,t)\|_{L_t^\infty L_x^\infty \cap L_t^\infty L_x^2} \|\psi_0\|_{L_x^2}^2 \\
\lesssim & \left(\|\langle x\rangle^{1+\delta} V_j(x,t)\|_{L_t^\infty H_x^1 \cap L_t^\infty W_x^{1,\infty}} \times \|\langle x\rangle^{1+\delta} V_l(x,t)\|_{L_t^\infty H_x^1 \cap L_t^\infty W_x^{1,\infty}} \right) \|\psi_0\|_{H_x^1}^2 \qquad (2.191)
\end{aligned}
$$

where we use that for $n \geq 5$

$$
\int_0^t ds_1 \int_0^t ds_2 \chi_1(s_1,s_2) \times \frac{1}{\langle s_1-s_2\rangle^{n/2}} \lesssim \int_0^t ds_1 \frac{1}{\langle s_1\rangle^{n/2-1}} \lesssim 1. \qquad (2.192)
$$

For $R_2(t)$, based on estimate

$$
\left\|\frac{1}{\langle P\rangle}\langle x\rangle^{-1-\delta} e^{is_1 v_j\cdot P} e^{i(s_2-s_1)H_0} e^{-is_2 v_j\cdot P} \langle x\rangle^{1+\delta} \frac{1}{\langle P\rangle}\right\|_{L_x^2 \cap L_x^1 \to L_x^\infty + L_x^2} \lesssim \langle s_2-s_1\rangle^{1+\delta-n/2} \qquad (2.193)
$$

which follows from the method of stationary phase and taking integration by parts, we have

$$
\begin{aligned}
|R_2(t)| \leq & \int_0^t ds_1 \int_0^t ds_2 \langle s_2-s_1\rangle^{1+\delta-n/2} \bar{\chi}_1(s_1,s_2) \\
& \|\langle x\rangle^{1+\delta} V_j(x,s_1) e^{is_1 v_j\cdot P}\psi(s_1)\|_{H_x^1 \cap W_x^{1,1}} \times \left\|\frac{1}{\langle x\rangle^{1+\delta}} V_l(x-s_2(v_l-v_j),s_2) e^{is_2 v_j\cdot P}\psi(s_2)\right\|_{H_x^1 \cap W_x^{1,1}} \\
& \lesssim \int_0^t ds_1 \int_0^t ds_2 \langle s_2-s_1\rangle^{1+\delta-n/2} \bar{\chi}_1(s_1,s_2) \\
& \|\langle x\rangle^{1+\delta} V_j(x,t)\|_{L_t^\infty H_x^1 \cap L_t^\infty W_x^{1,\infty}} \|\psi(s_1)\|_{H_x^1} \|\psi(s_2)\|_{H_x^1} \left\|\frac{1}{\langle x\rangle^{1+\delta}} V_l(x-s_2(v_l-v_j))\right\|_{H_x^1 \cap W_x^{1,\infty}} \\
& \lesssim \int_0^t ds_1 \int_0^t ds_2 \langle s_2-s_1\rangle^{1+\delta-n/2} \frac{1}{\langle s_2\rangle^{1+\delta}} \bar{\chi}_1(s_1,s_2) \|\psi_0\|_{H_x^1}^2 \\
& \lesssim \|\psi_0\|_{H_x^1}^2 \qquad (2.194)
\end{aligned}
$$

where we use

$$\|\frac{1}{\langle x\rangle^{1+\delta}}V_l(x-s_2(v_l-v_j))\|_{H^1_x\cap W^{1,1}_x}\lesssim\|\frac{1}{\langle x\rangle^{1+\delta}}\frac{1}{\langle x-s_2(v_l-v_j)\rangle^{1+\delta}}\|_{W^{1,1}_x}$$
$$\|\langle x-s_2(v_l-v_j)\rangle^{1+\delta}V_l(x-s_2(v_l-v_j))\|_{H^1_x\cap W^{1,1}_x}$$
$$\lesssim\frac{1}{\langle s_2\rangle^{1+\delta}}\|\langle x\rangle^{1+\delta}V_l(x)\|_{H^1_x\cap W^{1,1}_x}, \quad (2.195)$$

$$\int_0^t ds_1\int_0^t ds_2\langle s_2-s_1\rangle^{1+\delta-n/2}\frac{1}{\langle s_2\rangle^{1+\delta}}\bar{\chi}_1(s_1,s_2)\lesssim\int_0^t ds_1\langle s_1\rangle^{2+\delta-n/2}\frac{1}{\langle s_1\rangle^{2}}\lesssim 1 \quad (2.196)$$

and $\langle x\rangle^{1+\delta}V_j(x)\in W^{1,1}_x(\mathbb{R}^n)\cap H^1_x(\mathbb{R}^n)$.

Based on (2.191), (2.194), we get

$$|R(t)|\lesssim\|\psi_0\|^2_{H^1_x}. \quad (2.197)$$

So we prove (2.186) and finish the proof.

□

Now we prove Proposition 2.5.1 by using propagation estimates.

Proof of Proposition 2.5.1. We denote

$$\langle B\rangle_{\phi,t}:=(\phi(t),B(t)\phi(t))_{L^2_x(\mathbb{R}^n)}. \quad (2.198)$$

Unlike previous section, we replace $\psi(t)$ with $\phi(t)$ which is not a solution to (5.1). Choose

$$B(t)=F_c(\frac{|x|}{t^\alpha}-1)^2 \quad (2.199)$$

to be our observable and

$$\phi(t)=\psi_j(t). \quad (2.200)$$

Then

$$\partial_t[\langle B(t)\rangle_{\phi(t),t}]=\langle\partial_t[F_c(\frac{|x|}{t^\alpha}-1)^2]\rangle_{\phi(t),t}+$$
$$(\phi(t),F_c(\frac{|x|}{t^\alpha}-1)^2e^{itH_0}V_j(x-tv_j,t)\psi(t))_{L^2_x(\mathbb{R}^n)}+(F_c(\frac{|x|}{t^\alpha}-1)^2e^{itH_0}V_j(x-tv_j,t)\psi(t),\phi(t))_{L^2_x(\mathbb{R}^n)}$$
$$=:Q_1(t)+Q_2(t)+Q_3(t). \quad (2.201)$$

According to Lemma 2.5.2, Hölder's inequality, L^p decay estimates and unitarity of e^{itH_0}, we have

$$\begin{aligned} |Q_2(t)| &\lesssim \|\phi(t)\|_{L^2_x}\|F_c(\frac{|x|}{t^\alpha}-1)\|_{L^2_x\cap L^\infty_x}\|e^{itH_0}V_j(x-tv_j,t)\psi(t)\|_{L^2_x+L^\infty_x} \\ &\lesssim \|\psi(t)\|_{H^1_x}t^{\alpha n/2}\frac{1}{\langle t\rangle^{n/2}}\|V_j(x,t)\|_{L^1_x\cap L^2_x}\|\psi_0\|_{L^2_x} \\ &\lesssim \frac{1}{\langle t\rangle^{n/2(1-\alpha)}}\|V_j(x)\|_{L^\infty_t L^1_x\cap L^\infty_t L^2_x}\|\psi_0\|_{H^1_x}\|\psi_0\|_{L^2_x}\in L^1_t \end{aligned} \tag{2.202}$$

for $\alpha\in(0,1-\frac{2}{n})$. Similarly,

$$|Q_2(t)|\lesssim\frac{1}{\langle t\rangle^{n/2(1-\alpha)}}\|V_j(x,t)\|_{L^\infty_t L^1_x\cap L^\infty_t L^2_x}\|\psi_0\|_{H^1_x}\|\psi_0\|_{L^2_x}\in L^1_t. \tag{2.203}$$

Since

$$Q_1(t)\equiv 0, \tag{2.204}$$

we get (2.179) and finish the proof. □

Now we can prove Theorem 2.2.3.

Proof of Theorem 2.2.3. Based on Proposition 2.5.1, the assumptions in Proposition 2.4.1 are satisfied if we set

$$\phi_j(t)=\int_0^t ds e^{isH_0}V_j(x-sv_j,s)\psi(s). \tag{2.205}$$

Then we have

$$\|\bar F_2(|x-tv_j|\geq t^{1/2+\epsilon})e^{-itH_0}\bar F_c(\frac{|x|}{t^\alpha}>1)\int_0^t ds e^{isH_0}V_j(x-sv_j,s)\psi(s)\|_{L^2_x(\mathbb{R}^n)}\to 0. \tag{2.206}$$

Set

$$\phi_+=\psi_0+\sum_j\phi_{\alpha,j} \tag{2.207}$$

where

$$\phi_{\alpha,j}:=s\text{-}\lim_{t\to\infty}F_c(\frac{|x|}{t^\alpha}-1)\phi_j(t). \tag{2.208}$$

Set

$$\psi_{w,b,\epsilon,j}(t)=-i\bar F_2(\frac{|x-tv_j|}{t^{1/2+\epsilon}}<1)e^{-itH_0}\bar F_c(\frac{|x|}{t^\alpha}>1)\int_0^t ds e^{isH_0}V_j(x-sv_j,s)\psi(s). \tag{2.209}$$

Then we obtain (2.29). We finish the proof.

□

2.6 Applications

Proof for Lemma 2.2.1. Based on Corollary 2.5.1 and Corollary 2.4.2, we have that there exists $T > T_0$ such that for all $|t| \geq T$,

$$C C_0 \|\psi_r(t)\|_{H^a_x}^k =: r_\epsilon < 1, \tag{2.210}$$

where

$$\psi_r(t) := \psi(t) - e^{-itH_0}\Omega_{F,\epsilon}\psi_0. \tag{2.211}$$

It suffices to estimate $\psi_r(t)$ since $e^{-itH_0}\Omega_{\alpha,\epsilon}\psi_0$ enjoys Strichartz estimates. Then by Duhamel's formula, we have that for $t \geq T$,

$$\begin{aligned} \|\chi(t \geq T)\psi_r(t)\|_{L^2_tL^6_x} \leq \|e^{-itH_0}\psi_0\|_{L^2_tL^6_x} + \|e^{-itH_0}\Omega_{F,\epsilon}\psi_0\|_{L^2_tL^6_x} + \\ C\|N(\psi(t)) - N(\psi_r(t))\|_{L^2_tL^{6/5}_x} + \\ C\|\chi(t \geq T)N(\psi_r(t))\|_{L^2_tL^{6/5}_x} + \|e^{-itH_0}\int_0^T ds e^{isH_0}N(\psi(t))\|_{L^2_tL^{6/5}_x} =: S_1+S_2+S_3+S_4+S_5. \end{aligned} \tag{2.212}$$

$S_1, S_2 < \infty$ due to Strichartz estimates for free flow. $S_3 < \infty$ since

$$\begin{aligned} \|N(\psi(t)) - N(\psi_r(t))\|_{L^2_tL^{6/5}_x} \leq \left\|\frac{N(\psi(t)) - N(\psi_r(t))}{|\psi(t) - \psi_r(t)|}\right\|_{L^\infty_t L^{3/2}_x}\|e^{-itH_0}\Omega_{F,\epsilon}\psi_0\|_{L^2_xL^6_x} \\ \lesssim_{N,\|\psi(t)\|_{H^a_x}} \|\psi_0\|_{L^2_x} \end{aligned} \tag{2.213}$$

For S_4, by using (2.31),

$$\begin{aligned} S_4 = C\|\chi(t \geq T)N(\psi_r(t))\|_{L^2_tL^{6/5}_x} \leq C\left\|\chi(t \geq T)\frac{N(\psi_r(t))}{|\psi_r(t)|}\right\|_{L^\infty_t L^{3/2}_x}\|\psi_r(t)\|_{L^2_tL^6_x} \\ \leq CC_0\|\chi(t \geq T)\psi_r(t)\|_{H^a_x}^k\|\psi(t)\|_{L^2_xL^6_x} = r_\epsilon\|\chi(t \geq T)\psi(t)\|_{L^2_tL^6_x}. \end{aligned} \tag{2.214}$$

For S_5,

$$S_5 \lesssim \left\|\int_0^T ds e^{isH_0}N(\psi(s))\right\|_{L^2_x} \lesssim_T \|\psi_0\|_{L^2_x}. \tag{2.215}$$

Thus, we have

$$(1 - r_\epsilon)\|\chi(t \geq T)\psi_r(t)\|_{L^2_tL^6_x} \lesssim_{N,\|\psi(t)\|_{H^a_x}} \|\psi_0\|_{L^2_x} \tag{2.216}$$

which implies

$$\|\chi(t \geq T)\psi_r(t)\|_{L^2_tL^6_x} \lesssim_{N,\|\psi(t)\|_{H^a_x}} \frac{1}{1 - r_\epsilon}\|\psi_0\|_{L^2_x} \tag{2.217}$$

Similarly, we have

$$\|\chi(t \geq T)\psi_r(t)\|_{L^2_t L^6_x} \lesssim_{N,\|\psi(t)\|_{H^a_x}} \frac{1}{1-r_\epsilon}\|\psi_0\|_{L^2_x}. \tag{2.218}$$

For $\chi(|t| \leq T)\psi(t)$, we use local Strichartz estimate

$$\|\chi(|t| \leq T)\psi(t)\|_{L^2_t L^6_x} \lesssim_{T,N,\|\psi(t)\|_{H^a_x}} \|\psi_0\|_{L^2_x}. \tag{2.219}$$

Based on (2.217), (2.218) and (2.219), we get Strichartz estimate for $\psi_r(t)$ and subsequently Strichartz estimate for $\psi(t)$. We finish the proof.

□

We end with an explicit class of NLS equations as an example. Consider the case where the interaction term of the NLS is of the form

$$N(\psi) = V(x,t) + c|\psi|^m, \ c > 0 \tag{2.220}$$

$$|x|^{3+0}|V(x,t)| \leq c < \infty, \ |x| > 1 \tag{2.221}$$

$$|x|^{3+0}|\nabla V(x,t)| \leq c < \infty, \ |x| > 1 \tag{2.222}$$

$$|\nabla V(x,t)| + |V(x,t)| \lesssim 1. \tag{2.223}$$

We let the initial data be any function in H^1. We take the power m to be inter-critical. Then, global existence together with the energy identity implies the H^1 is uniformly bounded for time independent V. In particular, in one dimension we conclude that the L^∞ norm is also bounded. Hence, in one dimension we can allow $V(x,t)$ to be ψ dependent. In all of these cases we conclude that the solution is asymptotic to a free wave plus a non-free remainder. Now, one can use the defocusing nature of the nonlinear term, to prove that the non-free part is in fact weakly localized. This follows by showing an exterior propagation estimate of the Morawetz type [58, 57], using the following propagation observable:

$$F_1\left(\frac{|x|}{t^\alpha} - 1\right)\gamma F_1\left(\frac{|x|}{t^\alpha} - 1\right).$$

Here $\alpha = 1/3 + 0$, $\gamma = g(x) \cdot \nabla + \nabla \cdot g(x)$. $g(x)$ is a smooth vector field equal to $x/|x|$ for $|x| > 2$. The resulting propagation estimate implies, that the solution decays in time in the region $|x| > t^\alpha$, and therefore the only localized part can be around the origin.

Chapter 3

Existence of weakly localized part

3.1 Introduction

The analysis of dispersive wave equations and systems is of critical importance in the study of evolution equations in Physics and Geometry.

It is well known that the asymptotic solutions of such equations, if they exist, show a dizzying zoo of possible solutions. Besides the "free wave", which corresponds to a solution of the equation without interaction terms, a multitude or other solutions may appear.

Such solutions are localized around possibly moving center of mass. They include nonlinear bound states, solitons, breathers, hedgehogs, vortices etc... The analysis of such equations is usually done on a case by case basis, due to this complexity. [85]

A natural question then follows: is it true that in general, solutions of dispersive equations converge in appropriate norm (L^2 or H^1) to a free wave and independently moving localized parts?

In fact this is precisely the statement of Asymptotic Completeness in the case of N-body Scattering[18, 32, 40, 41, 75, 76]. In this case the possible outgoing clusters are clearly identified, as bound states of subsystems.

But when the interaction term includes time dependent potentials (even localized in space) and more general nonlinear terms, we do not have an a-priory knowledge of the possible asymptotic states.

In fact, there are no general scattering results for localized time dependent potentials. The exceptions are charge transfer hamiltonians [104, 33, 101, 63, 72, 16], decaying in time potentials and small potentials [39, 71], time periodic potentials [103, 39] and random (in time) potentials [7]. See also[6, 5]. For potentials with asymptotic energy distribution more could be done [79].

A very recent progress for more general localized potentials without smallness assumptions is obtained in [87]. Some tools from this work will be used in this paper.

Turning to the nonlinear case, Tao [94, 96, 97] has shown that the asymptotic decomposition holds for NLS with inter-critical nonlinearities in 3 or higher dimensions, in the case of radial initial data.

In particular, in a very high dimension, and with an interaction that is a sum of smooth compactly supported potential and repulsive nonlinearity, Tao was able to show that the localized part is smooth and localized.

In other cases, Tao showed the localized part is only ***weakly localized*** and smooth.

Tao's work uses direct estimates of the incoming and outgoing parts of the solution to control the nonlinear part, via Duhamel representation. In a ceratin sense, it is in the spirit of Enss' work. See also [70].

A new approach due to Liu-Soffer [58, 57] is based on proving a-priory estimates on the full dynamics, which hold in suitably localized regions of the extended phase-space. In this way it was possible to show the asymptotic decomposition for general localized interactions, including time and space dependent interactions. Radial initial data is assumed.

More detailed information is obtained on the localized part of the solution. Besides being smooth, its expanding part (if it exists) can grow at most like $|x| \sim \sqrt{t}$, and furthermore, is concentrated in a thin set of the extended phase-space.

The free part of the solution concentrates on the ***propagation set*** where $x = vt, v = 2P$, and P being the dual to the space variable, the momentum, is given by the operator $-i\nabla_x$.

The weakly localized part is found be localized in the regions where

$$|x|/t^{\alpha} \sim 1 \quad \textbf{and} \quad |P| \sim t^{-\alpha}, \quad \forall\, 0 < \alpha \le 1/2.$$

It therefore shows that the spreading part follows a self similar pattern.

The question is therefore, does there exist solutions which are weakly localized but not localized? For equations which are not dispersive/hyperbolic many such solutions are known. But for hyperbolic/dispersive equations it is harder to see how such global solutions can emerge.

In the energy critical non-linear wave equation, the conformal symmetry implies a specific structure of scaling. Indeed in these models, it was possible to show that the asymptotic states also include self-similar solitons[19, 20]. See also cited references in [19, 20]. These solutions are global and preserve the energy (but have divergent L^2 norm).

In [92] it is shown that for small data there is a spreading weakly localized part for the KdV type equation.

In the case of NLS we have both energy and L^2 conservation, and no conformal symmetry. The aim of this work is to construct self-similar solutions for NLS type equations which are global and with non-zero mass (L^2 norm). We do that first for linear problems with time dependent potentials, which decay like $r^{\;2}$ at infinity, and then use it to show that the addition of nonlinear terms does not change much the situation.

We do not know yet if one can get such solutions for a purely nonlinear equation, but it is more plausible now.

Our approach to this problem can be described in the language of scattering theory: We construct a channel of scattering where the asymptotic dynamics is given by a scaling transformation, which is unitary in L^2, and a phase transformation. This is to be expected in view of the recent work [58, 57] showing that the asymptotic solutions (in the radial case) concentrates on thin sets of the phase-space, corresponding to self-similar solutions. This is the case for both the free wave and the weakly localized part.

3.1.1 Problem and Results

We consider the general class of Nonlinear Schrödinger type equations of the form:

$$\begin{cases} i\partial_t\psi - H_0\psi = V(x,t)\psi + \mathcal{N}(|\psi|)\psi \\ \psi(x,0) = \psi_0 \in H^a_x(\mathbb{R}^n) \end{cases}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{3.1}$$

with $H_0 := -\Delta_x$. Here H^a_x denotes some suitable Sobolev space.($a = 0$ when it comes to linear problem and $a = 1$ when it comes to nonlinear one.)

We start with a linear model

$$\begin{cases} i\partial_t\psi = H_0\psi + g(t)^{-2}V(\frac{x}{g(t)})\psi \\ \psi(x, g(t_0)) = e^{-iD\ln g(t_0)}\psi_b(x) \in L^2_x(\mathbb{R}^n) \end{cases}, \quad n \geq 3 \tag{3.2}$$

for some $t_0 > 0$ (t_0 will be chosen later), $g(t) \in C^2(\mathbb{R})$ satisfying that there exist two positive constants $c_g \in (0,1), \epsilon \in (0,1/2)$ such that

$$\begin{cases} \inf_{t\in\mathbb{R}} g(t) \gtrsim 1, \\ g(t) \sim \langle t\rangle^{\epsilon} \text{ as } t \to \infty \\ g(t) \gtrsim tg'(t) \gtrsim t^2g''(t) \text{ as } t \to \infty \\ \lim_{t\to\infty} \frac{g(t)-2tg'(t)}{g(t)} = c_g \end{cases}, \tag{3.3}$$

and $V(x)$ and $H := H_0 + V(x)$ satisfying that H has a unique normalized eigenstate $\psi_b(x)$ with an eigenvalue $\lambda < 0$ and

$$\begin{cases} 0 \text{ is regular for } H \\ \langle x\rangle D\psi_b(x) \in L^2_x \\ \langle x\rangle V(x) \in L^1_x, V(x) \in L^2_x \end{cases} \tag{3.4}$$

where $P_x := -i\nabla_x$, $D := \frac{1}{2}(x \cdot P_x + P_x \cdot x)$ and P_c denotes the projection on the continuous spectrum of H. We refer system (3.2) to mass critical system(**MCS**). Since $g(t)^{-2}V(\frac{x}{g(t)}) \in L^1_t L^2_x(\mathbb{R}^3 \times \mathbb{R})$ when $\inf_t g(t) > 0$, due to [91], the channel wave operator

$$\Omega_\alpha := s\text{-}\lim_{t\to\infty} e^{itH_0}F_c(\frac{|x-2tP_x|}{t^\alpha} \geq 1)U(t,0) \tag{3.5}$$

exists from $L^2_x(\mathbb{R}^3)$ to $L^2_x(\mathbb{R}^3)$ for all $\alpha \in (0,1/3)$, where F_c denotes a smooth characteristic function. When it comes to the case when $n > 3$, even though $V(x,t) \notin L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})$, by using a similar propagation estimate introduced in [91, 90], we can get the existence of the free channel wave operator defined by (3.5), see Section 3.2. In [91, 90], the weakly localized part $\psi_{w,l}(t)$ is defined by

$$\psi_{w,l}(t) = \bar{F}_c(\frac{|x - 2tP_x|}{t^\alpha} > 1)\psi(t) + \psi_e(t) \tag{3.6}$$

for some $\psi_e(t)$ satisfying

$$\lim_{t \to \infty} \|\psi_e(t)\|_{L^2_x} = 0. \tag{3.7}$$

Let

$$\psi_{w,l}(t) = c(t)e^{-iD\ln(g(t))}\psi_b(x) - \psi_{w,l,c}(t) \tag{3.8}$$

where

$$(e^{-iD\ln(g(t))}\psi_b(x), \psi_{w,l,c}(t))_{L^2_x} = 0. \tag{3.9}$$

Let $T: \mathbb{R} \to \mathbb{R}, t \mapsto s = \int_0^t du\, g(u)^{-2}$. Due to monotonicity of T, T^{-1} exists. Due to assumption (3.3) on $g(t)$,

$$s = T(t) = \int_0^t du\, g(u)^{-2} \gtrsim \int_0^t du \langle u\rangle^{-2\epsilon} \gtrsim \langle t\rangle^{1-2\epsilon} \to \infty \text{ as } t \to \infty. \tag{3.10}$$

We prove the weakly localized part $\psi_{w,l}(t)$ to (3.2) has a non-trivial self-similar part

$$\psi_{w,b}(t) := c(t)e^{-iD\ln(g(t))}\psi_b(x) \tag{3.11}$$

in the following sense:

1. Let

$$\tilde{a}(t) := (\psi_b(x), e^{iD\ln(g(t))}\psi(t))_{L^2_x}. \tag{3.12}$$

$$\tilde{A}(\infty) := \lim_{t \to \infty} e^{i\lambda T(t)}\tilde{a}(t) \tag{3.13}$$

exists.

2. Furthermore, there exists $t_0 > 0$ such that with an initial condition

$$\psi(t_0) = e^{-iD\ln(g(t_0))}\psi_b(x), \tag{3.14}$$

$$|\tilde{A}(\infty)| > 0 \tag{3.15}$$

which implies

$$\liminf_{t \to \infty} |c(t)| \gtrsim 1. \tag{3.16}$$

Indeed,

$$\tilde{a}(t) = (e^{-iD \ln(g(t))}\psi_b(x), \psi(t))_{L^2_x}. \tag{3.17}$$

Here we remind you that $e^{-iD \ln(g(t))}\psi_b(x)$ is a self-similar function

$$e^{-iD \ln(g(t))}\psi_b(x) = \frac{1}{g(t)^{n/2}}\psi_b(x/g(t)) \tag{3.18}$$

satisfying

$$\|e^{-iD \ln(g(t))}\psi_b(x)\|_{L^2_x(\mathbb{R}^n)} = \|\psi_b(x)\|_{L^2_x(\mathbb{R}^n)} = 1. \tag{3.19}$$

Remark 10. *Besides the free part of $\psi(t)$, (3.13) is the unique asymptotic part we can get from $e^{iD \ln(g(t))}\psi(t)$. Indeed, we find that $e^{iD \ln(g(t))}\psi(t)$ satisfies (3.71) and after changing variables from $t \mapsto s = T(t)$, $\phi(s) := e^{iD \ln(\langle T^{-1}(s)\rangle)}\psi(T^{-1}(s))$ enjoys (3.72), see section 3.1.2. Theorem 3.1.1 tells us that*

$$\lim_{s \to \infty} \|F_2(\frac{|x|}{T^{-1}(s)^{\beta}} - 1)e^{isH_0}\phi(s)\|_{L^2_x(\mathbb{R}^n)} = 0 \tag{3.20}$$

and

$$\lim_{s \to \infty} \|F_2(\frac{|x|}{T^{-1}(s)^{\beta}} - 1)e^{-iD \ln(\langle T^{-1}(s)\rangle)}e^{isH_0}\phi(s)\|_{L^2_x(\mathbb{R}^n)} = 0 \tag{3.21}$$

for some $\beta > 0$ when $n \geq 3$, which implies

$$w\text{-}\lim_{s \to \infty} e^{isH_0}\phi(s) = 0 \quad \text{in } L^2_x \tag{3.22}$$

and

$$w\text{-}\lim_{s \to \infty} e^{-iD \ln(\langle T^{-1}(s)\rangle)}e^{isH_0}\phi(s) = 0 \quad \text{in } L^2_x. \tag{3.23}$$

Here $e^{-isH_0}e^{iD \ln(\langle T^{-1}(s)\rangle)}$ is the dynamics of the system (See Lemma 3.3.1)

$$i\partial_s\phi_f(s) = (H_0 + f(s)D)\phi_f(s). \tag{3.24}$$

(3.20) implies that

$$w\text{-}\lim_{s \to \infty} P_c e^{isH}e^{iD \ln(\langle T^{-1}(s)\rangle)}\psi(T^{-1}(s)) = 0 \quad \text{in } L^2_x. \tag{3.25}$$

Remark 11. *So far the existence of free channel wave operator to* (3.1) *is missing in* 4 *or higher dimensions. We prove it in section 3.2 by using a similar argument introduced by [91].*

Remark 12. *The argument in the following context is applicable to the case when* $V(x)$ *is of short-range type with more than one bounded state and for such* $V(x)$*,* 0 *regular assumption on* H *is used.*

Theorem 3.1.1. *If* $V(x), H$ *satisfy* (3.4) *and* $g(t)$ *satisfies* (3.3)*, then when* $n \geq 3$ *and* $\beta \in (0, 1-2/n-2(n-2)\epsilon/n)$ *(for* $n \geq 3, \epsilon < 1/2$ *implies* $1-2/n-2(n-2)\epsilon/n > 0$*) ,* (3.20) *and* (3.21) *are true.*

Proof. It is equivalent to show

$$\lim_{t\to\infty} \| F_2(\frac{|x|}{t^\beta} \geq 1) e^{iT(t)H_0} e^{iD\ln(g(t))}\psi(t)\|_{L^2_x} = 0. \tag{3.26}$$

Now we prove (3.26). Since

$$e^{-iD\ln(g(t))} H_0 e^{iD\ln(g(t))} = g(t)^2 H_0, \tag{3.27}$$

which implies

$$e^{iT(t)H_0} e^{iD\ln(g(t))}\psi(t) = e^{iD\ln(g(t))} e^{iT(t)g(t)^2 H_0}\psi(t), \tag{3.28}$$

using Duhamel's formula, one has

$$\begin{aligned} &e^{iT(t)H_0} e^{iD\ln(g(t))}\psi(t) = e^{iD\ln(g(t))} e^{i(T(t)g(t)^2-t)H_0}\psi(t_0) + \\ &(-i)\int_{t_0}^t ds e^{iD\ln(g(t))} e^{i(T(t)g(t)^2-t+s)H_0} \frac{1}{g(s)^2} V(x/g(s))\psi(s) =: \psi_1(t) + \psi_2(t). \end{aligned} \tag{3.29}$$

Since

$$\begin{aligned} \liminf_{t\to\infty} \frac{T(t)g(t)^2}{t} &= \liminf_{t\to\infty} \frac{\int_0^t du g(u)^{-2} g(t)^2}{t} \\ \text{(L'Hôpital's rule)} &= \liminf_{t\to\infty} \frac{g(t)^{-2}}{g(t)^{-2} - 2tg'(t)/g(t)^3} \\ \text{(use } c_g \in (0,1)) &= \frac{1}{c_g} =: 1 + c_g' > 1, \end{aligned}$$

one has

$$T(t)g(t)^2 - t \geq \frac{c_g'}{2} t, \quad t \geq t_M \tag{3.30}$$

for some sufficiently large $t_M \geq t_0$. Hence, for $t \geq t_M$, $s \in (t_0, t)$,

$$T(t)g(t)^2 - (t - s) \geq T(t)g(t)^2 - t \geq \frac{c_g'}{2}t \tag{3.31}$$

and by using Hölder's inequality and L^∞_x decay for a free flow e^{-itH_0},

$$\begin{aligned}
\|F_2(\frac{|x|}{t^\beta} \geq 1)\psi_1(t)\|_{L^2_x(\mathbb{R}^n)} &\leq \|F_2(\frac{|x|}{t^\beta} \geq 1)\|_{L^2_x(\mathbb{R}^n)}\|\psi_1(t)\|_{L^\infty_x(\mathbb{R}^n)} \\
&\lesssim t^{n\beta/2} g(t)^{n/2}\|e^{i(T(t)g(t)^2 - t)H_0}\psi(t_0)\|_{L^\infty_x(\mathbb{R}^3)} \\
&\lesssim t^{n(\beta+\epsilon)/2} \frac{1}{t^{n/2}}\|\psi(t_0)\|_{L^1_x} \\
&\lesssim_{t_0} t^{-n/2(1-\epsilon-\beta)}\|\psi_b(x)\|_{L^1_x} \to 0
\end{aligned} \tag{3.32}$$

as $t \to \infty$ when $\beta < 1 - \epsilon$. Here we use $1 - 2/n - 2(n-2)\epsilon/n < 1 - \epsilon$. For $\psi_2(t)$, similarly, by using Hölder's inequality and L^∞_x decay for a free flow e^{-itH_0},

$$\begin{aligned}
\|F_2(\frac{|x|}{t^\beta} \geq 1)\psi_2(t)\|_{L^2_x(\mathbb{R}^n)} &\leq \|F_2(\frac{|x|}{t^\beta} \geq 1)\|_{L^2_x(\mathbb{R}^n)}\|\psi_2(t)\|_{L^\infty_x(\mathbb{R}^n)} \\
&\lesssim t^{n\beta/2} g(t)^{n/2}\int_{t_0}^t ds \frac{1}{t^{n/2}}\|\frac{1}{g(s)^2}V(x/g(s))\|_{L^2_x}\|\psi(s)\|_{L^2_x} \\
&\lesssim t^{n(\beta+\epsilon)/2} \int_{t_0}^t ds \frac{1}{t^{n/2}}\frac{1}{g(s)^{(2-n/2)}}\|V(x)\|_{L^2_x}\|\psi_b(x)\|_{L^2_x} \\
&\lesssim \frac{1}{t^{(n-2)/2-(n-2)\epsilon-n\beta/2}}\|V(x)\|_{L^2_x}\|\psi_b(x)\|_{L^2_x} \to 0
\end{aligned} \tag{3.33}$$

as $t \to \infty$ when $n \geq 3, \epsilon \in (0, 1/2)$ and $\beta \in (0, 1 - 2/n - 2(n-2)\epsilon/n)$. Hence, we get (3.26). Similarly, since

$$\|e^{-iD\ln(g(t))}e^{iT(t)H_0}e^{iD\ln(g(t))}\psi(t)\|_{L^\infty_x(\mathbb{R}^n)} \leq \frac{1}{g(t)^{n/2}}\|e^{iT(t)H_0}e^{iD\ln(g(t))}\psi(t)\|_{L^\infty_x(\mathbb{R}^n)}, \tag{3.34}$$

we get

$$\begin{aligned}
&\|F_2(\frac{|x|}{T^{-1}(s)^\beta} \geq 1)e^{-iD\ln(g(\langle T^{-1}(s)\rangle))}e^{isH_0}e^{iD\ln(g(T^{-1}(s)))}\psi(T^{-1}(s))\|_{L^2_x(\mathbb{R}^n)} \\
&\lesssim_{t_0} \frac{1}{g(t)^{n/2}}\frac{1}{t^{(n-2)/2-(n-2)\epsilon-n/2\beta}}\|V(x)\|_{L^2_x}\|\psi_b(x)\|_{L^2_x} + \frac{1}{g(t)^{n/2}}t^{-n/2(1-\beta-\epsilon)}\|\psi_b(x)\|_{L^1_x} \to 0.
\end{aligned} \tag{3.35}$$

We finish the proof. □

Theorem 3.1.2. *Let* $\tilde{a}(t)$ *be as in* (3.12)*. If* $V(x), H$ *satisfy* (3.4) *and* $g(t)$ *satisfies* (3.3)*, then when* $n \geq 3$*,*

$$\tilde{A}(\infty) := \lim_{t \to \infty} e^{i\lambda T(t)}\tilde{a}(t) \tag{3.36}$$

exists and

$$\psi_{w,l}(x,t) = c(t)e^{-iD\ln(g(t))}\psi_b(x) - \psi_c(x,t) \tag{3.37}$$

where

$$c(t) := (e^{-iD\ln(g(t))}\psi_b(x), \psi_{w,l}(x,t))_{L^2_x}, \tag{3.38}$$

$$(e^{-iD\ln(g(t))}\psi_b(x), \psi_c(x,t))_{L^2_x} = 0 \tag{3.39}$$

with $c(t)$ *satisfying* (3.16)*. Moreover, based on* (3.25)*, the* $g(t)$*-self-similar channel wave operator*

$$\Omega_g\psi(0) := w\text{-}\lim_{s \to \infty} e^{isH}e^{iD\ln(g(T^{-1}(s)))}\psi(T^{-1}(s)) \tag{3.40}$$

exists in L^2_x *and*

$$\Omega_g\psi(0) = \tilde{A}(\infty)\psi_b(x). \tag{3.41}$$

Based on (3.2), we also consider a class of mixture models

$$\begin{cases} i\partial_t\psi = H_0\psi + W(x)\psi + g(t)^{-2}V(\frac{x}{g(t)})\psi \\ \psi(x,t_0) = \psi_d(x) + e^{-iD\ln(g(t_0))}\psi_b(x) \in H^1_x(\mathbb{R}^n) \\ \sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1_x} \lesssim 1 \end{cases} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{3.42}$$

$W(x)$ satisfies that

$$\begin{cases} H_0 + W(x) \text{ has a normalized eigenvector } \psi_d(x) \text{ with an eigenvalue } \lambda_0 < 0 \\ W(x) \in L^2_x(\mathbb{R}^n) \end{cases}. \tag{3.43}$$

We show that the weakly localized part defined in (3.6) asymptotically has at least two bubbles: a non-trivial self-similar part and a non-trivial localized part near the origin.

Theorem 3.1.3. *Let* $\tilde{a}(t)$ *be as in* (3.12)*. If* $W(x), V(x), H$ *satisfy* (3.42) *and* $g(t)$ *satisfies* (3.3)*, then when* $n \geq 5$*,* $\epsilon \in (2/n, 1/2)$*,*

$$\tilde{A}(\infty) := \lim_{t \to \infty} e^{i\lambda T(t)}\tilde{a}(t) \tag{3.44}$$

exists and

$$\psi_{w,l}(x,t) = c(t)e^{-iD\ln(g(t))}\psi_b(x) - \psi_c(x,t) \tag{3.45}$$

$$c(t) := (e^{-iD\ln(g(t))}\psi_b(x), \psi_{w,l}(x,t))_{L^2_x}, \tag{3.46}$$

$$(e^{-iD\ln(g(t))}\psi_b(x), \psi_c(x,t))_{L^2_x} = 0, \tag{3.47}$$

where $c(t)$ *satisfies* (3.16) *and there exists* $M > 1$ *such that*

$$\liminf_{t\to\infty} |(\psi_c(x,t), \psi_d(x))_{L^2_x}| \geq c. \tag{3.48}$$

Moreover, based on (3.25)*, the* $g(t)$*-self-similar channel wave operator*

$$\Omega_g\psi(0) := w\text{-}\lim_{s\to\infty} e^{isH}e^{iD\ln(g(T^{-1}(s)))}\psi(T^{-1}(s)) \tag{3.49}$$

exists in L^2_x *and*

$$\Omega_g\psi(0) = \tilde{A}(\infty)\psi_b(x). \tag{3.50}$$

As an application of Theorem 3.4.1, consider a focusing nonlinear Schrödinger equation

$$\begin{cases} i\partial_t\psi = H_0\psi + g(t)^{-2}V(\frac{x}{g(t)})\psi + \mathcal{N}(|\psi|)\psi \\ \psi(x,t_0) = \psi_s(x) + e^{-iD\ln(g(t_0))}\psi_b(x) \in H^1_x(\mathbb{R}^n) \\ \text{There is a global } H^1_x \text{ solution } \psi(t) \\ \sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1_x} \lesssim_{\psi(t_0)} 1 \\ \text{Both } \psi(t_0) \text{ and } V(x) \text{ are radial in } x \\ \mathcal{N} \leq 0 \end{cases}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{3.51}$$

when $n \geq 5$. Assume that $V(x), \psi_b(x)$ satisfy (3.4)($H := H_0 + V(x)$) and

$$\begin{cases} \|D\psi_b(x)\|_{L^1_x} \lesssim 1 \\ \|\frac{1}{\lambda - H}P_c\|_{L^1_x \to L^1_x} \lesssim 1 \end{cases}, \tag{3.52}$$

and $\psi_s(x)$ is a soliton of

$$i\partial_t\phi = H_0\phi + \mathcal{N}_{F,0}(|\phi|)\phi, \tag{3.53}$$

where

$$\mathcal{N}_{F,0}(k) := \int_0^k dq\, q\mathcal{N}(q)/k^2 < 0, \tag{3.54}$$

that is,

$$(H_0 + 2N_{F,0}(|\psi_s(x)|))\psi_s(x) = E\psi_s(x), \text{ for some } E < 0. \tag{3.55}$$

If the nonlinearity satisfies that

1. there exists $T \geq 1$ such that for all $t_0 \geq T$,

$$(\psi(t_0), (H_0 + \frac{1}{g(t_0)^2} V(\frac{x}{g(t_0)}) + 2N_{F,0}(|\psi(t_0)|))\psi(t_0))_{L^2_x} \leq \frac{E}{2}\|\psi_s(x)\|^2_{L^2_x} \tag{3.56}$$

with $\psi(t_0) = \psi_s(x) + e^{-iD\ln(g(t_0))}\psi_b(x)$,

2. N satisfies that

$$|N_{F,0}(k)| \lesssim |k|^{\beta}, \text{ for some } \beta > 0 \tag{3.57}$$

and for $f \in H^1_x$,

$$\begin{cases} \|N(|f(x)|)\|_{L^2_x} \leq C(\|f(x)\|_{H^1_x}) \\ \|N(|f(x)|)f(x)\|_{L^2_x} \leq C(\|f(x)\|_{H^1_x}) \end{cases}, \tag{3.58}$$

then there are at least two bubbles in $\psi(t)$, a solution to system (3.51).

Remark 13. *The radial assumption implies that*

$$|\psi(t)| \lesssim \frac{1}{|x|^{\frac{n-1}{2}}} \sup_k \|\psi(k)\|_{H^1_x}, \quad |x| \geq 1. \tag{3.59}$$

Based on (3.57) *and* (3.59), *one has that*

$$\left|(\psi(t), N_{F,0}(|\psi(t)|)\psi(t))_{L^2_x}\right| \leq \frac{1}{M^{\frac{(n-1)\beta}{2}}} C(\|\psi(t)\|_{H^1_x}) + \|\chi(|x| \leq M)\psi(t)\|_{L^2_x} C(\|\psi(t)\|_{H^1_x}) \tag{3.60}$$

for all $t \geq t_0$ and t_0, sufficiently large.

Remark 14. *Here*

$$\|\frac{1}{\lambda - H} P_c\|_{L^\infty_x \to L^\infty_x} \lesssim 1 \tag{3.61}$$

is true when the wave operator, associated with a pair H_0, H, from high frequency cut-off L^p space to L^p, is bounded for $p = \infty$. See [87] for more details about such potentials.

We show that the weakly localized part of (3.51) has at least two bubbles: a non-trivial self-similar part and a non-trivial localized part near the origin.

Theorem 3.1.4. *Let $\tilde{a}(t)$ be as in* (3.12)*. If $N, V(x), H$ satisfy* (3.51) *and $g(t)$ satisfies* (3.3)*, then when $n \geq 5$, $\epsilon > (2/n, 1/2)$,*

$$\tilde{A}(\infty) := \lim_{t\to\infty} e^{i\lambda T(t)}\tilde{a}(t) \tag{3.62}$$

exists and

$$\psi_{w,l}(x,t) = c(t)e^{-iD\ln(g(t))}\psi_b(x) - \psi_c(x,t) \tag{3.63}$$

$$c(t) := (e^{-iD\ln(g(t))}\psi_b(x), \psi_{w,l}(x,t))_{L^2_x}, \tag{3.64}$$

$$(e^{-iD\ln(g(t))}\psi_b(x), \psi_c(x,t))_{L^2_x} = 0, \tag{3.65}$$

with $c(t)$ satisfying (3.16)*. Furthermore, there exists some large number $M \gg 1$ such that*

$$\lim_{t\to\infty}\inf \|\chi(|x| \leq M)\psi(t)\|_{L^2_x} \geq c' \tag{3.66}$$

for some $c' > 0$. Moreover, based on (3.25)*, the $g(t)$-self-similar channel wave operator*

$$\Omega_g\psi(0) := w\text{-}\lim_{s\to\infty} e^{isH}e^{iD\ln(g(T^{-1}(s)))}\psi(T^{-1}(s)) \tag{3.67}$$

exists in L^2_x and

$$\Omega_g\psi(0) = \tilde{A}(\infty)\psi_b(x). \tag{3.68}$$

Typical example of Theorem 3.1.4 is

$$N(|\psi(t)|) = -\lambda\frac{|\psi(t)|}{1+|\psi(t)|^2}, \quad g(t) = \langle t\rangle^{\epsilon} \tag{3.69}$$

for some $\epsilon \in (2/5, 1/2)$ and some sufficiently large $\lambda > 0$ in 5 space dimensions. In this case, by taking $\lambda > 0$ large enough, there is a soliton to (3.53). By using standard iteration scheme, there is a global L^2 solution to (3.51) for any initial H^1_x data. The H^1_x norm of the solution is uniformly bounded in t since this system has an asymptotic energy. See Lemma 3.5.1 in Section 3.5 for more details.

3.1.2 Outline of the proof

For the linear problem, the proof scheme of Theorem 3.1.2 is first to set

$$\tilde{\phi}(t) := e^{iD\ln(g(t))}\psi(x,t), \tag{3.70}$$

with $\tilde{\phi}(t)$ satisfying

$$\begin{cases} i\partial_t\tilde{\phi} = g(t)^{-2}H\tilde{\phi} - (\partial_t[g(t)]g(t)^{-1})D\tilde{\phi} \\ \tilde{\phi}(t_0) = \psi_b(x) \end{cases}, \tag{3.71}$$

see Lemma 3.3.1. Secondly, using change of variables from t to $s = T(t)$, (3.71) can be rewritten as

$$\begin{cases} i\partial_s\phi = H\phi + f(s)D\phi \\ \phi(s_0) = \psi_b(x) \end{cases} \tag{3.72}$$

by setting

$$\phi(s) := \tilde{\phi}(T^{-1}(s)), \quad t_0 := T^{-1}(s_0) \tag{3.73}$$

where

$$f(s) := -(\partial_t[g(t)]g(t))|_{t=T^{-1}(s)}. \tag{3.74}$$

Based on (3.3), we have

$$f(s) \lesssim \frac{1}{\langle s\rangle} \quad \text{and} \quad f'(s) \lesssim \frac{1}{\langle s\rangle^2}, \tag{3.75}$$

see Lemma ??. So up to here, the problem is reduced to study the ionization problem(for ionization problem, see [83]). To be precise, it is reduced to study the asymptotic behavior of $a(s)$ with

$$a(s) := (\psi_b, \phi(s))_{L^2_x}. \tag{3.76}$$

Indeed

$$\tilde{a}(t) = a(T(t)). \tag{3.77}$$

Let

$$A(s) := e^{i\lambda s}a(s). \tag{3.78}$$

In the end, we show that the limit

$$A(\infty) := \lim_{s\to\infty} e^{i\lambda s}a(s) \tag{3.79}$$

exists which implies

$$\tilde{A}(\infty) = A(\infty) \tag{3.80}$$

exists since

$$\tilde{A}(T^{-1}(s)) = A(s). \tag{3.81}$$

And if we choose s_0 wisely(large enough),

$$|A(\infty)| \geq \frac{1}{2} > 0 \tag{3.82}$$

and finish the proof.

For the nonlinear problem or the mixture problem, it is like the linear one except for the nonlinear term and $W(x)$ term. For these terms, we use

$$|(g(t)^2 e^{-iD\ln(g(t))}\psi_b(x), N(|\psi|)\psi)_{L^2_x}| \lesssim g(t)^{-(n/2-2)}\|\psi_b(x)\|_{L^\infty_x}\|N(|\psi|)\psi\|_{L^1_x}, \tag{3.83}$$

$$|(g(t)^2 e^{-iD\ln(g(t))}\psi_b(x), W(x)\psi)_{L^2_x}| \lesssim g(t)^{-(n/2-2)}\|\psi_b(x)\|_{L^\infty_x}\|W(x)\psi\|_{L^1_x}, \tag{3.84}$$

and

$$g(T^{-1}(u))^{-(n/2-2)} \lesssim \langle u\rangle^{-(n/2-2)\epsilon/(1-2\epsilon)} \in L^1_u[1,\infty) \tag{3.85}$$

when $n \geq 5$ and $\epsilon \in (2/n, 1/2)$.

This follows stability analysis of coherent structures [84, 81].

In order to prove the existence of another bubble near the origin, we use the fact that for these systems, there is an asymptotic energy which is negative. Since the self-similar part carries no energy, there must be a part of the solution localized on the support of the potential W.

3.2 Existence of free channel wave operator for a MCS in 4 or higher space dimensions

In this section, we prove the existence of free channel wave operator in system

$$\begin{cases} i\partial_t\psi(t) = (H_0 + V(x,t))\psi(t) \\ \psi(0) = \psi_0 \in L^2_x(\mathbb{R}^n) \end{cases} \quad n \geq 4 \tag{3.86}$$

provided that $\langle t\rangle^{n/2+2}V(x,t) \in L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})$.

Theorem 3.2.1. *If* $\langle t\rangle^{-n/2+2}V(x,t) \in L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})$, *then when* $n \geq 4$, *the free channel wave operator of* (3.86)

$$\Omega_\alpha\psi(0) := s\text{-}\lim_{t\to\infty} e^{itH_0}F_c(\frac{|x-2tP|}{t^\alpha} \leq 1)\psi(t), \tag{3.87}$$

exists in $L^2_x(\mathbb{R}^n)$ *for* $\alpha \in (0, 2/n)$.

Proof. Let

$$\Omega_\alpha(t)\psi(0) := e^{itH_0}F_c(\frac{|x-2tP|}{t^\alpha} \leq 1)\psi(t). \tag{3.88}$$

Since

$$e^{itH_0}F_c(\frac{|x-2tP|}{t^\alpha} \leq 1)\psi(t) = F_c(\frac{|x|}{t^\alpha} \leq 1)e^{itH_0}\psi(t), \tag{3.89}$$

then by using Fundamental Theorem in Calculus,

$$\begin{aligned}\Omega_\alpha(t)\psi(0) &= \Omega_\alpha(1)\psi(0) + \int_1^t ds\partial_s[e^{isH_0}F_c(\frac{|x-2sP|}{s^\alpha} \leq 1)\psi(s)] \\ &= \Omega_\alpha(1)\psi(0) + \int_1^t ds\partial_s[F_c(\frac{|x|}{s^\alpha} \leq 1)]e^{isH_0}\psi(s) + \int_1^t dsF_c(\frac{|x|}{s^\alpha} \leq 1)e^{isH_0}V(x,s)\psi(s) \\ &=: \Omega_\alpha(1)\psi(0) + \psi_1(t) + \psi_2(t). \end{aligned} \tag{3.90}$$

$\psi_2(\infty) \in L^2_x(\mathbb{R}^n)$ since for $T \geq t \geq 1$,

$$\begin{aligned}&\int_t^T ds\|F_c(\frac{|x|}{s^\alpha} \leq 1)e^{isH_0}V(x,s)\psi(s)\|_{L^2_x(\mathbb{R}^n)} \\ &\text{(Hölder's inequality)} \leq \int_t^T ds\|F_c(\frac{|x|}{s^\alpha} \leq 1)\|_{L^2_x}\|e^{isH_0}V(x,s)\psi(s)\|_{L^\infty_x} \\ &(L^\infty_x \text{ decay estimates of free flow}) \lesssim \int_t^T ds s^{n/2\alpha}\frac{1}{s^{n/2}}\langle s\rangle^{n/2-2}\|\langle t\rangle^{-n/2+2}V(x,t)\|_{L^\infty_t L^2_x}\|\psi(s)\|_{L^2_x} \\ &\lesssim \int_t^T ds\frac{1}{s^{2-n/2\alpha}}\|\langle t\rangle^{-n/2+2}V(x,t)\|_{L^\infty_t L^2_x}\|\psi(s)\|_{L^2_x} \\ &\lesssim_\alpha \frac{1}{t^{1-n/2\alpha}}\|\langle t\rangle^{-n/2+2}V(x,t)\|_{L^\infty_t L^2_x}\|\psi(s)\|_{L^2_x} \to 0 \end{aligned} \tag{3.91}$$

as $t \to \infty$ when $\alpha \in (0, 2/n)$.

For $\psi_1(t)$, we use Propagation estimates, see [91]. To be precise, choose

$$B(t) = F_c(\frac{|x|}{t^\alpha} \leq 1) \tag{3.92}$$

as our observable. Observe

$$\langle B(t)\rangle := (e^{itH_0}\psi(t), B(t)e^{itH_0}\psi(t))_{L^2_x}. \tag{3.93}$$

Compute $\partial_t \langle B(t)\rangle$

$$\partial_t \langle B(t)\rangle = \langle \partial_t [B(t)]\rangle + (-i)(e^{itH_0}\psi(t), B(t)e^{itH_0}V(x,t)\psi(t))_{L^2_x}+$$
$$i(e^{itH_0}V(x,t)\psi(t), B(t)e^{itH_0}\psi(t))_{L^2_x} =: A_1(t)+A_2(t)+A_3(t). \quad (3.94)$$

$A_2(t), A_3(\infty) \in L^1_t[1,\infty)$ due to (3.91). Since $A_1(t) \ge 0$ and

$$|\langle B(t)\rangle| \le \|\psi(0)\|^2_{L^2_x}, \quad (3.95)$$

one has that for $T \ge t \ge 1$,

$$\int_t^T dsA_1(s) \le 2\|\psi(0)\|^2_{L^2_x} + \|A_2(s)\|_{L^1_s[1,\infty)} + \|A_3(s)\|_{L^1_s[1,\infty)} \quad (3.96)$$

which implies $A_1(t) \in L^1_t[1,\infty)$. Then by using Hölder's inequality in s variable, for $T \ge t \ge 1$,

$$\|\psi_1(T) - \psi_1(t)\|_{L^2_x} \lesssim \left(\int d^3x \int_t^T ds|\partial_s[F_c(\frac{|x|}{s^\alpha} \le 1)]||e^{isH_0}\psi(s)|^2\right)^{1/2}$$
$$\lesssim \left(\int_t^T dsA_1(s)\right)^{1/2} \to 0 \quad (3.97)$$

as $t \to \infty$, which implies $\psi_1(\infty)$ exists in L^2_x. Thus, that $\psi_1(\infty), \psi_2(\infty) \in L^2_x$ implies that $\Omega_\alpha\psi(0)$ exists in L^2_x and we finish the proof.

□

Since

$$\sup_{t\in\mathbb{R}} \|\langle t\rangle^{-n/2+2} g(t)^{-2} V(x/g(t))\|_{L^2_x} = \sup_{t\in\mathbb{R}} \left(\frac{g(t)}{\langle t\rangle}\right)^{n/2-2} \|V(x)\|_{L^2_x} \quad (3.98)$$
$$\lesssim \sup_{t\in\mathbb{R}} \left(\frac{1}{\langle t\rangle^{1-\epsilon}}\right)^{n/2-2} \|V(x)\|_{L^2_x} \quad (3.99)$$
$$\lesssim \|V(x)\|_{L^2_x} \quad (3.100)$$

for $\epsilon \in (0,1]$, in (3.2), (3.42) and (3.51) with space dimension $n \ge 4$, the free channel wave operator

$$\Omega_a := s\text{-}\lim_{t\to\infty} e^{itH_0} F_c(\frac{|x-2tP|}{t^\alpha} \le 1)U(t,0) \quad (3.101)$$

exists from L^2_x to L^2_x.

3.3 Proof of Theorem 3.1.2: Self-similar Time-dependent Linear Problem

3.3.1 Tool box

Lemma 3.3.1. *Let* $\tilde{\phi}$ *be as in* (3.70) *and* ψ *denote the solution to* (3.1). *Then* $\tilde{\phi}$ *satisfies* (3.71).

Proof. Let $U(t,0)$ denote the solution operator to (3.1). Then $\psi(x,t)$ can be rewritten as

$$\psi(x,t) = [U(t,0)\psi_0](x,t). \tag{3.102}$$

Thus, based on the definition of $\tilde{\phi}$, it can be rewritten as

$$\tilde{\phi}(x,t) = e^{iD\ln g(t)}\psi(t) = g(t)^{n/2}[U(t,0)\psi_0](g(t)x,t). \tag{3.103}$$

Using Chain rule, compute $i\partial_t[\tilde{\phi}(x,t)]$

$$\begin{aligned} i\partial_t[\tilde{\phi}(x,t)] &= e^{iD\ln g(t)}(H_0 + g(t)^{-2}V(x/g(t)))\psi(t) - (\partial_t[g(t)]g(t)^{-1})D\tilde{\phi} \\ &= g(t)^{-2}H\tilde{\phi} - (\partial_t[g(t)]g(t)^{-1})D\tilde{\phi}. \end{aligned} \tag{3.104}$$

We finish the proof.

□

Lemma 3.3.2. *If* $g(t)$ *satisfies* (3.3), (3.75) *is true.*

Proof. Based on the definition of T, we have

$$s = \int_0^t du\, g(u)^{-2} \sim \langle t\rangle^{1-2\epsilon}. \tag{3.105}$$

Since as $t \to \infty$,

$$|f(s)| = |g'(t)g(t)| \lesssim \frac{1}{t}g(t)^2 \sim \frac{1}{\langle t\rangle^{1-2\epsilon}}, \tag{3.106}$$

we get

$$|f(s)| \lesssim \frac{1}{\langle s\rangle}, \tag{3.107}$$

Now we show

$$f'(s) \lesssim \frac{1}{\langle s\rangle^2}. \tag{3.108}$$

Since

$$f'(s) = \quad (g'(t)^2 + g''(t)g(t)) \quad \frac{dt}{ds} = \quad (g'(t)^2 + g''(t)g(t))g(t)^2 = \quad f(s)^2 \quad g''(t)g(t)^3, \tag{3.109}$$

and since due to (3.3), (3.105),

$$g''(t)g(t)^3 \quad \frac{1}{\langle t\rangle^2} \quad \langle t\rangle^{4\epsilon} \quad \frac{1}{\langle s\rangle^2}, \tag{3.110}$$

according to (3.107), we get

$$f'(s) \quad f(s)^2 + g''(t)g(t)^3 \quad \frac{1}{\langle s\rangle^2}. \tag{3.111}$$

We finish the proof. □

Lemma 3.3.3. *Let* λ, H, V, g *be as in* (3.4).

$$\|D\frac{1}{\lambda \quad H}P_c\langle x\rangle^{\ 1}\|_{L^2_x \to L^2_x} \lesssim_\lambda 1. \tag{3.112}$$

Proof. Using second resolvent identity, we have

$$D\frac{1}{\lambda \quad H}P_c\langle x\rangle^{\ 1} = D\frac{1}{\lambda \quad H_0}P_c\langle x\rangle^{\ 1} \quad D\frac{1}{\lambda \quad H}V(x)\frac{1}{\lambda \quad H}P_c\langle x\rangle^{\ 1}. \tag{3.113}$$

(6.104) follows from

$$\|D\frac{1}{\lambda \quad H_0}\langle x\rangle^{\ 1}\|_{L^2_x \to L^2_x} \lesssim_\lambda 1, \tag{3.114}$$

$$\|\langle x\rangle P_c\langle x\rangle^{\ 1}\|_{L^2_x \to L^2_x} \lesssim 1, \tag{3.115}$$

and

$$\|\langle x\rangle V(x)\frac{1}{\lambda \quad H}P_c\|_{L^2_x \to L^2_x} \lesssim_\lambda \|\langle x\rangle V\|_{L^{\ 1}_x}. \tag{3.116}$$

□

3.3.2 Ionization Problem

Based on (3.76) and (3.72), we write out an equation of a

$$i\partial_s[a(s)] = \lambda a(s) + f(s)(\psi_b, D\psi_b)_{L^2_x}a(s) + f(s)(\psi_b, DP_c\phi(s))_{L^2_x} \tag{3.117}$$

where P_c denotes the projection on the continuous spectrum of H. Without loss of generality, we can choose $\psi_b(x)$ real. Then

$$(\psi_b, D\psi_b)_{L^2_x} = 1/2 \int d^3x P_x \, (x|\psi_b|^2) = 0. \tag{3.118}$$

Let

$$A(s) := e^{i\lambda s} a(s). \tag{3.119}$$

$A(s)$ satisfies

$$i\partial_s[A(s)] = e^{i\lambda s} f(s)(\psi_b, DP_c\phi(s))_{L^2_x}. \tag{3.120}$$

Thus,

$$A(s) = A(s_0) + (-i)\int_{s_0}^{s} du e^{i\lambda u} f(u)(\psi_b, DP_c\phi(u))_{L^2_x}. \tag{3.121}$$

Theorem 3.3.1. *Let $A(s)$ be as defined above. $A(\infty)$ exists and if $s_0 > 0$ large enough,*

$$|A(\infty)| \geq \frac{1}{2} > 0. \tag{3.122}$$

Proof. Writing $e^{i\lambda u}P_c\phi(u)$ as

$$e^{i\lambda u}P_c\phi(u) = e^{i\lambda u}e^{-iuH}P_c e^{iuH}\phi(u) \tag{3.123}$$

$$= \frac{P_c}{i(\lambda - H)}\partial_u[e^{i\lambda u}e^{-iuH}P_c]e^{iuH}\phi(u) \tag{3.124}$$

and taking integration by parts in u variable, we obtain that for $s > s_1 \geq s_0$

$$A(s) = A(s_1) + (-1)f(u)(\psi_b, D\frac{P_c}{\lambda - H}e^{i\lambda u}P_c\phi(u))_{L^2_x}|_{u=s_1}^{u=s} + \tag{3.125}$$

$$\int_{s_1}^{s} du f'(u)(\psi_b, D\frac{P_c}{\lambda - H}e^{i\lambda u}P_c\phi(u))_{L^2_x} + \tag{3.126}$$

$$\int_{s_1}^{s} du f(u)^2(\psi_b, D\frac{P_c}{\lambda - H}e^{i\lambda u}P_c D\phi(u))_{L^2_x} \tag{3.127}$$

$$=: A(s_1) + \sum_{j=1}^{3} A_j(s). \tag{3.128}$$

Here

$$(\psi_b, D\frac{1}{\lambda - H}e^{i\lambda u}P_c D\phi(u))_{L^2_x} \tag{3.129}$$

is understood in weak sense, that is,

$$(\psi_b, D\frac{1}{\lambda - H}e^{i\lambda u}P_c D\phi(u))_{L^2_x} = (P_c\frac{1}{\lambda - H}D\psi_b, De^{i\lambda u}\phi(u))_{L^2_x}. \tag{3.130}$$

For $s \geq s_1 \geq s_0$, using L^2_x conservation law, Hölder's inequality and Lemma 3.3.2,

$$|A_1(s)| \lesssim |f(s_1)| \|D\psi_b(x)\|_{L^2_x} \left\|\frac{1}{\lambda - H}P_c\right\|_{L^2_x \to L^2_x} \|\phi(s_0)\|_{L^2_x} \lesssim |f(s_1)| \lesssim \frac{1}{\langle s_1 \rangle}, \tag{3.131}$$

$$|A_2(s)| \lesssim \int_{s_1}^{s} du |f'(u)| \|D\psi_b(x)\|_{L^2_x} \left\|\frac{1}{\lambda - H}P_c\right\|_{L^2_x \to L^2_x} \|\phi(s_1)\|_{L^2_x} \lesssim \frac{1}{\langle s_1 \rangle}, \tag{3.132}$$

and due to Lemma 3.3.2 and Lemma 3.3.3,

$$|A_3(s)| \lesssim \int_{s_1}^{s} du f(u)^2 \|\langle x \rangle D\psi_b(x)\|_{L^2_x} \left\|D\frac{1}{\lambda - H}P_c \langle x \rangle^{-1}\right\|_{L^2_x \to L^2_x} \|\phi(s_0)\|_{L^2_x} \lesssim \frac{1}{\langle s_1 \rangle}. \tag{3.133}$$

So $\{A(s)\}_{s \geq s_0}$ is Cauchy and therefore $A(\infty)$ exists with

$$|A(s)| \geq |A(s_0)| - C\frac{1}{\langle s_0 \rangle} = 1 - C\frac{1}{\langle s_0 \rangle} \geq \frac{1}{2}, \text{ for all } s \in [s_0, \infty] \tag{3.134}$$

if we choose s_0 large enough. We finish the proof. □

Lemma 3.3.4. *Let V, H, g be as in 3.4. For $t \geq t_0$ and t_0 large enough,*

$$|(\psi_b(x), \tilde{\phi}(x,t))_{L^2_x}| \geq 1/2. \tag{3.135}$$

Proof. Taking $t_0 = T^{-1}(s_0)$ for s_0 satisfying (3.134), we have

$$|(\psi_b(x), \tilde{\phi}(x,t))_{L^2_x}| = |(\psi_b(x), \phi(x,s))_{L^2_x}| = |A(s)| \geq 1/2 \tag{3.136}$$

with $s = T(t)$.

□

3.3.3 Linear Problem

Now we prove Theorem 3.1.2.

Proof. Express $\psi(x,t)$ in terms of $\tilde{\phi}(x,t)$

$$\psi(x,t) = e^{-iD\ln g(t)}\tilde{\phi}(x,t) = c(t)e^{-iD\ln g(t)}\psi_b(x) + \psi_c(x,t) \tag{3.137}$$

where

$$(\psi_c(x,t), e^{-iD\ln g(t)}\psi_b(x))_{L^2_x} = 0. \tag{3.138}$$

Then due to Lemma 3.3.4,

$$|\tilde{a}(t)| \geq \frac{1}{2}, \quad \text{for all } t \geq t_0 \tag{3.139}$$

for some sufficiently large $t_0 > 0$. Let

$$\psi_{g,b}(x,t) := e^{-iD\ln g(t)}\psi_b(x). \tag{3.140}$$

Using L^p decay estimates for the free flow, we obtain

$$\begin{aligned} \|F_c(\frac{|x-2tP_x|}{t^\alpha}\leq 1)\psi_{g,b}(x,t)\|_{L^2_x} &= \|F_c(\frac{|x|}{t^\alpha}\leq 1)e^{itH_0}\psi_{g,b}(x,t)\|_{L^2_x} \\ &\lesssim \|F_c(\frac{|x|}{t^\alpha}\leq 1)\|_{L^2_x}\|e^{itH_0}\psi_{g,b}(x,t)\|_{L^\infty_x} \\ &\lesssim t^{-n(1-\alpha)/2}\|\psi_{g,b}\|_{L^1_x} \lesssim \frac{g(t)^{n/2}}{t^{n(1-\alpha)/2}} \lesssim \frac{\langle t\rangle^{n\epsilon/2}}{t^{n(1-\alpha)/2}} \to 0 \end{aligned} \tag{3.141}$$

as $t\to\infty$ when $1-\alpha-\epsilon>0$. Given t_0 satisfying (3.139),

$$\begin{aligned} |c(t)| = |(\psi_{g,b}(x,t),\psi_{w,l}(t))_{L^2_x}| &= |a(t)-(\psi_{g,b}(x,t),F_c(\frac{|x-2tP|}{t^\alpha}\leq 1)(\psi(t)-\psi_e(x,t))_{L^2_x}| \\ &\geq \frac{1}{2}-\frac{1}{8}-\frac{1}{8}=\frac{1}{4} \end{aligned} \tag{3.142}$$

when $t\geq t_M$ for some sufficiently large $t_M\geq t_0$, which implies (3.16). We finish the proof.

□

3.4 Proof of Theorem 3.4.1: the Linear mixture Problem

In this section, we keep using

$$s = T(t) := \int_0^t du\, g(u)^{-2}. \tag{3.143}$$

3.4.1 Tool box

Lemma 3.4.1. *Let* ψ *be the global solution to* (3.42) *and* $f(s)$ *be as in* (3.75). *Let*

$$\tilde{\phi}(x,t) := e^{iD\ln g(t)}\psi(x,t), \tag{3.144}$$

and

$$\phi(x,s) := \tilde{\phi}(x,T^{-1}(s)). \tag{3.145}$$

Then with $s_0 = T(t_0),\ s = T(t)$,

$$\begin{cases} i\partial_s \phi = H\phi + g(T^{-1}(s))^2 W(g(T^{-1}(s))x)\phi + f(s)D\phi \\ \phi(s_0) = g(t)^{n/2}\psi_d(g(t)x) + \psi_b(x) \end{cases} \tag{3.146}$$

where

$$\psi_g(t) := \psi(g(t)x, t). \tag{3.147}$$

Proof. It follows from the same proof for Lemma 3.3.1 by replacing $V(x,t) = g(t)^{-2}V(\frac{x}{g(t)})$ with $V(x,t) = g(t)^{-2}V(\frac{x}{g(t)}) + W(x)$ and then change variable from t to $s = T(t)$.

□

3.4.2 Ionization Problem of a mixture

Let

$$a(s) := (\psi_b(x), \phi(s))_{L^2_x}. \tag{3.148}$$

As what we did for linear problem, we derive an equation for $a(t)$ first. Compute $i\partial_s[a(s)](s = T(t))$

$$i\partial_s[a(s)] = \lambda a(s) + f(s)(\psi_b(x), DP_c\phi(s))_{L^2_x} + \tag{3.149}$$

$$g(T^{-1}(s))^2(\psi_b(x), W(g(T^{-1}(s))x)\phi(s))_{L^2_x} \tag{3.150}$$

where we use (3.118). Set

$$A(s) := e^{is\lambda}a(s). \tag{3.151}$$

Then $A(s)$ satisfies

$$i\partial_s[A(s)] = e^{is\lambda}f(s)(D\psi_b(x), P_c\phi(s))_{L^2_x} + e^{is\lambda}g(T^{-1}(s))^2(\psi_b(x), W(g(T^{-1}(s))x)e^{is\lambda}\phi(s))_{L^2_x}. \tag{3.152}$$

Theorem 3.4.1. *Let* $A(s)$ *be as defined above. If* $g(t) \sim \langle t\rangle^{\epsilon}$ *with* $\epsilon \in (\frac{2}{n}, \frac{1}{2}), W(x) \in L^2_x$, *when* $n \geq 5$, $A(\infty)$ *exists and if* $s_0 > 0$ *large enough,*

$$|A(s)| \geq \frac{1}{2} > 0 \quad \text{for all } s \geq s_0. \tag{3.153}$$

Proof. Set

$$N_g(x,u) := W(g(T^{-1}(u))x). \tag{3.154}$$

$$\begin{aligned} A(s) =& A(s_0) + (-i)\int_{s_0}^{s} due^{iu\lambda} f(u) e^{iu\lambda}(D\psi_b(x), P_c\phi(u))_{L_x^2} + & (3.155)\\ & (-i)\int_{s_0}^{s} e^{iu\lambda} g(T^{-1}(u))^2(\psi_b(x), N_g(x,u)e^{iu\lambda}\phi(u))_{L_x^2} & (3.156)\\ =:& A(s_0) + A_1(s) + A_2(s). & (3.157)\end{aligned}$$

For $A_1(s)$, take integration by parts in s variable by setting

$$e^{i\lambda s} P_c\phi(s) = \frac{1}{i(\lambda - H)} P_c \partial_s [e^{i\lambda s} e^{-isH}] e^{isH}\phi(s) \tag{3.158}$$

and we obtain

$$\begin{aligned} A_1(s) =& f(u)(D\psi_b(x), \frac{1}{i(\lambda - H)} P_c e^{iu\lambda}\phi(u))_{L_x^2}|_{u=s_0}^{u=s} & (3.159)\\ & -\int_{s_0}^{s} duf'(u)(D\psi_b(x), \frac{1}{i(\lambda - H)} P_c e^{iu\lambda}\phi(u))_{L_x^2} & (3.160)\\ & -\int_{s_0}^{s} duf(u)^2(D\psi_b(x), \frac{1}{i(\lambda - H)} P_c D e^{iu\lambda}\phi(u))_{L_x^2} & (3.161)\\ & -\int_{s_0}^{s} dug(T^{-1}(u))^2 f(u)(D\psi_b(x), \frac{1}{i(\lambda - H)} P_c N_g(x,u) e^{iu\lambda}\phi(u))_{L_x^2} & (3.162)\\ =:& \sum_{j=1}^{4} \tilde{A}_{1j}(s). & (3.163)\end{aligned}$$

As what we did in linear case, for $s_2 \geq s_1 \geq s_0$,

$$|\tilde{A}_{1,j}(s_2) - \tilde{A}_{1,j}(s_1)| \lesssim \frac{1}{\langle s_1 \rangle}, \quad j = 1,2,3. \tag{3.164}$$

For $\tilde{A}_{1,4}$, let

$$\tilde{\psi}_b(x) := P_c \frac{1}{i(\lambda - H)} D\psi_b(x). \tag{3.165}$$

Then

$$\|\tilde{\psi}_b(x)\|_{L_x^1} = \|P_c \frac{1}{i(\lambda - H)} D\psi_b(x)\|_{L_x^1} \lesssim \|D\psi_b(x)\|_{L_x^1}. \tag{3.166}$$

Using change of variable from x to $y = \langle T^{-1}(u)\rangle^{\epsilon} x$, we have

$$\begin{aligned} &(\tilde{\psi}_b(x), N_g(x,u)e^{iu\lambda}\phi(u))_{L_x^2} = \\ &\qquad (\frac{1}{g(T^{-1}(u))^{n/2}} \tilde{\psi}_b(g(T^{-1}(u))^{-1}y), W(y)e^{iu\lambda}\psi(y, T^{-1}(u)))_{L_y^2}. & (3.167)\end{aligned}$$

So using Cauchy Schwarz inequality, due to (3.42), (3.43) and (3.3)($g(t) \sim \langle t\rangle^{\epsilon}$), we obtain

$$|\tilde{A}_{1,4}(s_2) - \tilde{A}_{1,4}(s_1)| \lesssim \int_{s_1}^{s_2} du|f(u)| \frac{1}{\langle T^{-1}(u)\rangle^{(\frac{n}{2}-2)\epsilon}} \|\tilde{\psi}_b(y)\|_{L_y^1} \|W(y)\psi(y, T^{-1}(u))\|_{L_y^1} \tag{3.168}$$

$$\lesssim \frac{\|W(x)\|_{L_x^2}\|\psi(0)\|_{L_x^2}}{\langle T^{-1}(s_1)\rangle^{(\frac{n}{2}-2)\epsilon}} \lesssim \frac{\|W(x)\|_{L_x^2}\|\psi(0)\|_{L_x^2}}{\langle s_1\rangle^{(n/2-2)\epsilon/(1-2\epsilon)}} \to 0 \tag{3.169}$$

as $s_1 \to \infty$ when $\epsilon \in (0, 1/2)$ and $n \geq 5$. Thus,

$$\tilde{A}_{1,j}(\infty) \text{ exists for all } j = 1, 2, 3, 4 \tag{3.170}$$

and therefore $A_1(\infty)$ exists. And for s_0 large enough,

$$|A_1(s)| \leq C\frac{(\|W(x)\|_{L_x^2}+1)\|\psi(0)\|_{L_x^2}}{\langle s_0\rangle^{\min(1,(n/2-2)\epsilon/(1-2\epsilon))}} \leq \frac{1}{4}. \tag{3.171}$$

For $A_2(s)$, using Cauchy Schwarz inequality, due to (3.42) and (3.3), we obtain that for $s_2 \geq s_1 \geq s_0$,

$$|A_2(s_2) - A_2(s_1)| \lesssim \int_{s_1}^{s_2} du \frac{1}{\langle T^{-1}(u)\rangle^{(\frac{n}{2}-2)\epsilon}} \|\tilde{\psi}_b(y)\|_{L_y^1} \|W(y)\psi(T^{-1}(u))\|_{L_y^1} \tag{3.172}$$

$$\lesssim \int_{s_1}^{s_2} du \frac{\|W(x)\|_{L_x^2}\|\psi(0)\|_{L_x^2}}{\langle u\rangle^{(n/2-2)\epsilon/(1-2\epsilon)}} \lesssim_{\epsilon} \frac{\|W(x)\|_{L_x^2}\|\psi(0)\|_{L_x^2}}{\langle s_1\rangle^{[(n/2-2)\epsilon/(1-2\epsilon)]-1}} \to 0 \tag{3.173}$$

since when $\epsilon \in (\frac{2}{n}, \frac{1}{2})$, $n \geq 5$,

$$[(n/2-2)\epsilon/(1-2\epsilon)] - 1 > 0. \tag{3.174}$$

Therefore $A_2(\infty)$ exists. And for s_0 large enough,

$$|A_2(s)| \leq C\frac{\|W(x)\|_{L_x^2}\|\psi(0)\|_{L_x^2}}{\langle s_0\rangle^{(n/2-2)\epsilon/(1-2\epsilon)-1}} \leq \frac{1}{4}. \tag{3.175}$$

According to (3.171) and (3.175), we have that $A(\infty)$ exists and for s_0 large enough,

$$|A(s)| \geq |A(s_0)| - |A_1(s)| - |A_2(s)| \geq \frac{9}{10} - |A_1(s)| - |A_2(s)| \geq \frac{1}{2} \tag{3.176}$$

where we use that for large $t_0 > 0$,

$$\left|(g(t_0)^{n/2}\psi_d(g(t_0)x), \psi_b(x))_{L_x^2}\right| \geq \frac{1}{10}. \tag{3.177}$$

We finish the proof. □

Remark 15. *Based on the proof of Theorem 3.4.1, if we have* $W(x,t)$ *instead of* $W(x)$ *in* (3.42)*, in order to get the conclusion of Theorem 3.4.1, we only need that* $W(x,t) \in L_t^1 L_x^2(\mathbb{R}^n \times \mathbb{R})$ *for* $n \geq 5, \epsilon \in (\frac{2}{n}, \frac{1}{2})$.

Lemma 3.4.2. *Let* W, V, H, g *be as in 3.42. For* $t \geq t_0$ *and* t_0 *large enough,*

$$|(\psi_b(x), \tilde{\phi}(x,t))_{L_x^2}| \leq 1/2. \tag{3.178}$$

Proof. Taking $t_0 = T^{-1}(s_0)$ for s_0 satisfying (3.176), we have

$$|(\psi_b(x), \tilde{\phi}(x,t))_{L_x^2}| = |(\psi_b(x), \phi(x,s))_{L_x^2}| = |A(s)| \leq 1/2 \tag{3.179}$$

with $s = T(t)$.

□

Corollary 3.4.1. *Let* N, V, H, g *be as in* (3.51)*. For* $t \geq t_0$ *and* t_0 *large enough,*

$$|(\psi_b(x), \tilde{\phi}(x,t))_{L_x^2}| \leq 1/2. \tag{3.180}$$

Proof. Based on the proof of Theorem 3.4.1, all we need is that $N(|\psi(t)|) \in L_t^1 L_x^2$. Also, see Remark 15. So based on (3.51), we have (4.4). □

Proof of Theorem 3.1.3. Based on Lemma 3.4.2, we have $c(t)$ satisfies (3.16). (3.48) follows from that this system has an asymptotic energy

$$\partial_t(\psi(t), (H_0 + W(x) + g(t)^{-2}V(\frac{x}{g(t)}))\psi(t))_{L_x^2} = (\psi(t), \partial_t[g(t)^{-2}V(\frac{x}{g(t)})]\psi(t))_{L_x^2} \in L_t^1 \tag{3.181}$$

with

$$\int_{t_0}^{T} dt \left| \partial_t(\psi(t), (H_0 + W(x) + g(t)^{-2}V(\frac{x}{g(t)}))\psi(t))_{L_x^2} \right| \leq C \int_{t_0}^{T} \frac{dt}{\langle t\rangle^{1+2\epsilon}} \| |DV(x)| + |V(x)| \|_{L_x^1} \|\psi(t_0)\|_{L_x^2}^2 \leq \frac{|\lambda_0|}{2} \|\psi_d(x)\|_{L_x^2}^2 \tag{3.182}$$

for all $T \geq t_0$ if we choose t_0 large enough and recall that $g(t) \sim \langle t\rangle^{\epsilon}$. Since

$$(\psi(t_0), (H_0 + W(x) + g(t_0)^{-2}V(\frac{x}{g(t_0)}))\psi(t_0))_{L_x^2} = \lambda_0 \|\psi_d(x)\|_{L_x^2}^2 + g(t_0)^{-2}\lambda \|\psi_b(x)\|_{L_x^2}^2 + (\psi_d(x), g(t_0)^{-2}V(\frac{x}{g(t_0)})\psi_d(x))_{L_x^2} + (e^{-iD\ln g(t_0)}\psi_b(x), W(x)e^{-iD\ln g(t_0)}\psi_b(x))_{L_x^2} \leq \frac{3}{4}\lambda_0 \|\psi_d(x)\|_{L_x^2}^2 \tag{3.183}$$

if we choose t_0 large enough, then for all $T \geq t_0$,

$$(\psi(T), (H_0 + W(x) + g(T)^{-2}V(\frac{x}{g(T)}))\psi(T))_{L^2_x} \geq \frac{1}{4}\lambda_0\|\psi_d(x)\|^2_{L^2_x}. \tag{3.184}$$

If t_0 is large enough, for all $T \geq t_0$,

$$(\psi(T), (H_0 + W(x))\psi(T))_{L^2_x} \geq \frac{1}{8}\lambda_0\|\psi_d(x)\|^2_{L^2_x} \tag{3.185}$$

since

$$(\psi(t), g(t)^{-2}V(\frac{x}{g(t)}))\psi(t))_{L^2_x} \to 0 \tag{3.186}$$

as $t \to \infty$. Hence for $t \geq t_0$,

$$\frac{1}{\|\psi_d(x)\|_{L^2_x}} \left|(\psi(t), \psi_d(x))_{L^2_x}\right| \geq \frac{1}{2\sqrt{2}}, \tag{3.187}$$

which implies (3.48) since

$$\left|(g(t)^{-n/2}\psi_b(\frac{x}{g(t)}), \psi_d(x))_{L^2_x}\right| \lesssim \frac{1}{g(t)^{n/2}} \tag{3.188}$$

and

$$\left|(F_c(\frac{|x - 2tP|}{t^{\alpha}} \leq 1)\psi(t), \psi_d(x))_{L^2_x}\right| \lesssim \frac{1}{t^{n/2(1-\alpha)}}\|\psi_d(x)\|_{L^1_x}\|\psi(t_0)\|_{L^2_x}. \tag{3.189}$$

We finish the proof.

□

3.5 Nonlinear Problem

Proof of Theorem 3.1.4. The first part of Theorem 3.1.4 follows by using Corollary 3.4.1. For the second part of Theorem 3.1.4, prove by contradiction. Assume that (3.66) is not true. Then for any $M \geq 1$, given $t_0 \geq 1$, there exists $t_M \geq t_0$ such that

$$\|\chi(|x| \leq M)\psi(t_M)\|_{L^2_x} \geq \frac{1}{M}. \tag{3.190}$$

We will get contradiction from the fact that this system has an asymptotic energy

$$\partial_t(\psi(t), (H_0 + g(t)^{-2}V(\frac{x}{g(t)}) + \mathcal{N}(|\psi(t)|))\psi(t))_{L^2_x} =$$
$$(\psi(t), \partial_t[g(t)^{-2}V(\frac{x}{g(t)})]\psi(t))_{L^2_x} + (\mathcal{N}'_F(|\psi(t)|), \partial_t[|\psi(t)|])_{L^2_x} \tag{3.191}$$

where

$$N_F(k) := \int_0^k dq q^2 N'(q). \tag{3.192}$$

Then

$$\begin{aligned}
&(\psi(T), (H_0 + g(T)^{-2}V(\frac{x}{g(T)}) + N(|\psi(T)|))\psi(T))_{L^2_x} = \\
&(\psi(t_0), (H_0 + g(t_0)^{-2}V(\frac{x}{g(t_0)}) + N(|\psi(t_0)|))\psi(t_0))_{L^2_x} + \int_{t_0}^T ds g_0(s) + (G(T) - G(t_0))
\end{aligned} \tag{3.193}$$

where

$$g_0(t) := (\psi(t), \partial_t[g(t)^{-2}V(\frac{x}{g(t)})]\psi(t))_{L^2_x} \in L^1_t[1, \infty), \tag{3.194}$$

$$G(t) := \int d^n x N_F(|\psi(t)|) \tag{3.195}$$

$$=(\psi(t), N(|\psi(t)|)\psi(t))_{L^2_x} - 2(\psi(t), N_{F,0}(|\psi(t)|)\psi(t))_{L^2_x} \tag{3.196}$$

with

$$N_{F,0}(k) = \int_0^k dq q N(q)/k^2 < 0. \tag{3.197}$$

Then (3.193) can be rewritten as

$$\begin{aligned}
&(\psi(T), (H_0 + g(T)^{-2}V(\frac{x}{g(T)}) + N(|\psi(T)|))\psi(T))_{L^2_x} = \\
&\quad (\psi(t_0), (H_0 + g(t_0)^{-2}V(\frac{x}{g(t_0)}) + 2N_{F,0}(|\psi(t_0)|))\psi(t_0))_{L^2_x} + \int_{t_0}^T ds g_0(s) + G(T),
\end{aligned} \tag{3.198}$$

which is equivalent to

$$\begin{aligned}
&(\psi(T), (H_0 + g(T)^{-2}V(\frac{x}{g(T)}) + 2N_{F,0}(|\psi(T)|))\psi(T))_{L^2_x} = \\
&\quad (\psi(t_0), (H_0 + g(t_0)^{-2}V(\frac{x}{g(t_0)}) + 2N_{F,0}(|\psi(t_0)|))\psi(t_0))_{L^2_x} + \int_{t_0}^T ds g_0(s).
\end{aligned} \tag{3.199}$$

On the one hand, due to (3.56) and that $g(s) \in L^1_s$, we have that there exists $\tilde{t}_0 \geq 1$ such that for all $T \geq t_0 \geq \tilde{t}_0$,

$$(\psi(t_0), (H_0 + g(t_0)^{-2}V(\frac{x}{g(t_0)}) + 2N_{F,0}(|\psi(t_0)|))\psi(t_0))_{L^2_x} + \int_{t_0}^T ds g_0(s) \leq \frac{E}{4}\|\psi_s(x)\|^2_{L^2_x}. \tag{3.200}$$

On the other hand, based on assumption (3.190) and Remark 13, there exists $\tilde{M} \quad 1$ such that for all $M \quad \tilde{M}$, $t_M \quad t_0$, t_0 sufficiently large,

$$2|(\psi(t_M), N_{F,0}(|\psi(t_M)|)\psi(t_M))_{L^2_x}| \quad (\frac{1}{M^{\frac{(n \quad 1)\beta}{2}}} + \frac{1}{M})C(\sup_t \|\psi(t)\|_{H^1_x}) \quad \frac{E}{100}\|\psi_s(x)\|^2_{L^2_x} \tag{3.201}$$

and

$$|(\psi(t_M), g(t_M) \quad ^2 V(x/g(t_M))\psi(t_M))| \quad \langle t_M \rangle \quad ^{2\epsilon}\|V(x)\|_{L^1_x}\|\psi(t_0)\|^2_{L^2_x} \quad \frac{E}{100}\|\psi_s(x)\|^2_{L^2_x} \tag{3.202}$$

which implies that for all $M \quad \tilde{M}$, $t_M \quad t_0$, t_0 sufficiently large,

$$(\psi(t_M), (H_0 + g(t_M) \quad ^2 V(\frac{x}{g(t_M)}) + 2N_{F,0}(|\psi(t_M)|))\psi(t_M))_{L^2_x} \quad \frac{E}{50}\|\psi_s(x)\|^2_{L^2_x} \tag{3.203}$$

since

$$(\psi(t_M), H_0\psi(t_M))_{L^2_x} \quad 0. \tag{3.204}$$

Based on (3.199), (3.200) and (3.203), contradiction and we finish the proof. □

A typical example in this case is

$$N(k) = \quad \lambda\frac{k}{1+k^2}. \tag{3.205}$$

Lemma 3.5.1. *When*

$$N(k) = \quad \lambda\frac{k}{1+k^2} \tag{3.206}$$

for some su ciently large $\lambda > 0$, the assumption of Theorem 3.1.4 is satis es and we get at least two bubbles as t goes to in nity.

Proof. By taking $\lambda > 0$ large enough, there is a soliton $\psi_s(x)$ to (3.53). By using standard iteration scheme, there is a global L^2 solution to (3.51) for any initial H^1_x data for any V satisfying (3.4) and (3.52). Now we show that the H^1_x norm of the solution is uniformly bounded in t. Compute

$$\begin{aligned}\partial_t[(\psi(t), (\quad \Delta_x + \langle t \rangle \quad ^{2\epsilon}V(x/\langle t\rangle^\epsilon) \quad \lambda\frac{|\psi(t)|}{1+|\psi(t)|^2})\psi(t))_{L^2_x}] &= \\ (\psi(t), \partial_t[\langle t\rangle \quad ^{2\epsilon}V(x/\langle t\rangle^\epsilon)]\psi(t))_{L^2_x} \quad &\lambda(\psi(t), \partial_t[\frac{|\psi(t)|}{1+|\psi(t)|^2}]\psi(t))_{L^2_x} \\ =: g_0(t) \quad &\lambda(\psi(t), \partial_t[\frac{|\psi(t)|}{1+|\psi(t)|^2}]\psi(t))_{L^2_x}.\end{aligned} \tag{3.207}$$

$$\int_{t_0}^{T} dt|g_0(t)| \lesssim \frac{1}{\langle t\rangle^{1+2\epsilon}}(\|V(x)\|_{L^1_x} + \|x\cdot\nabla_x V(x)\|_{L^1_x})\|\psi(t_0)\|^2_{L^2_x}$$
$$\lesssim (\|V(x)\|_{L^1_x} + \|x\cdot\nabla_x V(x)\|_{L^1_x})\|\psi(t_0)\|^2_{L^2_x}. \quad (3.208)$$

Let

$$N_F(k) = \int_0^k dq q^2 N'(q) = \int_0^k dq q^2 \partial_q[\frac{q}{1+q^2}]. \quad (3.209)$$

Then

$$|N_F(k)| \lesssim k^2 \quad (3.210)$$

and

$$(\psi(t), \partial_t[\frac{|\psi(t)|}{1+|\psi(t)|^2}]\psi(t))_{L^2_x} = \int d^n x \partial_t[N_F(|\psi(t)|)]. \quad (3.211)$$

Thus, by using (3.210),

$$\left|\int_{t_0}^{T} dt(\psi(t), \partial_t[\frac{|\psi(t)|}{1+|\psi(t)|^2}]\psi(t))_{L^2_x}\right| =$$
$$\left|\int d^n x N_F(|\psi(T)|) - \int d^n x N_F(|\psi(t_0)|)\right|$$
$$\lesssim \|\psi(T)\|_{L^2_x} + \|\psi(t_0)\|_{L^2_x} \lesssim \|\psi(t_0)\|^2_{L^2_x}. \quad (3.212)$$

Based on (3.208) and (3.212), we have

$$(\psi(t), (-\Delta_x)\psi(t))_{L^2_x} \lesssim \|\psi(t_0)\|^2_{L^2_x} + (\psi(t), \langle t\rangle^{-2\epsilon}|V(x/\langle t\rangle^{\epsilon})|\psi(t))_{L^2_x} +$$
$$\lambda(\psi(t), \frac{|\psi(t)|}{1+|\psi(t)|^2}\psi(t))_{L^2_x} + \lambda(\psi(t_0), \frac{|\psi(t_0)|}{1+|\psi(t_0)|^2}\psi(t_0))_{L^2_x} +$$
$$(\psi(t_0), \langle t_0\rangle^{-2\epsilon}|V(x/\langle t_0\rangle^{\epsilon})|\psi(t_0))_{L^2_x} + (\psi(t_0), (-\Delta_x)\psi(t_0))_{L^2_x}$$
$$\lesssim \|\psi(t_0)\|^2_{L^2_x}(1 + \|V(x)\|_{L^1_x}) + \|\psi(t_0)\|^2_{H^1_x} \quad (3.213)$$

where we also use

$$\frac{|\psi(t)|}{1+|\psi(t)|^2} \lesssim 1. \quad (3.214)$$

Hence,

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1_x} \lesssim \|\psi(t_0)\|_{L^2_x}\sqrt{(1 + \|V(x)\|_{L^1_x})} + \|\psi(t_0)\|_{H^1_x}. \quad (3.215)$$

In addition, taking $\beta = 1 > 0$,

$$|N_{F,0}(k)| = |\int_0^k dq q N(q)/k^2| = \lambda\int_0^k dq \frac{q^2}{1+q^2}/k^2 \lesssim \lambda k^{\beta} \quad (3.216)$$

and for any $f \in H^1_x$,

$$\begin{cases} \|\frac{|f(x)|}{1+|f(x)|^2}\|_{L^2_x} \leq \|f(x)\|_{L^2_x} \leq \|f(x)\|_{H^1_x} \\ \|\frac{|f(x)|}{1+|f(x)|^2}f(x)\|_{L^2_x} \leq \frac{1}{2}\|f(x)\|_{L^2_x} \leq \frac{1}{2}\|f(x)\|_{H^1_x} \end{cases} \tag{3.217}$$

and (3.56) follows from that $g(t) \to \infty$ as $t \to \infty$ and $N_{F,0}(|\psi(t)|)$ is localized in x since we have radial symmetry. Thus, the assumption of Theorem 3.1.4 is satisfied and we get that the solution $\psi(t)$ to system

$$\begin{cases} i\partial_t\psi = (H_0 + \langle t\rangle^{-2\epsilon}V(x/\langle t\rangle^{\epsilon}) - \lambda\frac{|\psi(t)|}{1+|\psi(t)|^2})\psi \\ \psi(x,t_0) = \psi_s(x) + e^{-iD\ln(g(t_0))}\psi_b(x) \end{cases}, \tag{3.218}$$

has at least two bubbles with different patterns. We finish the proof. □

3.6 Existence of Self-similar weakly localized part in some purely nonlinear Schrödinger equation

In this section, we consider following nonlinear Schrödinger equation

$$\begin{cases} i\partial_t\psi(t) = (-\Delta_x - \frac{|\psi(t)|^{4/3}}{1+\delta|\psi(t)|^2})\psi(t) \\ \psi(t_0) = e^{\mu iD\ln(\langle t_0\rangle)}\psi_s(x) \in H^1_x(\mathbb{R}^3) \end{cases}, \quad \delta > 0. \tag{3.219}$$

Via the transformation introduced in previous sections, set

$$\phi(s) := e^{\mu iD\ln(\langle T^{-1}(s)\rangle)}\psi(T^{-1}(s)) \tag{3.220}$$

and (3.219) is equivalent to

$$\begin{cases} i\partial_s\phi(s) = (-\Delta_x - \frac{|\phi(s)|^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}|\phi(s)|^2} + f(s)D)\phi(s) \\ \phi(s_0) = e^{\mu iD\ln(\langle T^{-1}(s_0)\rangle)}\psi(T^{-1}(s_0)) \end{cases} \tag{3.221}$$

with $s_0 = T(t_0)$. Unlike blow-up phenomenon in mass-critical nonlinear Schrödinger equations(the power of nonlinearity is $4/n$ in n space dimensions), (3.221) has a global solution. We prove that the weakly localized part of $\psi(t)$ has a non-trivial self-similar part which survives as $t \to \infty$ if we take $\phi(s_0) = \psi_s(x)$ with $\psi_s(x)$ satisfying

$$\begin{cases} (H_0 - \frac{3}{5}|\psi_s(x)|^{4/3})\psi_s(x) = \lambda\psi_s(x), \quad \text{for some } \lambda < 0 \\ \psi_s(x) \text{ is radial in } x \end{cases}. \tag{3.222}$$

Theorem 3.6.1. *Given $\delta > 0$, there exists $\mu > 0$ such that if we take t_0 sufficiently large, the weakly localized part of $\psi(t)$ has a non-trivial self-similar part in the following sense:*

$$\liminf_{s \to \infty}(\phi(s), (H_0 - |\phi(s)|^{4/3})\phi(s))_{L^2_x} < 0. \tag{3.223}$$

Proof. Assume $\mu \in (0, 1/2)$. Let $\partial_r := \partial_{|x|}$. Let

$$H(s) := -\Delta_x - \frac{|\phi(s)|^{4/3}}{1 + \delta \langle T^{-1}(s) \rangle^{-3\mu} |\phi(s)|^2} + f(s)D. \tag{3.224}$$

Let us observe the energy of system (3.221) near the origin at time s

$$E_\alpha(s) = (F_2(|x| - \langle s \rangle^\alpha)\phi(s), H(s)F_2(|x| - \langle s \rangle^\alpha)\psi(s))_{L^2_x} \tag{3.225}$$

for some $\alpha \in (6/7, 1 - \mu)$, $\mu \in (0, 1/7)$, which will be determined later. At time $s = s_0$, if we choose s_0 large enough,

$$E_\alpha(s_0) + (\phi(s_0), F_2 \frac{|\phi(s_0)|^{4/3}}{1 + \delta \langle T^{-1}(s_0) \rangle^{-3\mu} |\phi(s_0)|^2} F_2 \phi(s_0))_{L^2_x}$$
$$- \frac{3}{5}(\phi(s_0), F_2 |\phi(s_0)|^{4/3} F_2 \phi(s_0))_{L^2_x} = (\phi(s_0), (H_0 + f(s)D - \frac{3}{5}|\phi(s_0)|^{4/3})\phi(s_0))_{L^2_x} \le -\frac{\lambda}{2}. \tag{3.226}$$

Now we estimate $E_\alpha(s), s \ge s_0$. Let $H_0(s) := -\Delta_x + f(s)D$ and $H_0 := -\Delta_x$. Compute $\partial_s[E_\alpha(s)]$

$$\partial_s[E_\alpha(s)] = (\phi(s), \partial_s[F_2 H(s) F_2]\phi(s))_{L^2_x} + (\phi(s), i[H(s), F_2 H(s) F_2]\phi(s))_{L^2_x} \tag{3.227}$$

$$=: G_1(s) + (\phi(s), i[H(s), F_2 H(s) F_2]\phi(s))_{L^2_x} \tag{3.228}$$

$$= G_1(s) + (\phi(s), i[H(s), F_2] H(s) F_2 \phi(s))_{L^2_x} + (\phi(s), i F_2 H(s)[H(s), F_2]\phi(s))_{L^2_x} \tag{3.229}$$

$$= G_1(s) + (\phi(s), i[H_0(s), F_2] H(s) F_2 \phi(s))_{L^2_x} + (\phi(s), i F_2 H(s)[H_0(s), F_2]\phi(s))_{L^2_x} \tag{3.230}$$

$$=: G_1(s) + G_2(s) + G_3(s). \tag{3.231}$$

Estimate for $G_1(s)$: For $G_1(s)$, break it into four pieces

$$G_1(s) = (\phi(s), \partial_s[F_2]H(s)F_2\phi(s))_{L^2_x} + (\phi(s), F_2H(s)\partial_s[F_2]\phi(s))_{L^2_x}+$$
$$(\phi(s), F_2\partial_s[\frac{|\phi(s)|^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}|\phi(s)|^2}]F_2\phi(s))_{L^2_x} + \partial_s[f(s)](\phi(s), F_2DF_2\phi(s))_{L^2_x}$$
$$=: G_{11}(s) + G_{12}(s) + G_{13}(s) + G_{14}(s). \quad (3.232)$$

For $G_{11}(s)$, write it as a sum of a positive number and a term which is integrable in s

$$G_{11}(s) = (\phi(s), F_2\partial_s[F_2]H_0\phi(s))_{L^2_x} + (\phi(s), \partial_s[F_2][H_0, F_2]\phi(s))_{L^2_x}+$$
$$(\phi(s), \partial_s[F_2](f(s)D + \frac{|\phi(s)|^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}|\phi(s)|^2})F_2\phi(s))_{L^2_x}$$
$$= (\phi(s), \sqrt{F_2\partial_s[F_2]}H_0\sqrt{F_2\partial_s[F_2]}\phi(s))_{L^2_x} + \Big\{ (\phi(s), \sqrt{F_2\partial_s[F_2]}[H_0, \sqrt{F_2\partial_s[F_2]}]\phi(s))_{L^2_x}+$$
$$(\phi(s), \partial_s[F_2][H_0, F_2]\phi(s))_{L^2_x} + (\phi(s), \partial_s[F_2](f(s)D + \frac{|\phi(s)|^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}|\phi(s)|^2})F_2\phi(s))_{L^2_x}\Big\}$$
$$=: \tilde{G}^+_{11}(s) + G^r_{11}(s) \quad (3.233)$$

where

$$G^r_{11}(s) := \Big\{ (\phi(s), \sqrt{F_2\partial_s[F_2]}[H_0, \sqrt{F_2\partial_s[F_2]}]\phi(s))_{L^2_x}+$$
$$(\phi(s), \partial_s[F_2][H_0, F_2]\phi(s))_{L^2_x} + (\phi(s), \partial_s[F_2](f(s)D + \frac{|\phi(s)|^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}|\phi(s)|^2})F_2\phi(s))_{L^2_x}\Big\}.$$
$$(3.234)$$

For $\tilde{G}^+_{11}(s)$, since

$$\partial_s[F_2(|x| \geq \langle s\rangle^{\alpha})] = -\frac{\alpha s|x|}{\langle s\rangle^2}\partial_r[F_2] \geq 0, \quad (3.235)$$

rewrite $\tilde{G}_{11}^{+}(s)$ as

$$
\begin{aligned}
\tilde{G}_{11}^{+}(s) = & \sum_{j=1}^{3}(\phi(s), \sqrt{F_2|\partial_r[F_2]|}P_j F_2(\frac{|x|}{100\langle s\rangle^{\alpha}} \leq 1)\frac{\alpha s|x|}{\langle s\rangle^2}P_j\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L_x^2}+ \\
& \sum_{j=1}^{3}(\phi(s), \sqrt{F_2|\partial_r[F_2]|}P_j \bar{F}_2(\frac{|x|}{100\langle s\rangle^{\alpha}} > 1)\frac{\alpha s|x|}{\langle s\rangle^2}P_j\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L_x^2}+ \\
& \sum_{j=1}^{3}(\phi(s), \sqrt{F_2|\partial_r[F_2]|}[\sqrt{\frac{\alpha s|x|}{\langle s\rangle^2}}, P_j]\sqrt{\frac{\alpha s|x|}{\langle s\rangle^2}}P_j\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L_x^2}+ \\
& \sum_{j=1}^{3}(\phi(s), \sqrt{F_2|\partial_r[F_2]|}\sqrt{\frac{\alpha s|x|}{\langle s\rangle^2}}P_j[P_j, \sqrt{\frac{\alpha s|x|}{\langle s\rangle^2}}]\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L_x^2} \\
& =: G_{11}^{+}(s) + G_{11}^{r0}(s) + G_{11}^{r1}(s) + G_{11}^{r2}(s) \qquad (3.236)
\end{aligned}
$$

with

$$
|G_{11}^{rj}(s)| \lesssim \frac{1}{\langle s\rangle^{1+\alpha}}\|\phi(s)\|_{H_x^1}\|\phi(s)\|_{L_x^2} \lesssim \frac{\langle t\rangle^{\mu}}{\langle s\rangle^{1+\alpha}}\|\phi(s)\|_{L_x^2}\sup_k\|\psi(k)\|_{H_x^1}, \quad j = 1, 2, \qquad (3.237)
$$

$$
|G_{11}^{r0}(s)| \lesssim_N \frac{1}{\langle s\rangle^{N}}\left(\sup_k\|\psi(k)\|_{H_x^1}\right)^2 \qquad (3.238)
$$

and

$$
\begin{aligned}
G_{11}^{+}(s) \quad & 200\alpha \quad \frac{1}{\langle s\rangle^{1-\alpha}}\sum_{j=1}^{3}\|P_j\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|_{L_x^2}^2 \\
& \lesssim \frac{1}{\langle s\rangle^{1-\alpha}}\|P\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|_{L_x^2}. \qquad (3.239)
\end{aligned}
$$

In this proof, we assume $t = T^{-1}(s)$. For $G_{11}^{r}(s)$, we have

$$
\begin{aligned}
|G_{11}^{r}(s)| \lesssim & \frac{1}{\langle s\rangle^{1+\alpha}}\|\phi(s)\|_{L_x^2}\|\phi(s)\|_{H_x^1} + \frac{1}{\langle s\rangle^{2-\alpha}}\|\phi(s)\|_{L_x^2}\|\phi(s)\|_{H_x^1}+ \\
& \frac{\langle t\rangle^{2/3\mu}}{\langle s\rangle^{1+4/3\alpha}}\left(\sup_{k\in\mathbb{R}}\|\psi(k)\|_{H_x^1}\right)^{4/3}\|\phi(s)\|_{L_x^2}^2 \\
\lesssim & \left(\frac{\langle t\rangle^{\mu}}{\langle s\rangle^{1+\alpha}} + \frac{\langle t\rangle^{\mu}}{\langle s\rangle^{2-\alpha}} + \frac{\langle t\rangle^{\mu}}{\langle s\rangle^{1+4/3\alpha}}\right)\|\phi(s)\|_{L_x^2} \quad (\sup_k\|\psi(k)\|_{H_x^1} + 1)^{7/3} \\
& (\text{ use } \alpha > 1/2) \lesssim \frac{\langle t\rangle^{\mu}}{\langle s\rangle^{2-\alpha}}\|\phi(s)\|_{L_x^2} \quad (\sup_k\|\psi(k)\|_{H_x^1} + 1)^{7/3} \qquad (3.240)
\end{aligned}
$$

since

$$
[H_0, F_2] = 2\partial_r[F_2]\frac{x}{|x|} \cdot P, \quad [H_0, \sqrt{F_2\partial_s[F_2]}] = 2\partial_r[\sqrt{F_2\partial_s[F_2]}]\frac{x}{|x|} \cdot P \qquad (3.241)
$$

and since (3.59) implies

$$
\begin{aligned}
&\left|\partial_s[F_2]\frac{|\phi(s)|^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}|\phi(s)|^2}\right| \lesssim |\partial_s[F_2]|\langle t\rangle^{2\mu}|\psi(t,\langle t\rangle^{\mu}x)^{4/3}| \lesssim \\
&\quad \frac{\langle t\rangle^{2/3\mu}}{\langle s\rangle^{1+4/3\alpha}}\left(\sup_{|x|\geq 1, t\in\mathbb{R}}\| |x|\psi(t)\|_{L^\infty_x}\right)^{4/3} \lesssim \frac{\langle t\rangle^{2/3\mu}}{\langle s\rangle^{1+4/3\alpha}}\left(\sup_{k\in\mathbb{R}}\|\psi(k)\|_{H^1_x}\right)^{4/3}. \quad (3.242)
\end{aligned}
$$

Based on (3.240), (3.237) and (3.238), if we choose $\mu \in (0,1/7)$ small enough, since $\alpha > 6/7$, one has

$$
|G^r_{11}(s)+\sum_{j=0}^{2}G^{rj}_{11}(s)| \leq \frac{C}{\langle s\rangle^{2-11/10\alpha}}\|\phi(s)\|_{L^2_x}\times(\sup_k\|\psi(k)\|_{H^1_x}+1)^{7/3} \quad (3.243)
$$

Hence,

$$
G_{11}(s) \leq G^{+}_{11}(s)+\frac{C}{\langle s\rangle^{2-11/10\alpha}}\|\psi(t_0)\|_{L^2_x}\times(\sup_k\|\psi(k)\|_{H^1_x}+1)^{7/3}. \quad (3.244)
$$

Since

$$
G_{12}(s) = G_{11}(s)\ , \quad (3.245)
$$

one has

$$
G_{12}(s) \leq G^{+}_{11}(s)+\frac{C}{\langle s\rangle^{2-11/10\alpha}}\|\psi(t_0)\|_{L^2_x}\times(\sup_k\|\psi(k)\|_{H^1_x}+1)^{7/3}. \quad (3.246)
$$

For $G_{13}(s)$, take integration by parts in $s$in $\int_{s_0}^{S} dsG_{13}(s)$ and rewrite it as

$$
\begin{aligned}
\int_{s_0}^{S} ds(\phi(s), F_2\partial_s[&\frac{|\phi(s)|^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}|\phi(s)|^2}]F_2\phi(s))_{L^2_x} \\
&= -\int d^3x F_2^2\frac{|\phi(s)|^{10/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}|\phi(s)|^2}|_{s=s_0}^{s=S}+ \\
&+\int_{s_0}^{S} ds\int d^3x F_2^2(2|\phi(s)|\partial_s[|\phi(s)|])\frac{|\phi(s)|^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}|\phi(s)|^2}+ \\
&\int_{s_0}^{S} ds\int d^3x 2F_2\partial_s[F_2]\frac{|\phi(s)|^{10/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}|\phi(s)|^2} \\
&=: G_{13,s_0}(S)+G^{m}_{13,s_0}(S)+G^{r}_{13,s_0}(S). \quad (3.247)
\end{aligned}
$$

For $G^m_{13,s_0}(S)$, rewrite it as

$$\begin{aligned} G^m_{13,s_0}(S) &= \int_{s_0}^S ds \int d^3x F_2^2 \partial_s[|\phi(s)|]\partial_k[\int_0^k du 2u - \frac{u^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}u^2}]|_{k=|\phi(s)|} \\ &= \int_{s_0}^S ds \int d^3x \partial_s[F_2^2 \int_0^k du 2u - \frac{u^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}u^2}|_{k=|\phi(s)|}] \\ &\quad - \int_{s_0}^S ds \int d^3x \partial_s[F_2^2] \int_0^k du 2u - \frac{u^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}u^2}|_{k=|\phi(s)|} \\ &\quad - \int_{s_0}^S ds \int d^3x F_2^2 \int_0^k du 2u - \partial_s[\frac{u^{4/3}}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}u^2}]|_{k=|\phi(s)|} \\ &=: G^{mm}_{13,s_0}(S) - G^{mr1}_{13,s_0}(S) - G^{mr2}_{13,s_0}(S). \end{aligned} \tag{3.248}$$

For $G^{mr1}_{13,s_0}(S)$, one has

$$\begin{aligned} |G^{mr1}_{13,s_0}(S)| &\lesssim \int_{s_0}^S ds \int d^3x |\partial_s[F_2^2]||\phi(s)|^{1+1+4/3} \\ &\lesssim \int_{s_0}^S ds \int d^3x |\partial_s[F_2^2]| \, |\langle t\rangle^{3/2\mu}\psi(t,\langle t\rangle^\mu x)|^{4/3}|\phi(s)|^2 \\ &\lesssim \int_{s_0}^S ds \int d^3x |\partial_s[F_2^2]| \, \frac{\langle t\rangle^{2/3\mu}}{\langle x\rangle^{4/3}} \left(\sup_k \langle x\rangle|\psi(k)|\right)^{4/3} |\phi(s)|^2 \\ (\text{Set } \mu > 0 \text{ sufficiently small}) &\lesssim \|\phi(s_0)\|^2_{L^2_x} \int_{s_0}^S ds \frac{1}{\langle s\rangle} \frac{1}{\langle s\rangle^\alpha} \left(\sup_k \langle x\rangle|\psi(k)|\right)^{4/3} \\ &\lesssim \frac{1}{\langle s_0\rangle^\alpha}\|\phi(s_0)\|^2_{L^2_x} \left(\sup_k \langle x\rangle|\psi(k)|\right)^{4/3}. \end{aligned} \tag{3.249}$$

For $G^{mr2}_{13,s_0}(S)$, we have

$$G^{mr2}_{13,s_0}(S) \geq 0 \tag{3.250}$$

since

$$\partial_s[\frac{1}{1+\delta\langle T^{-1}(s)\rangle^{-3\mu}u^2}] = \frac{3\mu\delta\langle T^{-1}(s)\rangle^{-3\mu-1}u^2}{(1+\delta\langle T^{-1}(s)\rangle^{-3\mu}u^2)^2} \frac{dt}{ds} \geq 0 \tag{3.251}$$

due to

$$\frac{dt}{ds} = 1/(\frac{ds}{dt}) = \langle t\rangle^{2\mu} \geq 0, \tag{3.252}$$

see (3.10). Based on (3.249) and (3.250), one has

$$G^m_{13,s_0}(S) \leq G^{mm}_{13,s_0}(S) + \frac{C}{\langle s_0\rangle^\alpha}\|\phi(s_0)\|^2_{L^2_x}\left(\sup_k \langle x\rangle\psi(k)\right)^{4/3} \tag{3.253}$$

for some constant $C > 0$. For $G^r_{13,s_0}(S)$, according to the estimate for $G^{mr1}_{13,s_0}(S)$, similarly, one has

$$|G^r_{13,s_0}(S)| \lesssim \frac{1}{\langle s_0\rangle^\alpha}\|\phi(s_0)\|^2_{L^2_x}\left(\sup_k \langle x\rangle|\psi(k)|\right)^{4/3}. \tag{3.254}$$

Based on (3.253) and (3.254), we have

$$\int_{s_0}^{S} ds G_{13}(s) \leq G_{13,s_0}(S) + G^{mm}_{13,s_0}(S) + \frac{C}{\langle s_0\rangle^{\alpha}}\|\phi(s_0)\|^2_{L^2_x}\left(\sup_k \langle x\rangle |\psi(k)|\right)^{4/3} \tag{3.255}$$

for some constant $C > 0$. For $G_{14}(s)$, if we choose $\mu \in (0, 1/2)$ small enough, one has

$$\begin{aligned} |G_{14}(s)| \lesssim{}& \frac{1}{\langle s\rangle^2}\langle s\rangle^{\alpha}\|\phi(s)\|_{L^2_x}\|\phi(s)\|_{H^1_x} \lesssim \\ & \frac{\langle t\rangle^{\mu}}{\langle s\rangle^{2-\alpha}}\left(\sup_k \|\psi(k)\|_{H^1_x}\right)^2 \lesssim \frac{1}{\langle s\rangle^{2-11\alpha/10}}\left(\sup_k \|\psi(k)\|_{H^1_x}\right)^2 \end{aligned} \tag{3.256}$$

where we use

$$\|F_2 D\phi(s)\|_{L^2_x} \lesssim s^{\alpha}\|\phi(s)\|_{H^1_x} \tag{3.257}$$

and

$$\|\phi(s)\|_{H^1_x} \lesssim \langle t\rangle^{\mu}\sup_k \|\psi(k)\|_{H^1_x}. \tag{3.258}$$

Based on (3.256), (3.246), (3.244), (3.234) and (3.246), if we choose $\mu \in (0, 1/7)$ small enough, one has

$$\begin{aligned} \int_{s_0}^{S} G_1(s) \leq{}& G_{13,s_0}(S) + G^{mm}_{13,s_0}(S) + 2\int_{s_0}^{S} ds G^{+}_{11}(s) + \\ & \frac{C}{\langle s_0\rangle^{1-11/10\alpha}}\|\psi(t_0)\|_{L^2_x} \times (\sup_k \|\psi(k)\|_{H^1_x} + 1)^{7/3}. \end{aligned} \tag{3.259}$$

Estimate for $G_2(s)$: For $G_2(s)$, break it into three pieces

$$G_2(s) =(\phi(s), i[H_0, F_2]H_0F_2\phi(s))_{L^2_x} + f(s)(\phi(s), i[H_0, F_2]DF_2\phi(s))_{L^2_x} + \tag{3.260}$$

$$(\phi(s), i[f(s)D, F_2]H(s)F_2\phi(s))_{L^2_x} \tag{3.261}$$

$$=: G_{21}(s) + G_{22}(s) + G_{23}(s). \tag{3.262}$$

For $G_{23}(s)$, compute

$$i[D, F_2] = |x|\partial_r[F_2]. \tag{3.263}$$

$G_{23}(s)$ can be rewritten as

$$\begin{aligned}
G_{23}(s) =& f(s)(\phi(s), |x|\partial_r[F_2]H_0F_2\phi(s))_{L^2_x}+ && (3.264)\\
& f(s)(\phi(s), |x|\partial_r[F_2](H(s)-H_0)F_2\phi(s))_{L^2_x} && (3.265)\\
=& \sum_{j=1}^{3} f(s)(\phi(s), \sqrt{|\partial_r[F_2]|F_2}P_j|x|P_j\sqrt{|\partial_r[F_2]|F_2}\phi(s))_{L^2_x} && (3.266)\\
& \sum_{j=1}^{3} f(s)(\phi(s), \sqrt{|\partial_r[F_2]|F_2}[|x|, P_j]P_j\sqrt{|\partial_r[F_2]|F_2}\phi(s))_{L^2_x} && (3.267)\\
& f(s)(\phi(s), |x|\sqrt{|\partial_r[F_2]|}[H_0, \sqrt{F_2}]\sqrt{|\partial_r[F_2]|F_2}\phi(s))_{L^2_x} && (3.268)\\
& f(s)(\phi(s), |x|\sqrt{|\partial_r[F_2]|}[\sqrt{|\partial_r[F_2]|}, H_0]F_2\phi(s))_{L^2_x}+ && (3.269)\\
& f(s)(\phi(s), |x|\partial_r[F_2](H(s)-H_0)F_2\phi(s))_{L^2_x} && (3.270)\\
=:& G^{+}_{23}(s) + G^{r1}_{23}(s) + G^{r2}_{23}(s) + G^{r3}_{23}(s) + G^{r4}_{23}(s). && (3.271)
\end{aligned}$$

Due to $\partial_r[F_2]$, since $f(s) \sim \frac{1}{\langle s\rangle}$,

$$|G^{+}_{23}(s)| \lesssim \frac{1}{\langle s\rangle}\|P\sqrt{|\partial_r[F_2]|F_2}\phi(s)\|^2_{L^2_x}. \tag{3.272}$$

Since $f(s) \sim \frac{1}{\langle s\rangle}$, by choosing $\mu \in (0, 1/7)$ small enough, one has

$$\begin{aligned}
|G^{rj}_{23}(s)| \lesssim & \frac{1}{\langle s\rangle^{1+\alpha}}\|\phi(s)\|_{L^2_x}\|\phi(s)\|_{H^1_x}\\
& \lesssim \frac{\langle t\rangle^{\mu}}{\langle s\rangle^{1+\alpha}}\|\psi(t_0)\|_{L^2_x}\sup_k\|\psi(k)\|_{H^1_x}\\
& \lesssim \frac{1}{\langle s\rangle^{1+10/11\alpha}}\|\psi(t_0)\|_{L^2_x}\sup_k\|\psi(k)\|_{H^1_x}
\end{aligned} \tag{3.273}$$

for $j = 1, 2, 3, 4$. Based on (3.272) and (3.273), one has

$$|G_{23}(s)| \lesssim \frac{1}{\langle s\rangle}\|P\sqrt{|\partial_r[F_2]|F_2}\phi(s)\|^2_{L^2_x} + \frac{1}{\langle s\rangle^{1+10/11\alpha}}\|\psi(t_0)\|_{L^2_x}\sup_k\|\psi(k)\|_{H^1_x}. \tag{3.274}$$

Compute

$$i[H_0, F_2] = 2\partial_r[F_2(|x| - \langle s\rangle^{\alpha})]\hat{x}\cdot P. \tag{3.275}$$

$G_{22}(s)$ can be rewritten as

$$G_{22}(s) = 2f(s)(\phi(s), \partial_r[F_2]\hat{x}\quad PDF_2\phi(s))_{L^2_x} \tag{3.276}$$

$$= \quad 2f(s)(\phi(s), \sqrt{|\partial_r[F_2]|}\hat{x}\quad PD\sqrt{|\partial_r[F_2]|}F_2\phi(s))_{L^2_x} \tag{3.277}$$

$$2f(s)(\phi(s), \sqrt{|\partial_r[F_2]|}[\sqrt{|\partial_r[F_2]|}, \hat{x}\quad PD]F_2\phi(s))_{L^2_x} \tag{3.278}$$

$$= \quad 2f(s)(\phi(s), \sqrt{F_2|\partial_r[F_2]|}\hat{x}\quad PD\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L^2_x} \tag{3.279}$$

$$2f(s)(\phi(s), \sqrt{|\partial_r[F_2]|}[\hat{x}\quad PD, \sqrt{F_2}]\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L^2_x} \tag{3.280}$$

$$2f(s)(\phi(s), \sqrt{|\partial_r[F_2]|}[\sqrt{|\partial_r[F_2]|}, \hat{x}\quad PD]F_2\phi(s))_{L^2_x} \tag{3.281}$$

$$=: G^{+}_{22}(s) + G^{r1}_{22}(s) + G^{r2}_{22}(s). \tag{3.282}$$

For $G^{+}_{22}(s)$, since $f(s)\quad \frac{1}{\langle s\rangle}$, one has

$$\begin{aligned}
|G^{+}_{22}(s)|\quad & |2f(s)(\phi(s), \sqrt{F_2|\partial_r[F_2]|}P\quad \hat{x}x\quad P\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L^2_x}| + \\
& |2f(s)(\phi(s), \sqrt{F_2|\partial_r[F_2]|}[\hat{x}, P]x\quad P\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L^2_x}| + \\
& |2f(s)(\phi(s), \sqrt{F_2|\partial_r[F_2]|}\hat{x}\quad P(x\quad P\quad D)\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L^2_x}| \\
& \lesssim \|P\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|^2_{L^2_x} + \frac{1}{\langle s\rangle^{1+\alpha}}\|\phi(s)\|_{L^2_x}\|\phi(s)\|_{H^1_x} \\
& \lesssim \|P\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|^2_{L^2_x} + \frac{\langle t\rangle^{\mu}}{\langle s\rangle^{1+\alpha}}\|\psi(t_0)\|_{L^2_x}\sup_k \|\psi(k)\|_{H^1_x} \\
& \lesssim \|P\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|^2_{L^2_x} + \frac{1}{\langle s\rangle^{1+10/11\alpha}}\|\psi(t_0)\|_{L^2_x}\sup_k \|\psi(k)\|_{H^1_x}.
\end{aligned} \tag{3.283}$$

Similarly, one has

$$|G^{rj}_{22}(s)| \lesssim \frac{1}{\langle s\rangle^{1+10/11\alpha}}\|\psi(t_0)\|_{L^2_x}\sup_k \|\psi(k)\|_{H^1_x}, \quad \text{for } j = 1, 2. \tag{3.284}$$

Based on (3.283) and (3.284), one has

$$|G_{22}(s)| \lesssim \frac{1}{\langle s\rangle}\|P\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|^2_{L^2_x} + \frac{1}{\langle s\rangle^{1+10/11\alpha}}\|\psi(t_0)\|_{L^2_x}\sup_k \|\psi(k)\|_{H^1_x}. \tag{3.285}$$

$G_{21}(s)$ can be rewritten as

$$G_{21}(s) = -2(\phi(s), |\partial_r[F_2(|x| - \langle s\rangle^{\alpha})]|\hat{x} \cdot PH_0 F_2\phi(s))_{L^2_x} \tag{3.286}$$

$$= -2(\phi(s), \sqrt{F_2|\partial_r[F_2]|}|P|^3\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L^2_x} + \tag{3.287}$$

$$-2(\phi(s), \sqrt{F_2|\partial_r[F_2]|}(\hat{x} \cdot P - |P|)H_0\sqrt{F_2|\partial_r[F_2]|}\phi(s))_{L^2_x} + \tag{3.288}$$

$$-2(\phi(s), \sqrt{F_2|\partial_r[F_2]|}[\hat{x} \cdot PH_0, \sqrt{F_2|\partial_r[F_2]|}]\phi(s))_{L^2_x} + \tag{3.289}$$

$$-2(\phi(s), |\partial_r[F_2]|[\hat{x} \cdot PH_0, F_2]\phi(s))_{L^2_x} \tag{3.290}$$

$$=: G_{21}(s) + G_{21}^{in}(s) + G_{21}^{r1}(s) + G_{21}^{r2}(s). \tag{3.291}$$

For $G_{21}(s)$, according to (3.239), we have

$$G_{21}(s) + G_{11}^{+}(s) + |G_{22}(s)| + |G_{23}(s)|$$

$$\leq -\frac{2}{\langle s\rangle^{\epsilon}}\sum_{j=1}^{3}\|\chi(|P| \geq \frac{1}{\langle s\rangle^{\epsilon}})P_j\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|^2_{L^2_x} + \frac{C}{\langle s\rangle^{1-\alpha}}\sum_{j=1}^{3}\|P_j\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|^2_{L^2_x}$$

$$\text{(Choose } s_0 \text{ large enough)} \leq -\frac{1}{\langle s\rangle^{\epsilon}}\|\chi(|P| \geq \frac{1}{\langle s\rangle^{\epsilon}})|P|\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|^2_{L^2_x} +$$

$$\frac{C}{\langle s\rangle^{1-\alpha}}\|\chi(|P| < \frac{1}{\langle s\rangle^{\epsilon}})|P|\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|^2_{L^2_x}$$

$$\leq -\frac{1}{\langle s\rangle^{\epsilon}}\|\chi(|P| \geq \frac{1}{\langle s\rangle^{\epsilon}})|P|\sqrt{F_2|\partial_r[F_2]|}\phi(s)\|^2_{L^2_x} + \frac{C}{\langle s\rangle}\frac{1}{\langle s\rangle^{2\epsilon}}\|\phi(s)\|^2_{L^2_x} \tag{3.292}$$

for any $\epsilon \in (0, 1-\alpha)$. Take $\epsilon = (1-\alpha)/2$ and one has

$$G_{21}(s) + G_{11}^{+}(s) + |G_{22}(s)| + |G_{23}(s)| \leq \frac{C}{\langle s\rangle^{2-\alpha}}\|\phi(s)\|^2_{L^2_x}. \tag{3.293}$$

For $, G_{21}^{r1}(s), G_{21}^{r2}(s)$, by choosing $\mu \in (0, 1/7)$ small enough, we have

$$|G_{21}^{rj}(s)| \lesssim \frac{1}{\langle s\rangle^{2\alpha}}\|\phi(s)\|^2_{H^1_x} \lesssim \frac{\langle t\rangle^{2\mu}}{\langle s\rangle^{2\alpha}}\left(\sup_{k\in\mathbb{R}}\|\psi(k)\|_{H^1_x}\right)^2$$

$$\lesssim \frac{1}{\langle s\rangle^{21/11\alpha}}\left(\sup_{k\in\mathbb{R}}\|\psi(k)\|_{H^1_x}\right)^2, \quad j = 1, 2, \tag{3.294}$$

by using

$$\|\langle P\rangle^{-1}[\hat{x} \cdot PH_0, \sqrt{F_2|\partial_r[F_2]|}]\|_{H^1_x \to L^2_x} \lesssim \frac{1}{\langle s\rangle^{3/2\alpha}} \tag{3.295}$$

and

$$\|\langle P\rangle^{-1}[\hat{x} \cdot PH_0, F_2]\|_{H^1_x \to L^2_x} \lesssim \frac{1}{\langle s\rangle^{\alpha}}. \tag{3.296}$$

For $G_{21}^{in}(s)$, we need following lemmas.

Lemma 3.6.1. *For $u \in [0,\langle s\rangle], \alpha > 6/7$, $f \in L^1_x \cap L^2_x$ with $f(x)$ radial in x,*

$$\left| \int d^3x \frac{e^{i|x||q|}}{|x|} \sqrt{F_2|\partial_r[F_2]|} e^{\mu i D \ln(\langle t\rangle)} e^{-iuH_0} \chi(|x| < \langle s\rangle^{\alpha+\mu}) f(x) \right| \lesssim_N$$

$$\|f(x)\|_{L^1_x \cap L^2_x} \frac{1}{|q|^3 \langle |q| \rangle^N} \frac{1}{\langle t\rangle^{\mu/2}} \frac{1}{\langle s\rangle^{7/2\alpha}}. \quad (3.297)$$

Lemma 3.6.2. *For $\alpha > 6/7$, choosing $\mu \in (0,1/7)$ small enough,*

$$|G^{in}_{21}(s)| \lesssim \frac{1}{\langle s\rangle^{7/6\alpha}} \left(\sup_k \|\psi(k)\|_{H^2_x} + 1 \right) \left(\sup_k \|\psi(k)\|_{H^1_x} + 1 \right)^{7/3}. \quad (3.298)$$

We defer the proof of Lemma 3.6.1 and Lemma 3.6.2 to the end of this section. Based on Lemma 3.6.2, (3.293) and (3.294), one has

$$\Re(G_2(s)) + G^{+}_{11}(s) \leq \frac{C}{\langle s\rangle^{7/6\alpha}} \left(\sup_k \|\psi(k)\|_{H^2_x} + 1 \right) \left(\sup_k \|\psi(k)\|_{H^1_x} + 1 \right)^{7/3} + \frac{C}{\langle s\rangle^{2-\alpha}} \|\phi(s)\|^2_{L^2_x}. \quad (3.299)$$

Since $G_3(s) = (G_2(s))^*$, we have

$$\Re(G_3(s)) + G^{+}_{11}(s) \leq \frac{C}{\langle s\rangle^{7/6\alpha}} \left(\sup_k \|\psi(k)\|_{H^2_x} + 1 \right) \left(\sup_k \|\psi(k)\|_{H^1_x} + 1 \right)^{7/3} + \frac{C}{\langle s\rangle^{2-\alpha}} \|\phi(s)\|^2_{L^2_x}. \quad (3.300)$$

Based on (3.259), (3.299) and (3.300), one has

$$\begin{aligned} |E_\alpha(S) - E_\alpha(s_0) - G_{13,s_0}(S) - G^{mm}_{13,s_0}(S)| &\leq \frac{C}{\langle s_0\rangle^{7/6\alpha-1}} \left(\sup_k \|\psi(k)\|_{H^2_x} + 1 \right) \left(\sup_k \|\psi(k)\|_{H^1_x} + 1 \right)^{7/3} + \\ &\frac{C}{\langle s_0\rangle^{1-\alpha}} \|\psi(t_0)\|^2_{L^2_x} + \frac{C}{\langle s_0\rangle^{1-11/10\alpha}} \|\psi(t_0)\|_{L^2_x} \times (\sup_k \|\psi(k)\|_{H^1_x} + 1)^{7/3} \\ &(\text{Choose } s_0 \text{ small enough}) \leq \frac{|\lambda|}{10} \end{aligned} \quad (3.301)$$

which implies that for all $S \geq s_0$,

$$\begin{aligned} &(F_2(|x| - \langle S\rangle^{\mu})\phi(S), (H_0 - \frac{3}{5}|\phi(S)|^4 + f(S)D)F_2(|x| - \langle S\rangle^{\alpha})\phi(S))_{L^2_x} < \\ &(F_2(|x| - \langle S\rangle^{\mu})\phi(S), (H_0 - \bar{N}(|\phi(S)|, S) + f(S)D)F_2(|x| - \langle S\rangle^{\alpha})\phi(S))_{L^2_x} < \frac{\lambda}{4} \end{aligned} \quad (3.302)$$

with

$$\bar{N}(k,S) := \int_0^k du2u \quad \frac{u^{4/3}}{1+\delta\langle T \quad 1(S)\rangle^{3\mu}u^2}. \tag{3.303}$$

(3.302) also implies

$$(F_2(|x| \quad \langle S\rangle^{\mu})\phi(S),(H_0 \quad \frac{3}{5}|\phi(S)|^4)F_2(|x| \quad \langle S\rangle^{\alpha})\phi(S))_{L^2_x} < \frac{\lambda}{8} \tag{3.304}$$

if we take $S \quad S_0$ for some sufficiently large $S_0 \quad s_0$. We finish the proof. □

Proof of Lemma 3.6.1. Set

$$H(x,u) := e^{\quad iuH_0}\chi(|x| < \langle s\rangle^{\alpha+\mu})f(x). \tag{3.305}$$

Use Fourier representation to rewrite $H(x)$

$$\begin{aligned}
H(x,u) &= c_3 \int d^3\xi e^{ix\ \xi} e^{\quad iu\xi^2} F_x[\chi(|x| < \langle s\rangle^{\alpha+\mu})f(x)](\xi) \\
&= (2\pi)^2 c_3^2 \int_0^1 (d|\xi|)\frac{e^{i|x||\xi|}}{i|x|} e^{\quad iu|\xi|^2} \int_0^1 |y|(d|y|)(\frac{e^{i|y||\xi|}}{i} \quad \frac{e^{\quad i|y||\xi|}}{i})\chi(|y| < \langle s\rangle^{\alpha+\mu})f(y) \\
&\quad (2\pi)^2 c_3^2 \int_0^1 (d|\xi|)\frac{e^{\quad i|x||\xi|}}{i|x|} e^{\quad iu|\xi|^2} \int_0^1 |y|(d|y|)\frac{e^{i|y||\xi|}}{i}\chi(|y| < \langle s\rangle^{\alpha+\mu})f(y) + \\
&\quad (2\pi)^2 c_3^2 \int_0^1 (d|\xi|)\frac{e^{\quad i|x||\xi|}}{i|x|} e^{\quad iu|\xi|^2} \int_0^1 |y|(d|y|)\frac{e^{\quad i|y||\xi|}}{i}\chi(|y| < \langle s\rangle^{\alpha+\mu})f(y) \\
&=: H_{out}(x,u) \quad H_{in,1}(x,u) + H_{in,2}(x,u).
\end{aligned} \tag{3.306}$$

For $H_{out}(x,u)$, let

$$\begin{aligned}
O_{out}(|q| + \langle t\rangle^{\mu}|\xi|) &:= \int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D \ln(\langle t\rangle)}\frac{e^{i|x||\xi|}}{|x|} \\
&= 4\pi \int_0^1 (d|x|)e^{i|x|(|q|+\langle t\rangle^{\mu}|\xi|)} \quad \langle t\rangle^{1/2\mu}\sqrt{F_2(|x| \quad \langle s\rangle^{\alpha})|\partial_r[F_2(|x| \quad \langle s\rangle^{\alpha})]|}.
\end{aligned} \tag{3.307}$$

When $|q| \quad 1$, via taking integration by parts in $|x|$ variable for 4 many times by setting

$$e^{i|x|(|q|+\langle t\rangle^{\mu}|\xi|)} = \frac{1}{i(|q|+\langle t\rangle^{\mu}|\xi|)}\partial_{|x|}[e^{i|x|(|q|+\langle t\rangle^{\mu}|\xi|)}], \tag{3.308}$$

one has

$$\begin{aligned}
|O_{out}(|q| + \langle t\rangle^{\mu}|\xi)| &\lesssim \frac{1}{(|q|+\langle t\rangle^{\mu}|\xi|)^4}\int_0^1 d|x|\frac{\langle t\rangle^{1/2\mu}}{\langle s\rangle^{4\alpha}}\frac{1}{\langle s\rangle^{\alpha/2}}\chi(|x| \quad 100\langle s\rangle^{\alpha}) \\
&\lesssim \frac{1}{(|q|+\langle t\rangle^{\mu}|\xi|)^4} \quad \frac{\langle t\rangle^{1/2\mu}}{\langle s\rangle^{7/2\alpha}}.
\end{aligned} \tag{3.309}$$

Since

$$\int_0^\infty |y|(d|y|)|f(y)| \lesssim \|f(x)\|_{L^1_x \cap L^2_x}, \tag{3.310}$$

plug (3.309) into $H_{out}(x,u)$ and one gets

$$\begin{aligned} &\left|\int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D\ln(\langle t\rangle)}H_{out}(x,u)\right| \lesssim \\ &\quad \|f(x)\|_{L^1_x\cap L^2_x}\int_0^\infty d|\xi|\frac{1}{(|q|+\langle t\rangle^\mu|\xi|)^4}\frac{\langle t\rangle^{1/2\mu}}{\langle s\rangle^{7/2\alpha}} \lesssim \|f(x)\|_{L^1_x\cap L^2_x}\frac{1}{|q|^3}\frac{1}{\langle s\rangle^{7/2\alpha}}. \end{aligned} \tag{3.311}$$

When $|q| > 1$, taking integration by parts in $|x|$ for N times in the same way, one gets

$$\begin{aligned} &\left|\int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D\ln(\langle t\rangle)}H_{out}(x,u)\right| \lesssim_N \\ &\|f(x)\|_{L^1_x\cap L^2_x}\int_0^\infty d|\xi|\frac{1}{(|q|+\langle t\rangle^\mu|\xi|)^N}\frac{\langle t\rangle^{\mu/2}}{\langle s\rangle^{(N-1/2)\alpha}} \lesssim_N \|f(x)\|_{L^1_x\cap L^2_x}\frac{1}{|q|^{N-1}}\frac{1}{\langle s\rangle^{(N-1/2)\alpha}}. \end{aligned} \tag{3.312}$$

For $H_{in,1}(x,u)$, when $|y| \leq \langle s\rangle^{\alpha+\mu}/100$ and $|x| \geq \langle s\rangle^{\alpha+\mu}$, take integration by parts by setting

$$e^{-i(|x||\xi|+u|\xi|^2-|y||\xi|)} = \frac{1}{-i(|x|+2u|\xi|-|y|)}\partial_{|\xi|}[e^{-i(|x||\xi|+u|\xi|^2-|y||\xi|)}] \tag{3.313}$$

and one has

$$\begin{aligned} &H_{in,1}(x,u) = C_{in,1}\int_0^\infty |y|(d|y|)\frac{1}{(|x|-|y|)|x|}\chi(|y|<\langle s\rangle^{\alpha+\mu})f(y)+ \\ &C_{in,1}\int_0^\infty d|\xi|\int_0^\infty |y|(d|y|)\frac{e^{-i(|x||\xi|+u|\xi|^2-|y||\xi|)}}{|x|}\partial_{|\xi|}\left[\frac{1}{(|x|+2u|\xi|-|y|)}\right]\chi(|y|<\langle s\rangle^{\alpha+\mu})f(y) \\ &\qquad =: H^1_{in,1}(x,u)+H^2_{in,1}(x,u). \end{aligned} \tag{3.314}$$

For $H^1_{in,1}(x,u)$, let

$$\begin{aligned} O_{in,1}(|q|) &:= \int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D\ln(\langle t\rangle)}\frac{1}{|x|(|x|-|y|)} \\ &= \langle t\rangle^{3/2\mu}\int_0^\infty (d|x|)\frac{e^{i|x||q|}}{\langle t\rangle^\mu}\frac{1}{\langle t\rangle^\mu|x|-|y|}\sqrt{F_2(|x|-\langle s\rangle^\alpha)|\partial_r[F_2(|x|-\langle s\rangle^\alpha)]|}. \end{aligned} \tag{3.315}$$

When $|q| \geq 1$, via taking integration by parts in $|x|$ variable for 3 many times by setting

$$e^{i|x||q|} = \frac{1}{i|q|}\partial_{|x|}[e^{i|x||q|}], \tag{3.316}$$

one has

$$|O_{in,1}(|q|)| \lesssim \frac{1}{|q|^3}\frac{1}{\langle t\rangle^{\mu/2}}\int_0^{\infty} d|x|\frac{1}{\langle s\rangle^{4\alpha}}\frac{1}{\langle s\rangle^{\alpha/2}}\chi(|x| \leq 100\langle s\rangle^{\alpha}) \lesssim \frac{1}{|q|^3}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{7/2\alpha}}. \tag{3.317}$$

Plug (3.323) into $H^1_{in,1}(x,u)$ and one gets

$$\Big|\int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D\ln(\langle t\rangle)}H^1_{in1}(x,u)\Big| \lesssim \|f(x)\|_{L^1_x\cap L^2_x}\frac{1}{|q|^3}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{7/2\alpha}}. \tag{3.318}$$

When $|q|>1$, taking integration by parts in $|x|$ for N times in the same way, one gets

$$\Big|\int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D\ln(\langle t\rangle)}H^1_{in,1}(x,u)\Big| \lesssim_N \|f(x)\|_{L^1_x\cap L^2_x}\frac{1}{|q|^N}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{(1/2+N)\alpha}}. \tag{3.319}$$

Thus,

$$\Big|\int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D\ln(\langle t\rangle)}H^1_{in,1}(x,u)\Big| \lesssim_N \|f(x)\|_{L^1_x\cap L^2_x}\frac{1}{|q|^3\langle|q|\rangle^N}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{7/2\alpha}}. \tag{3.320}$$

For $H^2_{in,1}(x,u)$, since

$$\Big|\partial_{|\xi|}\Big[\frac{1}{|x|+2u|\xi| - |y|}\Big]\Big| \lesssim \min\Big(\frac{1}{\langle s\rangle^{2(\alpha+\mu)-1}}, \frac{1}{|\xi|}\frac{1}{\langle s\rangle^{\alpha+\mu}}\Big) \tag{3.321}$$

for $|x| \geq \langle s\rangle^{\mu+\alpha}, |y| \leq \langle s\rangle^{\mu+\alpha}/100, u \in [0,\langle s\rangle]$, keep taking integration by parts in the same way and we get that for $N \geq 3$,

$$\Big|\int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D\ln(\langle t\rangle)}H^2_{in,1}(x,u)\Big| \lesssim_N \|f(x)\|_{L^1_x\cap L^2_x}\frac{1}{|q|^3\langle|q|\rangle^N}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{(1/2+N)\alpha}}. \tag{3.322}$$

Based on (3.320) and (3.322), one has

$$\Big|\int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D\ln(\langle t\rangle)}H_{in,1}(x,u)\Big| \lesssim_N \|f(x)\|_{L^1_x\cap L^2_x}\frac{1}{|q|^3\langle|q|\rangle^N}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{7/2\alpha}}. \tag{3.323}$$

Similarly, one has

$$\Big|\int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D\ln(\langle t\rangle)}H_{in,2}(x,u)\Big| \lesssim_N \|f(x)\|_{L^1_x\cap L^2_x}\frac{1}{|q|^3\langle|q|\rangle^N}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{7/2\alpha}}. \tag{3.324}$$

According to (3.311), (3.323) and (3.324), one has

$$\left|\int d^3x \frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}e^{\mu i D\ln(\langle t\rangle)}H(x,u)\right| \lesssim_N \|f(x)\|_{L^1_x\cap L^2_x}\frac{1}{|q|^3\langle|q|\rangle^N}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{7/2\alpha}} \tag{3.325}$$

and finish the proof. □

Proof of Lemma 3.6.2. Decompose $\phi(s)$ first. By using Duhamel's formula, $\phi(s)$ can be rewritten as

$$\begin{aligned}\phi(s) &= e^{\mu i D\ln(\langle t\rangle)}\psi(t)|_{t=T^{-1}(s)}\\ &= e^{\mu i D\ln(\langle t\rangle)}e^{-i(t-t_0)H_0}\psi(t_0)|_{t=T^{-1}(s)}+(-i)\int_{t_0}^t du e^{\mu i D\ln(\langle t\rangle)}e^{-i(t-u)H_0}\left(\frac{|\psi(u)|^{4/3}}{1+\delta|\psi(u)|^2}\right)\psi(u).\end{aligned} \tag{3.326}$$

Break $\phi(s)$ into several pieces

$$\phi(s) = e^{\mu i D\ln(\langle t\rangle)}e^{-i(t-t_0)H_0}\chi(|x|\geq\langle s\rangle^{\alpha+\mu}/100)\psi(t_0)|_{t=T^{-1}(s)}+ \tag{3.327}$$

$$(-i)\int_{t_0}^t du e^{\mu i D\ln(\langle t\rangle)}e^{-i(t-u)H_0}\chi(|x|\geq\langle s\rangle^{\alpha+\mu}/100)\left(\frac{|\psi(u)|^{4/3}}{1+\delta|\psi(u)|^2}\right)\psi(u)+ \tag{3.328}$$

$$e^{\mu i D\ln(\langle t\rangle)}e^{-i(t-t_0)H_0}\chi(|x|<\langle s\rangle^{\alpha+\mu}/100)\psi(t_0)|_{t=T^{-1}(s)}+ \tag{3.329}$$

$$(-i)\int_{t_0}^t du e^{\mu i D\ln(\langle t\rangle)}e^{-i(t-u)H_0}\chi(|x|<\langle s\rangle^{\alpha+\mu}/100)\left(\frac{|\psi(u)|^{4/3}}{1+\delta|\psi(u)|^2}\right)\psi(u) \tag{3.330}$$

$$=:\phi_1(s)+\phi_2(s)+\phi_3(s)+\phi_4(s). \tag{3.331}$$

Recall that $\langle s\rangle \sim \langle t\rangle^{1-2\mu}$, see (3.10). Then for $\phi_1(s), \phi_2(s)$,

$$\|\phi_1(s)\|_{H^1_x} \lesssim \frac{\langle t\rangle^\mu}{\langle s\rangle^{4/3(\alpha+\mu)}}\|\langle x\rangle^{4/3}\psi(t_0)\|_{H^1_x} \tag{3.332}$$

and

$$\|\phi_2(s)\|_{H^1_x} \leq \int_{t_0}^t du\langle t\rangle^\mu\|\psi(u)\|_{H^1_x}\times\frac{1}{\langle s\rangle^{4/3(\alpha+\mu)}}\times\|\langle x\rangle^{4/3}\frac{|\psi(u)|^{4/3}}{1+\delta|\psi(u)|^2}\|_{L^\infty_x} \tag{3.333}$$

$$(\text{use } (3.59)) \lesssim \frac{1}{\langle s\rangle^{4/3(\alpha+\mu)-\frac{1}{1-2\mu}}}\times\left(\sup_k\|\psi(k)\|_{H^1_x}\right)^{7/3}. \tag{3.334}$$

Here we get factor $\langle t\rangle^\mu$ since

$$\|\phi(s)\|_{H^1_x} \lesssim \langle t\rangle^\mu\|\psi(t)\|_{H^1_x}. \tag{3.335}$$

Thus, based on (3.332) and (3.334), choosing $\mu \in (0,1/7)$ small enough, one has that for $j=1,2$,

$$
\begin{aligned}
&|(\phi(s), \sqrt{F_2|\partial_r[F_2]|}(\hat{x}\cdot P - |P|)H_0\sqrt{F_2|\partial_r[F_2]|}\phi_j(s))_{L^2_x}| \lesssim \|\phi(s)\|_{H^2_x}\frac{1}{\langle s\rangle^{\alpha}}\|\phi_j(s)\|_{H^1_x}\\
&\quad\lesssim \left(\sup_k\|\psi(k)\|_{H^2_x}\right)\frac{\langle t\rangle^{2\mu}}{\langle s\rangle^{\alpha}}\frac{1}{\langle s\rangle^{4/3(\alpha+\mu)-\frac{1}{1-2\mu}}}\left(\sup_k\|\psi(k)\|_{H^1_x}\right)^{7/3}\\
&\text{(choose } \mu \text{ small enough and use } \alpha>6/7) \lesssim \frac{1}{\langle s\rangle^{7/6\alpha}}\left(\sup_k\|\psi(k)\|_{H^2_x}\right)\left(\sup_k\|\psi(k)\|_{H^1_x}\right)^{7/3}
\end{aligned}
\tag{3.336}
$$

In addition, based on (3.332) and (3.334), we also have

$$
\|\phi_3(s)+\phi_4(s)\|_{L^2_x} \lesssim \|\phi(s)\|_{L^2_x}. \tag{3.337}
$$

For $\phi_3(s)$ and $\phi_4(s)$, since $\psi(s), \phi_j(s) (j=1,2,3,4)$ are radial in x, let

$$
g_j(|q|,s) := c_3\int d^3x e^{-iq\cdot x}\sqrt{F_2|\partial_r[F_2]|}\phi_j(s) \tag{3.338}
$$

$$
=4\pi c_3\int_0^\infty |x|d|x|\frac{\sin(|x||q|)}{|q|}\sqrt{F_2|\partial_r[F_2]|}\phi_j(s), \quad j=0,3,4 \tag{3.339}
$$

with $\phi_0(s)=\phi(s)$. Use Fourier transform to rewrite $(\hat{x}\cdot P-|P|)H_0\sqrt{F_2|\partial_r[F_2]|}\phi_j(s), j=0,3,4$,

$$
(\hat{x}\cdot P-|P|)H_0\sqrt{F_2|\partial_r[F_2]|}\phi_j(x,s) \tag{3.340}
$$

$$
=c_3\int d^3q e^{ix\cdot q}(\hat{x}\cdot q-|q|)|q|^2 g_j(|q|,s) \tag{3.341}
$$

$$
=c_3\int_0^\infty |q|^4 d|q|\int_{S^2} d\sigma(q)(\hat{x}\cdot q-|q|)e^{ix\cdot q}g_j(|q|,s) \tag{3.342}
$$

$$
=2\pi c_3\int_0^\infty |q|^4 d|q|\int_{-1}^{1} dt(t-1)|q|e^{i|x||q|t}g_j(|q|,s) \tag{3.343}
$$

$$
=2\pi c_3\int_0^\infty |q|^4 d|q| g_j(|q|,s)\frac{(t-1)e^{i|x||q|t}}{i|x|}\Big|_{t=-1}^{t=1} \tag{3.344}
$$

$$
-2\pi c_3\int_0^\infty |q|^3 d|q| g_j(|q|,s)\frac{2\sin(|x||q|)}{i|x|^2}, \tag{3.345}
$$

for some $c_3>0$, that is,

$$
(\hat{x}\cdot P-|P|)H_0\sqrt{F_2|\partial_r[F_2]|}\phi_j(s) \tag{3.346}
$$

$$
=4\pi c_3\int_0^\infty |q|^4 d|q| g_j(|q|,s)\frac{e^{-i|x||q|}}{i|x|}-2\pi c_3\int_0^\infty |q|^3 d|q| g_j(|q|,s)\frac{2\sin(|x||q|)}{i|x|^2} \tag{3.347}
$$

$$
=4\pi c_3\int_0^\infty |q|^4 d|q| g_j(|q|,s)\frac{e^{-i|x||q|}}{i|x|}+\frac{2\pi i}{|x|}H_0\sqrt{F_2|\partial_r[F_2]|}\phi_j(s). \tag{3.348}
$$

Hence,

$$(\sqrt{F_2|\partial_r[F_2]|}\phi(s),(\hat{x}\cdot P-|P|)H_0\sqrt{F_2|\partial_r[F_2]|}(\phi_3(s)+\phi_4(s)))_{L^2_x} \tag{3.349}$$

$$=4\pi c_3(-i)\int_0^\infty |q|^4(d|q|)g_0(|q|,s)\left(\int d^3x\frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}(\phi_3(s)+\phi_4(s))\right)+ \tag{3.350}$$

$$(\sqrt{F_2|\partial_r[F_2]|}\phi(s),\frac{2\pi i}{|x|}H_0\sqrt{F_2|\partial_r[F_2]|}(\phi_3(s)+\phi_4(s)))_{L^2_x} \tag{3.351}$$

$$=:A_1(s)+A_2(s). \tag{3.352}$$

For $A_1(s)$, by using Lemma 3.6.1, one has

$$\left|\int d^3x\frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}\phi_3(s)\right|\lesssim_N\|\psi(t_0)\|_{L^1_x\cap L^2_x}\frac{1}{|q|^3\langle|q|\rangle^N}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{7/2\alpha}} \tag{3.353}$$

and

$$\left|\int d^3x\frac{e^{i|x||q|}}{|x|}\sqrt{F_2|\partial_r[F_2]|}\phi_3(s)\right|\lesssim_N\int_{t_0}^t du\||\psi(u)|^{7/3}\|_{L^1_x\cap L^2_x}\frac{1}{|q|^3\langle|q|\rangle^N}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{7/2\alpha}}$$

$$(\psi(t)\in H^1_x\Longrightarrow\psi(t)\in L^6_x)\lesssim_N\left(1+\sup_k\|\psi(k)\|_{H^1_x}\right)^{7/3}\frac{1}{|q|^3\langle|q|\rangle^N}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{7/2\alpha-\frac{1}{1-2\mu}}} \tag{3.354}$$

which imply

$$\begin{aligned}|A_1(s)|\lesssim&\left(1+\sup_k\|\psi(k)\|_{H^1_x}\right)^{7/3}\int_0^\infty|q|^4(d|q|)|g_0(|q|,s)|\frac{1}{|q|^3\langle|q|\rangle^3}\frac{1}{\langle t\rangle^{\mu/2}}\frac{1}{\langle s\rangle^{7/2\alpha-\frac{1}{1-2\mu}}}\\ \lesssim&\left(1+\sup_k\|\psi(k)\|_{H^1_x}\right)^{10/3}\frac{1}{\langle s\rangle^{4\alpha-\frac{1}{1-2\mu}}}\lesssim\left(1+\sup_k\|\psi(k)\|_{H^1_x}\right)^{10/3}\frac{1}{\langle s\rangle^{4\alpha-\frac{1}{1-2\mu}}}\\ (\alpha>1/2)\lesssim&\left(1+\sup_k\|\psi(k)\|_{H^1_x}\right)^{10/3}\frac{1}{\langle s\rangle^{2\alpha-\frac{2\mu}{1-2\mu}}}\lesssim\left(1+\sup_k\|\psi(k)\|_{H^1_x}\right)^{10/3}\frac{1}{\langle s\rangle^{21/11\alpha}}.\end{aligned} \tag{3.355}$$

Here we also use

$$\int_0^\infty q^2dq|g(q,s)|^2\lesssim\|\phi(s)\|_{L^2_x}\frac{1}{\langle s\rangle^{\alpha/2}}. \tag{3.356}$$

For $A_2(s)$, according to (3.337), by choosing $\mu\in(0,1/2)$ small enough, one has

$$|A_2(s)|\lesssim\frac{\langle t\rangle^{2\mu}}{\langle s\rangle^{2\alpha}}(\sup_k\|\psi(k)\|_{H^1_x})^2\lesssim\frac{1}{\langle s\rangle^{21/11\alpha}}(\sup_k\|\psi(k)\|_{H^1_x})^2. \tag{3.357}$$

Based on (3.336), (3.355) and (3.357), we get (3.298) and finish the proof. □

Chapter 4

Long-time behavior of Klein-Gordon type equations

4.1 Introduction

The analysis of dispersive and hyperbolic wave equations and systems is of critical importance in the study of evolution equations in Physics and Geometry.

It is well known that the asymptotic solutions of such equations, if they exist, show a dizzying zoo of possible solutions. Besides the "free wave", which corresponds to a solution of the equation without interaction terms, a multitude or other solutions may appear.

Such solutions are localized around possibly moving center of mass. They include nonlinear bound states, solitons, breathers, hedgehogs, vortices etc... The analysis of such equations is usually done on a case by case basis, due to this complexity. [85]

A natural question then follows: is it true that the solutions of dispersive/hyperbolic equations converge in appropriate norm (L^2 or H^1) to a free wave and independently moving localized parts?

In fact this is precisely the statement of Asymptotic Completeness in the case of N-body Scattering [75, 33, 18, 17, 78, 77, 40, 41]. In the N-body case the possible outgoing clusters are clearly identified, as bound states of subsystems.

Another situation in which recently a major progress was achieved involves nonlinear, completely integrable equations in one dimension [10, 44]

But when the interaction term includes time dependent potentials (even localized in space) and more general nonlinear terms, we do not have an a-priory knowledge of the possible asymptotic states.

In the case of time independent interaction terms, one can use spectral theory.

The scattering states evolve from the continuous spectrum, and the localized part is formed by the point spectrum. Once the interaction is time dependent/nonlinear, that decomposition is not possible. In recent works on Schrödinger type problems, it was possible to obtain general results for time dependent and nonlinear interaction terms.[58, 57, 91]. In this work we initiate the study of hyperbolic equations based on the above new approach. We will focus here on the Klein-Gordon(KG for short) equation in arbitrary dimension, and with general interaction terms, including semi-linear interactions. These are the first results on large data multichannel scattering for nonlinear KG equations, which are not integrable.

For earlier works on time dependent potentials we mention: charge transfer Hamiltonians [104, 33, 101, 63, 72, 69, 16], decaying in time potentials and small potentials [39, 71], time periodic potentials [103, 39] and random (in time) potentials [7]. See also[6, 5]. For potentials with asymptotic energy distribution more could be done [79].

A recent progress for more general localized potentials without smallness assumptions is obtained in [87]. Some tools from this work will be used in this paper.

Turning to the nonlinear case, Tao [94, 96, 97] has shown that the asymptotic decomposition holds for NLS with inter-critical nonlinearities, in 3 or higher dimensions, in the case of radial initial data.

In particular, in a very high dimension, and with an interaction that is a sum of smooth compactly supported potential and repulsive nonlinearity, Tao was able to show that the localized part is smooth and localized.

In other cases, Tao showed the localized part is only weakly localized and smooth.

Tao's work uses direct estimates of the incoming and outgoing parts of the solution to control the nonlinear term, via Duhamel representation. In a certain sense, it is in the spirit of Enss work. See also [70].

For the critical power wave equations and wave map problems there has been a great progress in understanding the large time behavior (with large data). See e.g. [20, 19, 47, 13] and cited references.

In the case of nonlinear KG equation, there are no results on multichannel scattering with large data.

There are many major works on the stability of coherent structures e.g. [59, 27, 51, 84] and large literature on NLS, KdV and more. For the case of small data and long range type interactions, see [25, 26, 55, 56, 54, 43, 53, 52].

In contrast, the new approach of Liu-Soffer [58, 57] is based on proving a-priory estimates on the full dynamics, which hold in a suitably defined domains of the extended phase-space. That is, one proves propagation estimates in domains exterior to the support of the interaction. Similar propagation observables were used in many other works, mostly linear problems, with time independent potentials. See e.g. [42, 33, 18] and cited references. In this way it was possible to show the asymptotic decomposition for general localized interactions, including time and space dependent ones. Radial initial data is assumed, to ensure the localization of the nonlinear part of the interaction terms.

More detailed information can be obtained on the localized part of the solution. Besides being smooth, its expanding part (if it exists) can grow at most like $|x| \lesssim \sqrt{t}$, and furthermore, is concentrated in a thin set of the extended phase-space.

The free part of the solution concentrates on the ***propagation set*** where $x = vt, v = 2P$, and P being the dual to the space variable, the momentum, is given by the operator $-i\nabla_x$.

The weakly localized part is found to be localized in the regions where

$$|x|/t^{\alpha} \sim 1 \quad \textbf{and} \quad |P| \sim t^{-\alpha}, \quad \forall\, 0 \leq \alpha \leq 1/2.$$

It therefore shows that the spreading part follows a self similar pattern. See[58, 57].

The method of proof is based on three main parts: first, construct the Free Channel Wave Operator. Then prove localization of the remainder localized part, and use it to prove the smoothness of the localized part. Finally, by using further propagation estimates which are adapted to localized solutions, prove the concentration on thin sets of the phase-space corresponding to self similar solutions. In this work we will mainly do the first part, and some of the second part of [58, 57].

It should be emphasized that the spreading localized solutions, if they exist, were shown to have a non-small nuclei part around the origin. This is true for both the results of Tao and Liu-Soffer.

Therefore, these are not pure self-similar solutions, as they appear in the special cases of critical nonlinearities. See e.g. [92, 20]. We expect a similar behavior of the weakly localized part of the solutions of KG equations.

We will follow here this point of view. It was generalized in [91] to include non-radial data and interactions, and with localized interactions to arbitrary dimension, in the case of the Schrödinger problem. This generalization is based on refined localization of the channel wave operators. By localizing around the phase-space support of the free wave, we get a sharper decomposition of the localized and scattering parts. Therefore, we can avoid the need for localization of the interactions in some cases. The idea of sharp localization was used in other ways in the study of long-range scattering theory, e.g. in [74, 76].

Here we follow these constructions also for the KG case, mainly by viewing the dynamics of the KG equation as generated by a couple of Schrödinger type equations, with dispersion relation given by $\sqrt{p^2+m^2}$ and group velocity $v = p/\sqrt{p^2+m^2}$.

The extension to this case of the previous methods proceeds along similar ideas, at least when the interaction terms are localized in space.

When the interaction terms are not localized around a given point, but only satisfy L^p decay conditions (with $1 \le p \le 2$), the situation is more complicated.

The problem comes from the fact that L^p decay estimates for the KG equation require control of derivatives of the initial data. This is due to the poor dispersion for hyperbolic equations for frequencies near infinity.

We deal with this problem by introducing an extra cutoff of high frequencies into the construction of the channel wave operators, as we explain next.

The key tool from scattering theory that is used to study multichannel scattering is the notion of ***channel wave operator***, which we denote by

$$\Omega_a \equiv s\text{-}\lim_{t\to\infty} U_a(-t)U(t)u(0). \tag{4.1}$$

Here the limit is taken in the strong sense in a suitable Hilbert space.

$U(t)u(0)$ is the solution of the KG equation with initial data $u(0), \dot{u}(0)$ and dynamics (linear or nonlinear) $U(t)=U(t,0)$ generated by a hamiltonian $H(t)$ and the equation:

$$u_{tt}+Hu=0.$$

Typically, for KG equations, we choose

$$H=-\Delta+1+N_0.$$

N_0 may depend on u,t,x.

The asymptotic dynamics U_a is generated by a Hamiltonian H_a for a given channel denoted by a. In this work we will only construct the free channel, where $H_a=-\Delta+1$

A crucial observation is that one can modify the definition of the Channel wave operators to

$$\Omega_a \equiv s-\lim_{t\to\infty} U_a(-t)J_aU(t)\psi(0). \tag{4.2}$$

See [75].

4.1.1 Problem and Results

We consider a general class of Klein-Gordon type equations of the form:

$$\begin{cases}(\Box+1)u=-N(u,x,t)=-V(x,t)u-N_0(u)u \\ \vec{u}(0):=(u(x,0),\dot{u}(x,0))=(u_0(x),\dot{u}_0(x))\in S\end{cases},\quad (x,t)\in\mathbb{R}^n\times\mathbb{R}, \tag{4.3}$$

for a Hilbert space $S=H^1\oplus L^2$, with space dimension $n\geq 1$.

Here $\Box:=\partial_t^2-\Delta_x$ and $N_0(u)$ is real.

The term $N(u,x,t)$ includes a combination of the following cases:

1. (Theorem 4.1.3, $n\geq 1$) **Localized time-dependent potential** $N(u,x,t)=V(x,t)u$, such that either $V(x,t)\in L^\infty_t L^2_{\delta,x}(\mathbb{R}^n\times\mathbb{R})$ or $V(x,t)\in L^\infty_t L^\infty_{\delta+n/2,x}(\mathbb{R}^n\times\mathbb{R})$ for some $\delta>1$. In addition,

$$\|V(x,t)u(t)\|_{L^\infty_t L^2_x}\lesssim \sup_{t\in\mathbb{R}}\|u(t)\|_{H^1_x}. \tag{4.4}$$

Furthermore $\vec{u}(0)\in S$ and such that it leads to a global uniformly bounded solution:

$$C(\|\vec{u}(0)\|_S):=\sup_{t\in\mathbb{R}}\|\vec{u}(t)\|_S\lesssim_{\vec{u}(0)} 1. \tag{4.5}$$

2. (Theorem 4.1.1 and Theorem 4.1.2) When $n \geq 3$, $N(u, x, t) = V(x, t)u$, such that $V(x, t) \in L^1_t L^2_x(\mathbb{R}^n \times \mathbb{R})$. And $\vec{u}(0) \in H^1 \times L^2_x$ leads to a global solution with

$$C(\|\vec{u}(0)\|_S) := \sup_{t \in \mathbb{R}} \|\vec{u}(t)\|_S \lesssim_{\vec{u}(0)} 1. \tag{4.6}$$

Here $\langle \cdot \rangle : \mathbb{R}^n \to \mathbb{R}, x \mapsto \mathbb{R}$,

$$L^p_{\delta,x} := \{f(x) : \langle x \rangle^\delta f(x) \in L^p_x\}, \quad \text{for } 1 \leq p \leq \infty \tag{4.7}$$

and denote weighted Agmon-Sobolev space

$$H^\sigma_\delta := \{f(x) : \|\langle \nabla_x \rangle^\sigma f(x)\|_{L^2_{\delta,x}} < \infty\}. \tag{4.8}$$

Let S_δ denote the complex Hilbert space $H^1_\delta \times H^0_\delta$ of vector functions $\vec{v} = (v_1, v_2)$ with the norm

$$\|\vec{v}\|_{S_\delta} = \|v_1\|_{H^1_\delta} + \|v_2\|_{H^0_\delta} < \infty. \tag{4.9}$$

We use H^σ, S to denote, respectively, H^σ_0 and S_0 for simplicity.

Throughout the paper, we always assume that there is a global S solution $(u(t), \dot{u}(t))$ to (4.3).

Remark 16. *When there exists a nonlinearity $N(u) = N_0(u)u$, we regard $N_0(u)$ as a linear time-dependent perturbation.*

Remark 17. *Typical example for* (1) *is*

$$N(u, x, t) = V(x, t)u + a(x)u^2 + b(x)u^3, \quad \text{in 1 dimension} \tag{4.10}$$

provided that we have global existence in $H^1 \times H^0$(for notation, see (4.8)*). See Theorem 4.4.2 for more details.*

Typical examples for (2) *are*

$$N(u, x, t) = V(x, t)u + \lambda u^3 + \lambda' u^4, \quad \text{in 3 or higher dimensions.} \tag{4.11}$$

More generally, one can control

$$(\Box + 1 + V(x, t))u = f(u)u, \quad 1 + V(x, t) \geq v_0 > 0, \tag{4.12}$$

with

$$\sup_t \|f(u)\|_{L^2_x} < \infty\,. \tag{4.13}$$

Here $V(x,t)$ *can be of general charge transfer type, that is,* $V(x,t) = \sum_{j=1}^{N} V_j(x - g_j(t)v_j, t)$*. See Theorem 4.4.3 for more details.*

We write $X \lesssim Y, Y \gtrsim X$ to indicate $X \leq CY$ for some constant $C > 0$ and $X \lesssim_a Y$ to indicate $X \leq CY$ for some $C = C(a) > 0$. Let $\bar{F}_c(\lambda), F_j(\lambda)(j = 1,2)$ denote smooth characteristic functions of the interval $[1, +\infty)$ and

$$F_c(\lambda \leq a) := 1 - \bar{F}_c(\lambda/a), \quad F_j(\lambda > a) := F_j(\lambda/a), \quad j = 1,2, \tag{4.14}$$

$$\bar{F}_c(\lambda \geq a) := \bar{F}_c(\lambda/a), \quad \bar{F}_j(\lambda \leq a) := 1 - F_j(\lambda/a). \tag{4.15}$$

Let $U(t,0)$ denote the dynamical group of KG equation (4.3), that is, for $\vec{u} \in S$,

$$U(t,0)\vec{u} = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} =: (u(t), \dot{u}(t))^T \tag{4.16}$$

and $U_0(t,0)$, the dynamical group of the free KG equation. Let $P := -i\nabla_x$. Throughout the paper,

$$F_l \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} := \begin{pmatrix} F_l u(t) \\ F_l \dot{u}(t) \end{pmatrix}, \quad l := 1,2,c. \tag{4.17}$$

For $\vec{v} = (v_1(x), v_2(x))^T$, let

$$\langle P\rangle_1^{\delta}\vec{v} := \begin{pmatrix} \langle P\rangle^{\delta} v_1(x) \\ v_2(x) \end{pmatrix} \tag{4.18}$$

and

$$\langle P\rangle_2^{\delta}\vec{v} := \begin{pmatrix} v_1(x) \\ \langle P\rangle^{\delta} v_2(x) \end{pmatrix}. \tag{4.19}$$

Let

$$\Omega_+(t) := \begin{pmatrix} u_+(t) \\ \dot{u}_+(t) \end{pmatrix} := U_0(0,t)U(t,0)\vec{u}(0), \tag{4.20}$$

$$\begin{aligned} \Omega^{\beta,+}_{\alpha}(t) &:= F_c\left(\frac{|x|}{t^{\alpha}} \leq 1\right)\bar{F}_1(|P| \leq t^{\beta})\Omega(t) \\ &:= F_c\left(\frac{|x|}{t^{\alpha}} \leq 1\right)\bar{F}_1(|P| \leq t^{\beta})U_0(0,t)U(t,0) \end{aligned} \tag{4.21}$$

in 3 or higher dimensions. Here are our main results:

Theorem 4.1.1. *Let $(u(t), \dot{u}(t))$ be a global solution to equation (4.3) in S and $H_0 := -\Delta_x$. For $n \geq 3$, given $\alpha, \beta > 0$ satisfying*

$$\frac{n(1-\alpha-\beta)-3\beta}{2} > 1, \tag{4.22}$$

if $V(x,t) \in L_t^1 L_x^2(\mathbb{R}^n \times \mathbb{R})$, the channel wave operator

$$\Omega_\alpha^{\beta,} := s\text{-}\lim_{t\to\infty} \Omega_\alpha^{\beta,}(t) \tag{4.23}$$

exists from $S \to L_x^2 \times L_x^2$, that is,

$$u_+ := s\text{-}\lim_{t\to\infty} F_c\left(\frac{|x|}{t^\alpha} \leq 1\right)\bar{F}_1(|P| \leq t^\beta)u(t) \tag{4.24}$$

exists in L_x^2 and

$$\dot{u}_+ := s\text{-}\lim_{t\to\infty} F_c\left(\frac{|x|}{t^\alpha} \leq 1\right)\bar{F}_1(|P| \leq t^\beta)\dot{u}(t) \tag{4.25}$$

exists in L_x^2.

Based on Theorem 4.1.1, we have that as $t \to \infty$, for all $a \in [0,1)$,

$$\left\|F_c\left(\frac{|x|}{t^\alpha} \leq 1\right)u(t) - \tilde{u}_+\right\|_{H^a} \to 0, \tag{4.26}$$

and

$$\left\|\langle P\rangle^{-1}F_c\left(\frac{|x|}{t^\alpha} \leq 1\right)\dot{u}(t) - \dot{\tilde{u}}_+\right\|_{H^a} \to 0 \tag{4.27}$$

for some $\tilde{u}_+, \dot{\tilde{u}}_+ \in H^a$.

Theorem 4.1.2. *Let $(u(t), \dot{u}(t)), V$ be as in Theorem 4.1.1. Then given $\alpha > 0$ satisfying*

$$\frac{n(1-\alpha-\beta)-3\beta}{2} > 1, \tag{4.28}$$

for some $\beta > 0$, for all $a \in [0,1)$,

$$\left\|F_c\left(\frac{|x|}{t^\alpha} \leq 1\right)u(t) - \tilde{u}_+\right\|_{H^a} \to 0, \tag{4.29}$$

and

$$\left\|\langle P\rangle^{-1}F_c\left(\frac{|x|}{t^\alpha} \leq 1\right)\dot{u}(t) - \dot{\tilde{u}}_+\right\|_{H^a} \to 0 \tag{4.30}$$

for some $\tilde{u}_+, \dot{\tilde{u}}_+ \in H^a$.

Proof. Based on Theorem 4.1.1, for (4.29), it follows from that

$$F_c(\frac{|x|}{t^\alpha} \quad 1)u \ (t) = F_c(\frac{|x|}{t^\alpha} \quad 1)\bar{F}_1(|P| \quad t^\beta)u \ (t) + F_c(\frac{|x|}{t^\alpha} \quad 1)F_1(|P| > t^\beta)u \ (t) \tag{4.31}$$

with

$$\|F_1(|P| > t^\beta)u \ (t)\|_{H^a} = \|\frac{1}{\langle P\rangle}\langle P\rangle F_1(|P| > t^\beta)u \ (t)\|_{H^a}$$
$$\lesssim \frac{1}{t^{\beta(1 \ a)}}(\|u(t)\|_{H^1} + \|\dot{u}(t)\|_{L^2_x}) \lesssim \frac{1}{t^{\beta(1 \ a)}}C(\|\vec{u}(0)\|_S) \to 0 \tag{4.32}$$

as $t \to \infty$. For (4.30), it follows from that

$$F_c(\frac{|x|}{t^\alpha} \quad 1)\dot{u} \ (t) = F_c(\frac{|x|}{t^\alpha} \quad 1)\bar{F}_1(|P| \quad t^\beta)\dot{u} \ (t) + F_c(\frac{|x|}{t^\alpha} \quad 1)F_1(|P| > t^\beta)\dot{u} \ (t) \tag{4.33}$$

with

$$\|\langle P\rangle^{\ 1}F_1(|P| > t^\beta)\dot{u} \ (t)\|_{H^a} = \|\frac{1}{\langle P\rangle}F_1(|P| > t^\beta)\langle P\rangle^{1 \ a}\frac{1}{\langle P\rangle^{1 \ a}}\dot{u} \ (t)\|_{H^a}$$
$$\lesssim \frac{1}{t^{\beta(1 \ a)}}(\|u(t)\|_{H^1} + \|\dot{u}(t)\|_{L^2_x}) \lesssim \frac{1}{t^{\beta(1 \ a)}}C(\|\vec{u}(0)\|_S) \to 0 \tag{4.34}$$

as $t \to \infty$. $\tilde{u}_+, \dot{\tilde{u}}_+ \in H^a$ since $u \ (t) \in H^1_x$ and $\dot{u} \ (t) \in L^2_x$. □

Theorem 4.1.3. *Let* $(u(t), \dot{u}(t))$ *be a global solution to equation* (4.3) *in S. If* $u(t), V(x,t)$ *are as in 1(Localized time-dependent potential), then for* $n \quad 1$, $b \in [0, 1/2), \alpha \in (b, \min(1 \quad b, 1 \quad \frac{2 \ \delta}{n}))$,

1. *the free channel wave operator*

$$\Omega_{\alpha,b} := s\text{-}\lim_{t \to \infty} F_c(\frac{|x|}{t^\alpha} \quad 1)F_1(t^b|P| > 1)\Omega(t) \tag{4.35}$$

exists from $S \to S$. *In particular,*

$$u_+ := s\text{-}\lim_{t \to \infty} F_c(\frac{|x|}{t^\alpha} \quad 1)F_1(t^b|P| > 1)u \ (t) \tag{4.36}$$

exists in H^1_x *and*

$$\dot{u}_+ := s\text{-}\lim_{t \to \infty} F_c(\frac{|x|}{t^\alpha} \quad 1)F_1(t^b|P| > 1)\dot{u} \ (t) \tag{4.37}$$

exists in L^2_x.

2. *furthermore, given $e > 0$, if $\delta > 2$ and if α, b also satisfy*

$$e > 1 - b > \alpha > b \geq 0, \tag{4.38}$$

there exist $u^1_{+,e,\alpha,b}, u^2_{+,e,\alpha,b} \in H^1_x, \dot u^1_{+,e,\alpha,b}, \dot u^2_{+,e,\alpha,b} \in L^2_x$ such that we have the following asymptotic decomposition

$$\lim_{t\to\infty} \|u(t) - \cos(t\sqrt{H_0+1})u^1_{+,e,\alpha,b} - \frac{\sin(t\sqrt{H_0+1})}{\sqrt{H_0+1}}\dot u^1_{+,e,\alpha,b} - u_{w,e,\alpha,b}(t)\|_{H^1_x} = 0 \tag{4.39}$$

and

$$\lim_{t\to\infty} \|\dot u(t) + \sin(t\sqrt{H_0+1})\sqrt{H_0+1}u^2_{+,e,\alpha,b} - \cos(t\sqrt{H_0+1})\sqrt{H_0+1}\dot u^2_{+,e,\alpha,b} - v_{w,e,\alpha,b}(t)\|_{L^2_x} = 0 \tag{4.40}$$

where $u_{w,e,\alpha,b}, v_{w,e,\alpha,b}$ are the weakly localized parts of the solution, with the following property: It is weakly localized in the region $|x| \leq t^e$, in the following sense

$$(\langle P\rangle u_{w,e,\alpha,b}(t), |x|\langle P\rangle u_{w,e,\alpha,b}(t))_{L^2_x} \lesssim_b t^e C(\|\vec u(0)\|_S)^2, \tag{4.41}$$

and

$$(v_{w,e,\alpha,b}(t), |x| v_{w,e,\alpha,b}(t))_{L^2_x} \lesssim_b t^e C(\|\vec u(0)\|_S)^2. \tag{4.42}$$

Remark 18. $e = 1/2 + 0$ *if we choose α, b wisely.*

Remark 19. *When $n = 1$, (4.4) is satisfied when either $V(x,t) \in L^1_t L^2_{\delta,x}(\mathbb{R}\times\mathbb{R})$ or $V(x,t) \in L^1_t L^1_{\delta+1/2,x}(\mathbb{R}^n\times\mathbb{R})$ for some $\delta > 1$ and*

$$\sup_t \|u(t)\|_{H^1} \lesssim \|\vec u(0)\|_S. \tag{4.43}$$

4.2 Preliminaries

4.2.1 Free KG equations

Let $\vec u_0(t) := (u_0(t), \dot u_0(t))$ be the solution to a free KG equation

$$\begin{cases} (\Box + 1)u_0(t) = 0 \\ \vec u_0(0) = \vec u(0) = (u(x,0), \dot u(x,0)) \in S \end{cases}, \quad (x,t) \in \mathbb{R}^n\times\mathbb{R}. \tag{4.44}$$

Let $H_0 := -\Delta_x$. $u_0(t)$ and $\dot{u}_0(t)$ have following representation

$$u_0(t) = \cos(t\sqrt{H_0+1})u(0) + \frac{\sin(t\sqrt{H_0+1})}{\sqrt{H_0+1}}\dot{u}(0) \tag{4.45}$$

and

$$\dot{u}_0(t) = -\sin(t\sqrt{H_0+1})\sqrt{H_0+1}u(0) + \cos(t\sqrt{H_0+1})\dot{u}(0). \tag{4.46}$$

Let

$$A_0 := \begin{pmatrix} 0 & -1 \\ H_0+1 & 0 \end{pmatrix}. \tag{4.47}$$

(4.44) is equivalent to

$$\partial_t[\vec{u}_0(t)] = -A_0\vec{u}_0(t). \tag{4.48}$$

So $\vec{u}_0(t)$ has another representation

$$\vec{u}_0(t) = e^{-tA_0}\vec{u}(0), \tag{4.49}$$

that is,

$$U_0(t,0) = e^{-tA_0}. \tag{4.50}$$

In the following context, we need following standard dispersive decay estimate for the KG propagator, see for instance Hörmander [36](Corollary 7.2.4) for a proof.

Lemma 4.2.1. *We have uniformly for all $t \in \mathbb{R}$ that*

$$\|e^{-it\langle P\rangle}f\|_{L^\infty_x(\mathbb{R}^n)} \lesssim \frac{1}{\langle t\rangle^{n/2}}\|\langle P\rangle^{\frac{n+3}{2}}f\|_{L^1_x(\mathbb{R}^n)}. \tag{4.51}$$

4.2.2 Perturbed KG and its Duhamel's formulas

Let $(u(t), \dot{u}(t))$ be the solution to a perturbed KG equation

$$\begin{cases} (\Box+1)u(t) = -V(x,t)u(t) \\ \vec{u}(0) = (u(x,0), \dot{u}(x,0)) \in S \end{cases}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{4.52}$$

Let

$$\mathcal{V}(x,t) := \begin{pmatrix} 0 & 0 \\ V(x,t) & 0 \end{pmatrix}. \tag{4.53}$$

(4.52) implies

$$\partial_t[\vec{u}(t)] = \quad (A_0 + V(x,t))\vec{u}(t), \tag{4.54}$$

that is,

$$\partial_t[U(t,0)\vec{u}(0)] = \quad (A_0 + V(x,t))U(t,0)\vec{u}(0). \tag{4.55}$$

Based on (4.50) and (4.55), we derive a Duhamel's formula for $\vec{u}(t)$

$$\vec{u}(t) = U_0(t,0)\vec{u}(0) \quad \int_0^t dsU_0(t,s)V(x,s)\vec{u}(s) \tag{4.56}$$

where we use

$$\frac{d}{ds}[U_0(t,s)U(s,0)] = U_0(t,0)\frac{d}{ds}[U_0(0,s)U(s,0)] = \quad U_0(t,s)V(x,s)U(s,0) \tag{4.57}$$

and

$$\vec{u}(t) = U_0(t,0)\vec{u}(0) + \int_0^t ds\frac{d}{ds}[U_0(t,s)\vec{u}(s)], \tag{4.58}$$

that is,

$$\begin{aligned} u(t) = &\left(\cos(t\sqrt{H_0+1})u(0) + \frac{\sin(t\sqrt{H_0+1})}{\sqrt{H_0+1}}\dot{u}(0)\right) + \\ &\int_0^t ds\frac{e^{\ i(t\ \ s)\sqrt{H_0+1}}}{2i\sqrt{H_0+1}}V(s)u(s) \quad \int_0^t ds\frac{e^{i(t\ \ s)\sqrt{H_0+1}}}{2i\sqrt{H_0+1}}V(s)u(s) \end{aligned} \tag{4.59}$$

and

$$\begin{aligned} \dot{u}(t) = &\quad \sin(t\sqrt{H_0+1})\sqrt{H_0+1}u(0) + \cos(t\sqrt{H_0+1})\dot{u}(0) \\ &\int_0^t ds\frac{e^{i(t\ \ s)\sqrt{H_0+1}}}{2}V(s)u(s) \quad \int_0^t ds\frac{e^{\ i(t\ \ s)\sqrt{H_0+1}}}{2}V(s)u(s). \end{aligned} \tag{4.60}$$

Recall that

$$\Omega(t) \ \ \vec{u}(0) = U_0(0,t)\vec{u}(t) = \begin{pmatrix} u \ \ (t) \\ \dot{u} \ \ (t) \end{pmatrix}. \tag{4.61}$$

Based on (4.56), a Duhamel's formula for $\Omega(t) \ \ \vec{u}(0)$ is given by

$$\Omega(t) \ \ \vec{u}(0) = \vec{u}(0) \quad \int_0^t dsU_0(0,s)V(x,s)\vec{u}(s), \tag{4.62}$$

that is,

$$u \ \ (t) = u(0) + \int_0^t ds\frac{\sin(s\sqrt{H_0+1})}{\sqrt{H_0+1}}V(s)u(s) \tag{4.63}$$

and

$$\dot{u} \ \ (t) = \dot{u}(0) \quad \int_0^t ds\cos(s\sqrt{H_0+1})V(s)u(s). \tag{4.64}$$

4.2.3 Estimates for interaction terms

Lemma 4.2.2. *If* $V(x,t) \in L_t^1 L_x^2(\mathbb{R}^n \times \mathbb{R})$, *we have that for* $t \geq 1, n \geq 3$, *and for* $\alpha \in (0, 1-\frac{2}{n})$, $\beta > 0$,

$$\|F_c(\frac{|x|}{t^\alpha} \leq 1)\bar{F}_1(|P| \leq t^\beta)\partial_t[u(t)]\|_{L_x^2} \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-\beta}{2}}}\|V(x,t)\|_{L_t^1 L_x^2}C(\|\vec{u}(0)\|_S), \tag{4.65}$$

$$\|F_c\bar{F}_1(|P| \leq t^\beta)\partial_t[\dot{u}(t)]\|_{L_x^2} \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-3\beta}{2}}}\|V(x,t)\|_{L_t^1 L_x^2}C(\|\vec{u}(0)\|_S), \tag{4.66}$$

$$\|F_c\bar{F}_1(|P| \leq t^\beta)\partial_t[u(t)]\|_{L_x^2} \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-\beta}{2}}}\|V(x,t)\|_{L_t^1 L_x^2}C(\|\vec{u}(0)\|_S), \tag{4.67}$$

$$\|F_c\bar{F}_1(|P| \leq t^\beta)\partial_t[\dot{u}(t)]\|_{L_x^2} \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-3\beta}{2}}}\|V(x,t)\|_{L_t^1 L_x^2}C(\|\vec{u}(0)\|_S), \tag{4.68}$$

$$\left|(\bar{F}_1 u(t), F_c\bar{F}_1\partial_t[u(t)])_{L_x^2}\right| \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-\beta}{2}}}\|V(x,t)\|_{L_t^1 L_x^2}C(\|\vec{u}(0)\|_S)^2, \tag{4.69}$$

$$\left|(\bar{F}_1\langle P\rangle\partial_t[u(t)], F_c\bar{F}_1\langle P\rangle u(t))_{L_x^2}\right| \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-\beta}{2}}}\|V(x,t)\|_{L_t^1 L_x^2}C(\|\vec{u}(0)\|_S)^2, \tag{4.70}$$

$$\left|(\bar{F}_1\dot{u}(t), F_c\bar{F}_1\partial_t[\dot{u}(t)])_{L_x^2}\right| \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-3\beta}{2}}}\|V(x,t)\|_{L_t^1 L_x^2}C(\|\vec{u}(0)\|_S)^2, \tag{4.71}$$

and

$$\left|(\bar{F}_1\partial_t[\dot{u}(t)], F_c\bar{F}_1\dot{u}(t))_{L_x^2}\right| \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-3\beta}{2}}}\|V(x,t)\|_{L_t^1 L_x^2}C(\|\vec{u}(0)\|_S)^2. \tag{4.72}$$

Proof. Let

$$a(t) :=(\bar{F}_1(|P| \leq t^\beta)u(t), F_c\bar{F}_1(|P| \leq t^\beta)\partial_t[u(t)])_{L_x^2} \tag{4.73}$$

$$=(\bar{F}_1(|P| \leq t^\beta)u(t), F_c\bar{F}_1(|P| \leq t^\beta)\frac{\sin(t\sqrt{H_0+1})}{\langle P\rangle}V(t)u(t))_{L_x^2}. \tag{4.74}$$

Using Cauchy-Schwartz's inequality, Hölder's inequality and Lemma 4.2.1 in this order, we have that for $|t| \geq 1$,

$$|a(t)| \lesssim_n \|\bar{F}_1 u(t)\|_{L_x^2} \times \|F_c\bar{F}_1\frac{\sin(t\sqrt{H_0+1})}{\sqrt{H_0+1}}V(t)u(t)\|_{L_x^2}$$
$$\lesssim_n \|\bar{F}_1 u(t)\|_{L_x^2} \times \|F_c\|_{L_x^2}\frac{1}{|t|^{n/2}}\|\bar{F}_1\langle P\rangle^{\frac{n+1}{2}}V(x,t)u(t)\|_{L_t^1 L_x^1}$$
$$\lesssim_n C(\|\vec{u}(0)\|_S) \times \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-\beta}{2}}}\|V(x,t)\|_{L_t^1 L_x^2} \times C(\|\vec{u}(0)\|_S) \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-\beta}{2}}}\|V(x,t)\|_{L_t^1 L_x^2}C(\|\vec{u}(0)\|_S)^2. \tag{4.75}$$

Here we also use

$$\|\bar{F}_1 u\ (t)\|_{L^2_x} \le \|\vec{u}(t)\|_S \le C(\|\vec{u}(0)\|_S) \tag{4.76}$$

and

$$\|\langle P\rangle^{\frac{n+1}{2}}\bar{F}_1(|P|\ \ t^{\beta})\|_{L^1_x \to L^1_x} \lesssim \langle t\rangle^{(n+1)\beta/2}. \tag{4.77}$$

Here actually, we have

$$\|F_c\bar{F}_1(|P|\ \ t^{\beta})\partial_t[u\ (t)]\|_{L^2_x} \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-\beta}{2}}}\|V(x,t)\|_{L^1_t L^2_x} C(\|\vec{u}(0)\|_S) \tag{4.78}$$

and we get (4.65). Since

$$\left|(\bar{F}_1 u\ (t), F_c\bar{F}_1\partial_t[u\ (t)])_{L^2_x}\right| = \left|(\bar{F}_1\partial_t[u\ (t)], F_c\bar{F}_1 u\ (t))_{L^2_x}\right|, \tag{4.79}$$

we get (4.70). For (4.71), let

$$\dot{a}(t) := (\bar{F}_1\dot{u}\ (t), F_c\bar{F}_1\partial_t[\dot{u}\ (t)])_{L^2_x} \tag{4.80}$$

$$= \ (\bar{F}_1\dot{u}\ (t), F_c\bar{F}_1\cos(t\sqrt{H_0+1})V(t)u(t))_{L^2_x}. \tag{4.81}$$

According to (4.64), Cauchy-Schwarz inequality, Hölder's inequality and Lemma 4.2.1 in this order, we have that for $|t| \ge 1$,

$$\begin{aligned} |\dot{a}(t)| &\lesssim_n \|\bar{F}_1\dot{u}\ (t)\|_{L^2_x}\ \|F_c\bar{F}_1\cos(t\sqrt{H_0+1})V(t)u(t)\|_{L^2_x} \\ &\lesssim_n \|\bar{F}_1\dot{u}\ (t)\|_{L^2_x}\ \|F_c\|_{L^2_x}\frac{1}{|t|^{n/2}}\|\bar{F}_1\langle P\rangle^{\frac{n+3}{2}}V(x,t)u(t)\|_{L^1_t L^1_x} \\ &\lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-3\beta}{2}}}\|V(x,t)\|_{L^1_t L^2_x} C(\|\vec{u}(0)\|_S)^2. \end{aligned} \tag{4.82}$$

Here we use

$$\|\langle P\rangle^{\frac{n+3}{2}}\bar{F}_1(|P|\ \ t^{\beta})\|_{L^1_x \to L^1_x} \lesssim \langle t\rangle^{(n+3)\beta/2}. \tag{4.83}$$

Here actually, we have

$$\|F_c\bar{F}_1(|P|\ \ t^{\beta})\partial_t[\dot{u}\ (t)]\|_{L^2_x} \lesssim_n \frac{1}{|t|^{\frac{n(1-\alpha-\beta)-3\beta}{2}}}\|V(x,t)\|_{L^1_t L^2_x} C(\|\vec{u}(0)\|_S) \tag{4.84}$$

and we get (4.66). Since

$$\left|(\bar{F}_1\dot{u}\ (t), F_c\bar{F}_1\partial_t[\dot{u}\ (t)])_{L^2_x}\right| = \left|(\bar{F}_1\partial_t[\dot{u}\ (t)], F_c\bar{F}_1\dot{u}\ (t))_{L^2_x}\right|, \tag{4.85}$$

we get (4.72). (4.67), (4.68) follow by using (4.65), (4.66) and by using

$$\bar F_1(|P| \leq t^\beta) F_c(\frac{|x|}{t^\alpha} \leq 1) F_1(|P| > 100 t^\beta) = \bar F_1 [F_1(|P| > t^\beta) F_c(\frac{|x|}{t^\alpha} \leq 1)] F_1 \tag{4.86}$$

and

$$\|[F_1(|P| > t^\beta) F_c(\frac{|x|}{t^\alpha} \leq 1)]\|_{L^2_x \to L^2_x} \lesssim_N o(\frac{1}{t^N}). \tag{4.87}$$

□

Lemma 4.2.3. *If either* $V(x,t) \in L^1_t L^2_{\delta,x}(\mathbb{R}^n \times \mathbb{R})$ *or* $V(x,t) \in L^1_t L^1_{\delta+n/2,x}(\mathbb{R}^n \times \mathbb{R})$ *for some* $\delta > 1$ *and if*

$$\|V(x,t)u(t)\|_{L^1_t L^2_x} \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{H^1_x}, \tag{4.88}$$

then for $b \in [0, 1/2)$, $\alpha \in (0, \min(1-b, 1-\frac{2-\delta}{n}))$, $t \geq 1, n \geq 1$,

$$\|F_c F_1 \langle P\rangle u\ (t)\|_{L^2_x} \lesssim_{n,b,\alpha,\delta} \frac{1}{t^{1+\beta}} \|V(x,t)\|_{L^1_t L^2_{\delta,x}} C(\|\vec u(0)\|_S), \tag{4.89}$$

$$\|F_c F_1 \dot u\ (t)\|_{L^2_x} \lesssim_{n,b,\alpha,\delta} \frac{1}{t^{1+\beta}} \|V(x,t)\|_{L^1_t L^2_{\delta,x}} C(\|\vec u(0)\|_S), \tag{4.90}$$

$$\|F_1 F_c \langle P\rangle u\ (t)\|_{L^2_x} \lesssim_{n,b,\alpha,\delta} \frac{1}{t^{1+\beta}} \|V(x,t)\|_{L^1_t L^2_{\delta,x}} C(\|\vec u(0)\|_S), \tag{4.91}$$

$$\|F_1 F_c \dot u\ (t)\|_{L^2_x} \lesssim_{n,b,\alpha,\delta} \frac{1}{t^{1+\beta}} \|V(x,t)\|_{L^1_t L^2_{\delta,x}} C(\|\vec u(0)\|_S), \tag{4.92}$$

$$\left|(F_1 F_c F_1 \langle P\rangle \partial_t [u\ (t)], \langle P\rangle u\ (t))_{L^2_x}\right| \lesssim_{n,b,\alpha,\delta} \frac{1}{t^{1+\beta}} \|V(x,t)\|_{L^1_t L^2_{\delta,x}} C(\|\vec u(0)\|_S)^2, \tag{4.93}$$

$$\left|(\langle P\rangle u\ (t), F_1 F_c F_1 \langle P\rangle \partial_t [u\ (t)])_{L^2_x}\right| \lesssim_{n,b,\alpha,\delta} \frac{1}{t^{1+\beta}} \|V(x,t)\|_{L^1_t L^2_{\delta,x}} C(\|\vec u(0)\|_S)^2, \tag{4.94}$$

$$\left|(F_1 F_c F_1 \partial_t [\dot u\ (t)], \dot u\ (t))_{L^2_x}\right| \lesssim_{n,b,\alpha,\delta} \frac{1}{t^{1+\beta}} \|V(x,t)\|_{L^1_t L^2_{\delta,x}} C(\|\vec u(0)\|_S)^2, \tag{4.95}$$

and

$$\left|(\dot u\ (t), F_1 F_c F_1 \partial_t [\dot u\ (t)])_{L^2_x}\right| \lesssim_{n,b,\alpha,\delta} \frac{1}{t^{1+\beta}} \|V(x,t)\|_{L^1_t L^2_{\delta,x}} C(\|\vec u(0)\|_S)^2, \tag{4.96}$$

with

$$\beta := \min\{\frac{(1-\alpha)n}{2} + \frac{\tilde\delta}{2} - 1, \tilde\delta - 1\} > 0 \tag{4.97}$$

for $\tilde\delta = \frac{1+\delta}{2}$.

Proof. Write $F_1F_cF_1\langle P\rangle\partial_t[u_-(t)]$ as

$$F_1F_cF_1\langle P\rangle\partial_t[u_-(t)] = F_1F_cF_1\langle P\rangle \frac{\sin t\sqrt{H_0+1}}{\sqrt{H_0+1}}V(t)u(t) \tag{4.98}$$

$$= F_1F_cF_1\sin t\sqrt{H_0+1}V(t)u(t). \tag{4.99}$$

Based on assumption (4.88),

$$\|V(x,t)u(t)\|_{L^1_tL^2_{\delta,x}\cap L^1_tL^1_{\delta,x}} \lesssim \|V(x,t)\|_{L^1_tL^2_{\delta,x}}C(\|\vec{u}(0)\|_S) \tag{4.100}$$

when $V(x,t)\in L^1_tL^2_{\delta,x}$ and

$$\|V(x,t)u(t)\|_{L^1_tL^2_{\delta,x}\cap L^1_tL^1_{\delta,x}} \lesssim_\delta \|V(x,t)\|_{L^1_tL^1_{\delta+n/2,x}}C(\|\vec{u}(0)\|_S) \tag{4.101}$$

when $V(x,t)\in L^1_tL^2_{\delta+n/2,x}$. Let

$$F_1^< := F_1(t^b|P|>1)\bar{F}_1(|P|\leq 100) \tag{4.102}$$

and

$$F_1^> := F_1(|P|>100). \tag{4.103}$$

(4.89) to (4.96) follow from the following two estimates. One is that for $t\geq 1$,

$$\begin{aligned}
&\|F_c(\frac{|x|}{t^\alpha}\leq 1)F_1^<e^{-it\sqrt{H_0+1}}\langle x\rangle^{-\delta}\|_{L^1_x\to L^2_x}\lesssim\\
&\quad\|F_c(\frac{|x|}{10t^\alpha}\leq 1)\|_{L^2_x}\|F_c(\frac{|x|}{t^\alpha}\leq 1)F_1^<e^{-it\sqrt{H_0+1}}\langle x\rangle^{-\delta}\chi(|x|\geq\sqrt{t})\|_{L^1_x\to L^\infty_x}+\\
&\quad\|F_c(\frac{|x|}{10t^\alpha}\leq 1)\|_{L^2_x}\|F_c(\frac{|x|}{t^\alpha}\leq 1)F_1^<e^{-it\sqrt{H_0+1}}\langle x\rangle^{-\delta}\chi(|x|<\sqrt{t})\|_{L^1_x\to L^\infty_x}\\
&\qquad\lesssim_{N,n,b,\alpha}\frac{t^{\alpha n/2}}{t^{n/2+\delta/2}}+\frac{1}{t^N},
\end{aligned}\tag{4.104}$$

where we use that for $l=0,1$, Lemma 4.2.1 implies that

$$\begin{aligned}
&\|F_c(\frac{|x|}{t^\alpha}\leq 1)F_1^<e^{-it\sqrt{H_0+1}}\langle x\rangle^{-\delta}\chi(|x|\geq\sqrt{t})\|_{L^1_x\to L^\infty_x}\lesssim\\
&\qquad\|\bar{F}_1(|P|\leq 100)F_1^<e^{-it\sqrt{H_0+1}}\|_{L^1_x\to L^\infty_x}\frac{1}{t^{\delta/2}}\lesssim_n\frac{1}{t^{n/2+\delta/2}},
\end{aligned}\tag{4.105}$$

$$\|F_c(\frac{|x|}{t^\alpha}\leq 1)F_1^<e^{-it\sqrt{H_0+1}}\langle x\rangle^{-\delta}\chi(|x|<\sqrt{t})\|_{L^1_x\to L^\infty_x}\lesssim_{N,n,b,\alpha}\frac{1}{t^N}, \tag{4.106}$$

which follows from the method of non-stationary phase since

$$\begin{aligned}
&F_c(\frac{|x|}{t^\alpha}\leq 1)e^{ix\cdot q}e^{itq^2}e^{-iq\cdot y}\chi(|y|<\sqrt{t})=\\
&\qquad\frac{1}{i(x\cdot\hat{q}+2t|q|-y\cdot\hat{q})}\partial_{|q|}[F_c(\frac{|x|}{t^\alpha}\leq 1)e^{ix\cdot q}e^{itq^2}e^{-iq\cdot y}\chi(|y|<\sqrt{t})]
\end{aligned}\tag{4.107}$$

with

$$|x - \hat{q} + 2t|q| - y - \hat{q}| \gtrsim t^{1-b} \tag{4.108}$$

due to factors(Recall that $\alpha \in (0, \min(1 - \frac{2-\delta}{n}, 1 - b)))$

$$F_c(\frac{|x|}{t^\alpha} - 1), \chi(|y| < \sqrt{t}), F_1(t^b|q| > 1). \tag{4.109}$$

The other one is that

$$\begin{aligned} \|F_c(\frac{|x|}{t^\alpha} - 1)F_1^{>} e^{-it\sqrt{H_0+1}}\langle x\rangle^{-\delta}\|_{L^2_x \to L^2_x} \lesssim & \\ \|F_c(\frac{|x|}{t^\alpha} - 1)F_1^{>} e^{-it\sqrt{H_0+1}}\langle x\rangle^{-\delta}\chi(|x| \geq |t|/100)\|_{L^2_x \to L^2_x} + & \\ \|F_c(\frac{|x|}{t^\alpha} - 1)F_1^{>} e^{-it\sqrt{H_0+1}}\langle x\rangle^{-\delta}\chi(|x| < |t|/100\|_{L^2_x \to L^2_x} & \\ \lesssim_{N,n,\alpha} \frac{1}{t^\delta} + \frac{1}{t^N}, & \end{aligned} \tag{4.110}$$

where we use

$$\|\langle x\rangle^{-\delta}\chi(|x| \geq |t|/100)\|_{L^2_x \to L^2_x} \lesssim \frac{1}{\langle t\rangle^\delta} \tag{4.111}$$

and the method of non-stationary phase for the part with $\chi(|x| < |t|/100)$. We finish the proof.

□

4.2.4 Commutator estimate

Lemma 4.2.4. *For* $t \geq 1$, $b < \alpha \leq 1, \beta > 0$, $l = 0, 1$,

$$\|[F_c(\frac{|x|}{t^\alpha} - 1)), F_1^{(l)}(t^b|P| > 1)]\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha-b}}, \tag{4.112}$$

$$\|[F_c^{(l)}(\frac{|x|}{t^\alpha} - 1)), F_1(t^b|P| > 1)]\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha-b}} \tag{4.113}$$

$$\|[F_c(\frac{|x|}{t^\alpha} - 1)), \bar{F}_1^{(l)}(|P| - t^\beta)]\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha+\beta}}, \tag{4.114}$$

$$\|[F_c^{(l)}(\frac{|x|}{t^\alpha} - 1)), \bar{F}_1(|P| - t^\beta)]\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha+\beta}} \tag{4.115}$$

where

$$F_1^{(l)}(k) := \frac{d^l}{dk^l}[F_1], \tag{4.116}$$

and

$$F_c^{(l)}(k) := \frac{d^l}{dk^l}[F_c]. \tag{4.117}$$

In particular,

$$\|[\partial_t[F_c(\frac{|x|}{t^\alpha}-1))], F_1(t^b|P|>1)]\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha-b+1}}, \tag{4.118}$$

$$\|[F_c(\frac{|x|}{t^\alpha}-1)), [\partial_t F_1(t^b|P|>1)]]\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha-b+1}}, \tag{4.119}$$

$$\|[\partial_t[F_c(\frac{|x|}{t^\alpha}-1))], \bar{F}_1(|P|\leq t^\beta)]\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha+\beta+1}}, \tag{4.120}$$

and

$$\|[F_c(\frac{|x|}{t^\alpha}-1)), [\partial_t \bar{F}_1(|P|\leq t^\beta)]]\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha+\beta+1}}. \tag{4.121}$$

Proof. Let

$$\tilde{F} := F_1^{(l)}. \tag{4.122}$$

Write $[F_c(\frac{|x|}{t^\alpha}-1),\tilde{F}]$ as

$$\begin{aligned}
[F_c(\frac{|x|}{t^\alpha}-1),\tilde{F}] = \\
c_n\int d^n\xi \hat{\tilde{F}}(\xi)e^{it^bP\cdot\xi}\left[e^{-it^bP\cdot\xi}F_c(\frac{|x|}{t^\alpha}-1)e^{it^bP\cdot\xi}-F_c(\frac{|x|}{t^\alpha}-1)\right] \\
= c_n\int d^n\xi \hat{\tilde{F}}(\xi)e^{it^bP\cdot\xi}\left[F_c(\frac{|x-t^b\xi|}{t^\alpha}-1)-F_c(\frac{|x|}{t^\alpha}-1)\right].
\end{aligned} \tag{4.123}$$

Since

$$\frac{\left|F_c(\frac{|x-t^b\xi|}{t^\alpha}-1)-F_c(\frac{|x|}{t^\alpha}-1)\right|}{t^{b-\alpha}|\xi|} \lesssim \sup_{x\in\mathbb{R}^n}|F_c'(|x|-1)| \lesssim 1, \tag{4.124}$$

we have that for each $\psi\in L^2_x$,

$$\begin{aligned}
\|[F_c(\frac{|x|}{t^\alpha}-1)), F_1^{(l)}(t^b|P|>1)]\psi\|_{L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha-b}}\int d^n\xi|\hat{\tilde{F}}(\xi)||\xi|\|\psi(x)\|_{L^2_x(\mathbb{R}^n)} \\
\lesssim_n \frac{1}{t^{\alpha-b}}\|\psi\|_{L^2_x(\mathbb{R}^n)}.
\end{aligned} \tag{4.125}$$

Similarly, we have (4.113). For (4.114), since

$$\begin{aligned}
\|[F_c(\frac{|x|}{t^\alpha}-1)), \bar{F}_1^{(l)}(|P|\leq t^\beta)]\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} = \\
\|[F_c(\frac{|x|}{t^\alpha}-1)), F_1^{(l)}(|P|>t^\beta)]\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)},
\end{aligned} \tag{4.126}$$

(4.114) follows by taking $b=-\beta$ in (4.112). Similarly, we have (4.115). We finish the proof. □

Corollary 4.2.1. *For* $t \quad 1,\ 0 \quad b<\alpha \quad 1, \beta>0,$

$$\|[F_c(\frac{|x|}{t^\alpha} \quad 1)), \partial_t[\bar{F}_1(|P| \quad t^\beta)F_1(t^b|P|>1)]]\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{\alpha-b+1}}, \tag{4.127}$$

$$\|[\partial_t[F_c(\frac{|x|}{t^\alpha} \quad 1))], \bar{F}_1(|P| \quad t^\beta)F_1(t^b|P|>1)]\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{t^{1+\alpha-b}}. \tag{4.128}$$

Proof. Since

$$[F_c(\frac{|x|}{t^\alpha} \quad 1)), \partial_t[\bar{F}_1(|P| \quad t^\beta)]F_1(t^b|P|>1)] =$$
$$[F_c(\frac{|x|}{t^\alpha} \quad 1)), \partial_t[\bar{F}_1(|P| \quad t^\beta)]]F_1(t^b|P|>1)+\partial_t[\bar{F}_1(|P| \quad t^\beta)][F_c(\frac{|x|}{t^\alpha} \quad 1)), F_1(t^b|P|>1)], \tag{4.129}$$

and since

$$[F_c(\frac{|x|}{t^\alpha} \quad 1)), \bar{F}_1(|P| \quad t^\beta)\partial_t[F_1(t^b|P|>1)]] =$$
$$[F_c(\frac{|x|}{t^\alpha} \quad 1)), \bar{F}_1(|P| \quad t^\beta)]\partial_t[F_1(t^b|P|>1)]+\bar{F}_1(|P| \quad t^\beta)[F_c(\frac{|x|}{t^\alpha} \quad 1)), \partial_t[F_1(t^b|P|>1)]], \tag{4.130}$$

(4.127) follows by using Lemma 4.2.4. Similarly, we have (4.128) by using Lemma 4.2.4. □

4.3 Proof of the existence of Channel Wave Operator and properties of Weakly Localized part

Proof of Theorem 4.1.1. For $F_c(\frac{|x|}{t^\alpha} \quad 1)\bar{F}_1(|P| \quad t^\beta)u \ (t)$, according to (4.63), use Duhamel's formula to expand it

$$F_c(\frac{|x|}{t^\alpha} \quad 1)\bar{F}_1(|P| \quad t^\beta)u \ (t) = F_c(|x| \quad 1)\bar{F}_1(|P| \quad 1)u \ (1)+$$
$$\int_1^t ds \partial_s[F_c(\frac{|x|}{s^\alpha} \quad 1)]\bar{F}_1(|P| \quad s^\beta)u \ (s) + \int_1^t ds F_c(\frac{|x|}{s^\alpha} \quad 1)\partial_s[\bar{F}_1(|P| \quad s^\beta)]u \ (s)+$$
$$\int_1^t ds F_c(\frac{|x|}{s^\alpha} \quad 1)\bar{F}_1(|P| \quad s^\beta)\frac{\sin(s\sqrt{H_0+1})}{\sqrt{H_0+1}}V(s)u(s)$$
$$=: F_c(|x| \quad 1)\bar{F}_1(|P| \quad 1)u \ (1) + u_{\ ,1}(t) + u_{\ ,2}(t) + u_{\ ,3}(t). \tag{4.131}$$

For $u_{\ ,3}(t)$, by using Lemma 4.2.2, we have

$$u_{\ ,3}(\infty) := \lim_{t\to\infty} u_{\ ,3}(t) \text{ exists in } L^2_x. \tag{4.132}$$

Estimate for $u_{\ ,1}(t)$: For $u_{\ ,1}(t)$, if we can show

$$\int_1^\infty ds \left\| \sqrt{|\partial_s [F_c(\frac{|x|}{s^\alpha}\ \ 1)]|}\bar{F}_1(|P|\ \ s^\beta) u\ (s)\right\|^2_{L^2_x} \lesssim C(\|\vec{u}(0)\|_S)^2, \tag{4.133}$$

then for $t_2\ \ t_1 > 1$, using Hölder's inequality in s variable and then Fubini's theorem, one has

$$\begin{aligned} \|u_{\ ,1}(t_2)\ \ u_{\ ,1}(t_1)\|_{L^2_x} &= \left\| \int_{t_1}^{t_2} ds \partial_s [F_c(\frac{|x|}{s^\alpha}\ \ 1)]\bar{F}_1(|P|\ \ s^\beta) u\ (s)\right\|_{L^2_x} \\ &\left(\int_{t_1}^{t_2} ds \left\| \sqrt{|\partial_s [F_c(\frac{|x|}{s^\alpha}\ \ 1)]|}\bar{F}_1(|P|\ \ s^\beta) u\ (s)\right\|^2_{L^2_x}\right)^{1/2}\ \ g(t_1) \to 0 \text{ as } t_1 \to \infty, \end{aligned} \tag{4.134}$$

where

$$g(t) := \sqrt{\int_t^\infty ds \left\| \sqrt{|\partial_s [F_c(\frac{|x|}{s^\alpha}\ \ 1)]|}\bar{F}_1(|P|\ \ s^\beta) u\ (s)\right\|^2_{L^2_x}} \tag{4.135}$$

and $g(t) \to 0$ as $t \to \infty$ due to (4.133). Then we get the existence of $u_{\ ,1}(\infty)$ in L^2_x.

Now we prove (4.133) by using propagation estimates(for Propagation estimates, see [91]). To be precise, choose

$$B_1(t) := \bar{F}_1(|P|\ \ t^\beta) F_c(\frac{|x|}{t^\alpha}\ \ 1)\bar{F}_1(|P|\ \ t^\beta). \tag{4.136}$$

Let

$$\langle B_1(t) : u\ (t)\rangle := (u\ (t), B_1(t) u\ (t))_{L^2_x}. \tag{4.137}$$

Then

$$|\langle B_1(t) : u\ (t)\rangle|\ \ C(\|\vec{u}(0)\|_S)^2. \tag{4.138}$$

Let

$$\begin{aligned} R(t) := (u\ (t), \partial_t[\bar{F}_1(|P|\ \ t^\beta)] F_c \bar{F}_1(|P|\ \ t^\beta) u\ (t))_{L^2_x} + \\ (u\ (t), \bar{F}_1(|P|\ \ t^\beta) F_c \partial_t[\bar{F}_1(|P|\ \ t^\beta)] u\ (t))_{L^2_x} \\ 2(u\ (t), \sqrt{F_c}\bar{F}_1(|P|\ \ t^\beta)\partial_t[\bar{F}_1(|P|\ \ t^\beta)]\sqrt{F_c} u\ (t))_{L^2_x}. \end{aligned} \tag{4.139}$$

Compute $\partial_t \langle B_1(t) : u \ (t) \rangle$

$$\partial_t \langle B_1(t) : u \ (t) \rangle = (u \ (t), \bar{F}_1 \partial_t [F_c] \bar{F}_1 u \ (t))_{L^2_x} +$$
$$(\partial_t [u \ (t)], \bar{F}_1 F_c \bar{F}_1 u \ (t))_{L^2_x} + (u \ (t), \bar{F}_1 F_c \bar{F}_1 \partial_t [u \ (t)])_{L^2_x} + R(t) +$$
$$2(u \ (t), \sqrt{F_c} \bar{F}_1 (|P| \ \ t^\beta) \partial_t [\bar{F}_1 (|P| \ \ t^\beta)] \sqrt{F_c} u \ (t))_{L^2_x}$$
$$=: A_1(t) + A_2(t) + A_3(t) + R(t) + A_4(t). \quad (4.140)$$

Here $A_1(t), A_4(t) \quad 0$ for all t and $A_2(t), A_3(t), R(t) \in L^1_t[1, \infty)$ due to Lemma 4.2.2, Lemma 4.2.4 and our assumption on α, that is,

$$\frac{n(1 \quad \alpha \quad \beta) \quad 3\beta}{2} > 1. \quad (4.141)$$

Hence,

$$|A_1(t)| = A_1(t) \quad A_1(t) + A_4(t)$$
$$2\|u \ (t)\|^2_{L^2_x} + \|A_2(t)\|_{L^1_t[1,\infty)} + \|A_3(t)\|_{L^1_t[1,\infty)} + \|R(t)\|_{L^1_t[1,\infty)}, \quad (4.142)$$

which implies that $A_1(\infty)$ exists and

$$\int_1^\infty ds \left\| \sqrt{|\partial_s [F_c(\frac{|x|}{s^\alpha} \quad 1)]|} \bar{F}_1(|P| \quad s^\beta) u \ (s) \right\|^2_{L^2_x} = A_1(\infty) \lesssim C(\|\vec{u}(0)\|_S)^2. \quad (4.143)$$

Therefore,

$$u \ _{,1}(\infty) := \lim_{t \to \infty} u \ _{,1}(t) \text{ exists in } L^2_x. \quad (4.144)$$

Estimate for $u \ _{,2}(t)$: Write $u \ _{,2}(t)$ as

$$u \ _{,2}(t) = \int_1^t ds \partial_s [\bar{F}_1(|P| \quad s^\beta)] F_c(\frac{|x|}{s^\alpha} \quad 1) u \ (s)$$
$$\int_1^t ds [\partial_s [\bar{F}_1(|P| \quad s^\beta)], F_c(\frac{|x|}{s^\alpha} \quad 1)] u \ (s) =: u \ _{,21}(t) + u \ _{,22}(t). \quad (4.145)$$

For $u \ _{,22}(t)$, $u \ _{,22}(\infty)$ exits in L^2_x since by using Lemma 4.2.4 and $\alpha, \beta > 0$,

$$\|[\partial_s [\bar{F}_1(|P| \quad s^\beta)], F_c(\frac{|x|}{s^\alpha} \quad 1)] u \ (s)\|_{L^2_x} \lesssim_n \frac{1}{s^{1+\alpha+\beta}} \|u \ (s)\|_{L^2_x} \in L^1_s[1, \infty). \quad (4.146)$$

For $u \ _{,21}(t)$, we use propagation estimates. Choose

$$B_{11}(t) := F_c(\frac{|x|}{t^\alpha} \quad 1) \bar{F}_1(|P| \quad t^\beta) F_c(\frac{|x|}{t^\alpha} \quad 1) \quad (4.147)$$

and let

$$\langle B_{11}(t) : u(t)\rangle := (u(t), B_{11}(t)u(t))_{L_x^2}. \tag{4.148}$$

Let

$$R_1(t) := (u(t), F_c\bar{F}_1\partial_t[F_c]u(t))_{L_x^2}+ \\ (u(t), \partial_t[F_c]\bar{F}_1F_cu(t))_{L_x^2} \quad 2(u(t), \sqrt{\bar{F}_1}F_c\partial_t[F_c]\sqrt{\bar{F}_1}u(t))_{L_x^2}. \tag{4.149}$$

Compute $\partial_t\langle B_{11}(t) : u(t)\rangle$

$$\partial_t\langle B_{11}(t) : u(t)\rangle = (u(t), F_c\partial_t[\bar{F}_1]F_cu(t))_{L_x^2}+ \\ (\partial_t[u(t)], F_c\bar{F}_1F_cu(t))_{L_x^2} + (u(t), F_c\bar{F}_1F_c\partial_t[u(t)])_{L_x^2} + R_1(t)+ \\ 2(u(t), \sqrt{\bar{F}_1}F_c\partial_t[F_c]\sqrt{\bar{F}_1}u(t))_{L_x^2} \\ =: A_{11}(t) + A_{12}(t) + A_{13}(t) + R_1(t) + A_{14}(t). \tag{4.150}$$

$A_{11}(t), A_{14}(t) \quad 0$ and $A_{12}(t), A_{13}(t), R_1(t) \in L_t^1[1, \infty)$ due to Lemma 4.2.2 and Lemma 4.2.4. Then by using propagation estimates, we have that $A_{11}(\infty)$ exists which implies that $u_{,21}(\infty)$ exists in L_x^2. Hence,

$$u_{,2}(\infty) := \lim_{t\to\infty} u_{,2}(t) \text{ exists in } L_x^2. \tag{4.151}$$

Hence, based on (4.167), (4.174), (4.159), we have

$$u_+ := s\text{-}\lim_{t\to\infty} F_c(\frac{|x|}{t^\alpha} \quad 1)\bar{F}_1(|P| \quad t^\beta)u(t) \tag{4.152}$$

exists in L_x^2.

Similarly, for $F_c(\frac{|x|}{t^\alpha} \quad 1)\bar{F}_1(|P| \quad t^\beta)\dot{u}(t)$, use Duhamel's formula to expand it

$$F_c(\frac{|x|}{t^\alpha} \quad 1)\bar{F}_1(|P| \quad t^\beta)\dot{u}(t) = F_c(|x| \quad 1)\bar{F}_1(|P| \quad 1)\dot{u}(1)+ \\ \int_1^t ds\partial_s[F_c(\frac{|x|}{s^\alpha} \quad 1)]\bar{F}_1(|P| \quad s^\beta)\dot{u}(s) + \int_1^t dsF_c(\frac{|x|}{s^\alpha} \quad 1)\partial_s[\bar{F}_1(|P| \quad s^\beta)]\dot{u}(s) \\ \int_1^t dsF_c(\frac{|x|}{s^\alpha} \quad 1)\bar{F}_1(|P| \quad s^\beta)\cos(s\sqrt{H_0+1})V(s)u(s) \\ =: F_c(|x| \quad 1)\bar{F}_1(|P| \quad 1)\dot{u}(1) + \dot{u}_{,1}(t) + \dot{u}_{,2}(t) + \dot{u}_{,3}(t). \tag{4.153}$$

$\dot{u}_{,3}(\infty)$ exists in L^2_x due to Lemma 4.2.2. For $\dot{u}_{,2}(t)$, break it into two pieces

$$\dot{u}_{,2}(t) = \int_1^t ds \partial_s[\bar{F}_1(|P| - s^{\beta})]F_c(\frac{|x|}{s^{\alpha}} - 1)\dot{u}(s) - \int_1^t ds[\partial_s[\bar{F}_1(|P| - s^{\beta})], F_c(\frac{|x|}{s^{\alpha}} - 1)]\dot{u}(s) =: \dot{u}_{,21}(t) + \dot{u}_{,22}(t) \quad (4.154)$$

$\dot{u}_{,22}(\infty)$ exists in L^2_x due to Lemma 4.2.4. Both $\dot{u}_{,1}(\infty)$ and $\dot{u}_{,21}(\infty)$ exist in L^2_x by using Lemma 4.2.2, Lemma 4.2.4 and propagation estimates via observing

$$\begin{cases} \langle B_2(t) : \dot{u}_{,1}(t)\rangle \\ B_2(t) = B_1(t) \end{cases} \text{and} \begin{cases} \langle B_{21}(t) : \dot{u}_{,21}(t)\rangle \\ B_{21}(t) = B_{11}(t) \end{cases} \quad (4.155)$$

respectively. Then

$$\dot{u}_+ := s\text{-}\lim_{t\to\infty} F_c(\frac{|x|}{t^{\alpha}} - 1)\bar{F}_1(|P| - t^{\beta})\dot{u}(t) \quad (4.156)$$

exists in L^2_x and we finish the proof.

□

Proof. Proof of Theorem 4.1.3 It is equivalent to show the following free channel wave operator

$$\tilde{\Omega}_{\alpha,b} := s\text{-}\lim_{t\to\infty} \langle P\rangle_1^{-1} F_c(\frac{|x|}{t^{\alpha}} - 1)\langle P\rangle_1 F_1(t^b|P| > 1)\Omega(t) \quad (4.157)$$

exists from S to S. For $F_c(\frac{|x|}{t^{\alpha}} - 1)F_1(t^b|P| > 1)\langle P\rangle u(t)$, use Duhamel's formula to expand it

$$F_c(\frac{|x|}{t^{\alpha}} - 1)F_1(t^b|P| > 1)\langle P\rangle u(t) = F_c(|x| - 1)F_1(|P| > 1)\langle P\rangle u(1)+$$
$$\int_1^t ds\partial_s[F_c(\frac{|x|}{s^{\alpha}} - 1)]F_1(s^b|P| > 1)u(s) + \int_1^t dsF_c(\frac{|x|}{s^{\alpha}} - 1)\partial_s[F_1(s^b|P| > 1)]u(s)+$$
$$\int_1^t dsF_c(\frac{|x|}{s^{\alpha}} - 1)F_1(s^b|P| > 1)\sin(s\sqrt{H_0+1})V(s)u(s)$$
$$=: F_c(|x| - 1)\bar{F}_1(|P| - 1)u(1) + u_{,1}(t) + u_{,2}(t) + u_{,3}(t). \quad (4.158)$$

For $u_{,3}(t)$, by using Lemma 4.2.3, we have

$$u_{,3}(\infty) := \lim_{t\to\infty} u_{,3}(t) \text{ exists in } L^2_x. \quad (4.159)$$

Estimate for $u_{,1}(t)$: Choose

$$B_1(t) := F_1(t^b|P| > 1)F_c(\frac{|x|}{t^{\alpha}} - 1)F_1(t^b|P| > 1). \quad (4.160)$$

Let

$$\langle B_1(t) : \langle P\rangle u(t)\rangle := (\langle P\rangle u(t), B_1(t)\langle P\rangle u(t))_{L^2_x}. \tag{4.161}$$

Then

$$|\langle B_1(t) : \langle P\rangle u(t)\rangle| \leq C(\|\vec{u}(0)\|_S)^2. \tag{4.162}$$

Let

$$\begin{aligned} R(t) := {} & (\langle P\rangle u(t), \partial_t[F_1(t^b|P| > 1)]F_c F_1(t^b|P|>1)\langle P\rangle u(t))_{L^2_x} + \\ & (\langle P\rangle u(t), F_1(t^b|P|>1)F_c\partial_t[F_1(t^b|P|>1)]\langle P\rangle u(t))_{L^2_x} \\ & -2(\langle P\rangle u(t), \sqrt{F_c}F_1(t^b|P|>1)\partial_t[F_1(t^b|P|>1)]\sqrt{F_c}\langle P\rangle u(t))_{L^2_x}. \end{aligned} \tag{4.163}$$

Compute $\partial_t\langle B_1(t) : \langle P\rangle u(t)\rangle$

$$\begin{aligned} \partial_t\langle B_1(t) : \langle P\rangle u(t)\rangle = {} & (\langle P\rangle u(t), F_1\partial_t[F_c]F_1\langle P\rangle u(t))_{L^2_x} + \\ & (\langle P\rangle\partial_t[u(t)], F_1F_cF_1\langle P\rangle u(t))_{L^2_x} + (\langle P\rangle u(t), F_1F_cF_1\langle P\rangle\partial_t[u(t)])_{L^2_x} + R(t) + \\ & 2(\langle P\rangle u(t), \sqrt{F_c}F_1(t^b|P|>1)\partial_t[F_1(t^b|P|>1)]\sqrt{F_c}\langle P\rangle u(t))_{L^2_x} \\ =: {} & A_1(t) + A_2(t) + A_3(t) + R(t) + A_4(t). \end{aligned} \tag{4.164}$$

Here $A_1(t), A_4(t) \geq 0$ for all t and $A_2(t), A_3(t), R(t) \in L^1_t[1,\infty)$ due to Lemma 4.2.3, Lemma 4.2.4 and our assumption on α. Hence,

$$\begin{aligned} |A_1(t)| = A_1(t) \leq {} & A_1(t) + A_4(t) \\ \leq {} & 2\|\langle P\rangle u(t)\|^2_{L^2_x} + \|A_2(t)\|_{L^1_t[1,\infty)} + \|A_3(t)\|_{L^1_t[1,\infty)} + \|R(t)\|_{L^1_t[1,\infty)}, \end{aligned} \tag{4.165}$$

which implies that $A_1(\infty)$ exists and

$$\int_1^\infty ds\|\sqrt{|\partial_s[F_c(\frac{|x|}{s^\alpha} - 1)]|}F_1(s^b|P|>1)\langle P\rangle u(s)\|^2_{L^2_x} = A_1(\infty) \lesssim C(\|\vec{u}(0)\|_S)^2. \tag{4.166}$$

Therefore,

$$u_{,1}(\infty) := \lim_{t\to\infty} u_{,1}(t) \text{ exists in } L^2_x. \tag{4.167}$$

Estimate for $u_{,2}(t)$: Write $u_{,2}(t)$ as

$$\begin{aligned} u_{,2}(t) = {} & \int_1^t ds\partial_s[F_1(s^b|P|>1)]F_c(\frac{|x|}{s^\alpha} - 1)\langle P\rangle u(s) \\ & -\int_1^t ds[\partial_s[F_1(s^b|P|>1)], F_c(\frac{|x|}{s^\alpha} - 1)]\langle P\rangle u(s) =: u_{,21}(t) + u_{,22}(t). \end{aligned} \tag{4.168}$$

For $u_{\ ,22}(t)$, $u_{\ ,22}(\infty)$ exists in L^2_x since by using Lemma 4.2.4 and $\alpha > b$,

$$\|[\partial_s[\bar{F}_1(|P| \quad s^{\beta})], F_c(\frac{|x|}{s^{\alpha}} \quad 1)]\langle P\rangle u\ (s)\|_{L^2_x} \lesssim_n \frac{1}{s^{1+\alpha \quad b}}\|\langle P\rangle u\ (s)\|_{L^2_x} \in L^1_s[1, \infty). \tag{4.169}$$

For $u_{\ ,21}(t)$, we use propagation estimates. Choose

$$B_{11}(t) := F_c(\frac{|x|}{t^{\alpha}} \quad 1)F_1(t^b|P| > 1)F_c(\frac{|x|}{t^{\alpha}} \quad 1) \tag{4.170}$$

and let

$$\langle B_{11}(t) : \langle P\rangle u\ (t)\rangle := (\langle P\rangle u\ (t), B_{11}(t)\langle P\rangle u\ (t))_{L^2_x}. \tag{4.171}$$

Let

$$R_1(t) := (\langle P\rangle u\ (t), F_cF_1\partial_t[F_c]\langle P\rangle u\ (t))_{L^2_x} + (\langle P\rangle u\ (t), \partial_t[F_c]F_1F_c\langle P\rangle u\ (t))_{L^2_x} \quad 2(\langle P\rangle u\ (t), \sqrt{F_1}F_c\partial_t[F_c]\sqrt{F_1}\langle P\rangle u\ (t))_{L^2_x}. \tag{4.172}$$

Compute $\partial_t\langle B_{11}(t) : \langle P\rangle u\ (t)\rangle$

$$\begin{aligned}\partial_t\langle B_{11}(t) : \langle P\rangle u\ (t)\rangle &= (\langle P\rangle u\ (t), F_c\partial_t[F_1]F_c\langle P\rangle u\ (t))_{L^2_x} + \\ &(\langle P\rangle\partial_t[u\ (t)], F_cF_1F_c\langle P\rangle u\ (t))_{L^2_x} + (\langle P\rangle u\ (t), F_cF_1F_c\langle P\rangle\partial_t[u\ (t)])_{L^2_x} + R_1(t) + \\ &2(\langle P\rangle u\ (t), \sqrt{F_1}F_c\partial_t[F_c]\sqrt{F_1}\langle P\rangle u\ (t))_{L^2_x} \\ &=: A_{11}(t) + A_{12}(t) + A_{13}(t) + R_1(t) + A_{14}(t).\end{aligned} \tag{4.173}$$

$A_{11}(t), A_{14}(t) \quad 0$ and $A_{12}(t), A_{13}(t), R_1(t) \in L^1_t[1, \infty)$ due to Lemma 4.2.3 and Lemma 4.2.4. Then by using propagation estimates, we have that $A_{11}(\infty)$ exists which implies that $u_{\ ,21}(\infty)$ exists in L^2_x. Hence,

$$u_{\ ,2}(\infty) := \lim_{t\to\infty} u_{\ ,2}(t) \text{ exists in } L^2_x. \tag{4.174}$$

Hence, based on (4.167), (4.174), (4.159), we have

$$\tilde{u}_+ := s\text{-}\lim_{t\to\infty}\langle P\rangle^{\ 1}F_c(\frac{|x|}{t^{\alpha}} \quad 1)F_1(t^b|P| > 1)\langle P\rangle u\ (t) \tag{4.175}$$

exists in H^1 which is equivalent to the existence of

$$u_+ := s\text{-}\lim_{t\to\infty} F_c(\frac{|x|}{t^{\alpha}} \quad 1)F_1(t^b|P| > 1)u\ (t) \tag{4.176}$$

in H^1. Similarly, for $F_c(\frac{|x|}{t^\alpha} - 1)F_1(t^b|P| > 1)\dot u_-(t)$, use Duhamel's formula to expand it

$$
\begin{aligned}
&F_c(\frac{|x|}{t^\alpha} - 1)F_1(t^b|P| > 1)\dot u_-(t) = F_c(|x| - 1)F_1(|P| > 1)\dot u_-(1)+\\
&\int_1^t ds\partial_s[F_c(\frac{|x|}{s^\alpha} - 1)]F_1(s^b|P| > 1)\dot u_-(s) + \int_1^t dsF_c(\frac{|x|}{s^\alpha} - 1)\partial_s[F_1(s^b|P| > 1)]\dot u_-(s)\\
&\qquad - \int_1^t dsF_c(\frac{|x|}{s^\alpha} - 1)F_1(s^b|P| > 1)\cos(s\sqrt{H_0+1})V(s)u(s)\\
&\qquad =: F_c(|x| - 1)\bar F_1(|P| \le 1)\dot u_-(1) + \dot u_{-,1}(t) + \dot u_{-,2}(t) + \dot u_{-,3}(t). \quad (4.177)
\end{aligned}
$$

$\dot u_{-,3}(\infty)$ exists in L^2_x due to Lemma 4.2.3. For $\dot u_{-,2}(t)$, break it into two pieces

$$
\begin{aligned}
\dot u_{-,2}(t) = &\int_1^t ds\partial_s[\bar F_1(s^b|P| > 1)]F_c(\frac{|x|}{s^\alpha} - 1)\dot u_-(s)\\
&- \int_1^t ds[\partial_s[\bar F_1(s^b|P| > 1)], F_c(\frac{|x|}{s^\alpha} - 1)]\dot u_-(s) =: \dot u_{-,21}(t) + \dot u_{-,22}(t) \quad (4.178)
\end{aligned}
$$

$\dot u_{-,22}(\infty)$ exists in L^2_x due to Lemma 4.2.4. Both $\dot u_{-,1}(\infty)$ and $\dot u_{-,21}(\infty)$ exist in L^2_x by using Lemma 4.2.3, Lemma 4.2.4 and propagation estimates via observing

$$
\begin{cases}
\langle B_2(t) : \dot u_{-,1}(t)\rangle\\
B_2(t) = B_1(t)
\end{cases}
\text{and}
\begin{cases}
\langle B_{21}(t) : \dot u_{-,21}(t)\rangle\\
B_{21}(t) = B_{11}(t)
\end{cases}
\quad (4.179)
$$

respectively. Then

$$
\dot u_+ := s\text{-}\lim_{t\to\infty} F_c(\frac{|x|}{t^\alpha} - 1)F_1(t^b|P| > 1)\dot u_-(t) \quad (4.180)
$$

exists in L^2_x. We finish the proof for (1). Before going to the proof of the second part of Theorem 4.1.3, let us remind you of the Duhamel's formulas for $u(t), \dot u(t)$,

$$
\begin{aligned}
u(t) = &\left(\cos(t\sqrt{H_0+1})u(0) + \frac{\sin(t\sqrt{H_0+1})}{\sqrt{H_0+1}}\dot u(0)\right) +\\
&\int_0^t ds\frac{e^{-i(t-s)\sqrt{H_0+1}}}{2i\sqrt{H_0+1}}V(s)u(s) - \int_0^t ds\frac{e^{i(t-s)\sqrt{H_0+1}}}{2i\sqrt{H_0+1}}V(s)u(s)\\
&=: u_f(t) + u_+(t) + u_-(t). \quad (4.181)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot u(t) = &-\sin(t\sqrt{H_0+1})\sqrt{H_0+1}u(0) + \cos(t\sqrt{H_0+1})\dot u(0)\\
&- \int_0^t ds\frac{e^{i(t-s)\sqrt{H_0+1}}}{2}V(s)u(s) - \int_0^t ds\frac{e^{-i(t-s)\sqrt{H_0+1}}}{2}V(s)u(s)\\
&=: \dot u_f(t) + \dot u_+(t) + \dot u_-(t). \quad (4.182)
\end{aligned}
$$

For the second part of Theorem 4.1.3, it follows from following theorem and we defer its proof to the end of this section:

Theorem 4.3.1. *Let e, α, b as in Theorem 4.1.3 and $V(x,t)$ as in Theorem 4.1.3 for some $\delta > 2$, there exist $u_{e,\alpha,b}, v_{e,\alpha,b} \in H^1_x, \dot{u}_{e,\alpha,b}, \dot{v}_{e,\alpha,b} \in L^2_x$ such that*

$$\lim_{t\to\infty} \| u(t) - \cos(t\sqrt{H_0+1})u_{e,\alpha,b} - \frac{\sin(t\sqrt{H_0+1})}{\sqrt{H_0+1}}\dot{u}_{e,\alpha,b} - u_{w,e,\alpha,b}(t)\|_{H^1_x} = 0 \quad (4.183)$$

$$\lim_{t\to\infty} \| \dot{u}(t) + \sin(t\sqrt{H_0+1})\sqrt{H_0+1}v_{e,\alpha,b} - \cos(t\sqrt{H_0+1})\dot{v}_{e,\alpha,b} - v_{w,e,\alpha,b}(t)\|_{L^2_x} = 0 \quad (4.184)$$

where $u_{w,e,\alpha,b}(t), v_{w,e,\alpha,b}(t)$ are weakly localized parts in the following sense

$$(u_{w,e,\alpha,b}(t), |x| u_{w,e,\alpha,b}(t))_{L^2_x} \lesssim_b t^e C(\|\vec{u}(0)\|_S)^2, \quad (4.185)$$

$$(v_{w,e,\alpha,b}(t), |x| v_{w,e,\alpha,b}(t))_{L^2_x} \lesssim_b t^e C(\|\vec{u}(0)\|_S)^2. \quad (4.186)$$

Based on Theorem 4.3.1, we get the second part of Theorem 4.1.3 by setting

$$u^1_{+,e,\alpha,b} := u^+_{e,\alpha,b} + u_{e,\alpha,b}, \quad \dot{u}^1_{+,e,\alpha,b} := \dot{u}^+_{e,\alpha,b} + \dot{u}_{e,\alpha,b}, \quad u_{w,e,\alpha,b}(t) := u^+_{w,e,\alpha,b} + u_{w,e,\alpha,b}(t), \quad (4.187)$$

$$u^2_{+,e,\alpha,b} := v^+_{e,\alpha,b} + v_{e,\alpha,b}, \quad \dot{v}^1_{+,e,\alpha,b} := \dot{v}^+_{e,\alpha,b} + \dot{v}_{e,\alpha,b}, \quad v_{w,e,\alpha,b}(t) := v^+_{w,e,\alpha,b} + v_{w,e,\alpha,b}(t) \quad (4.188)$$

and finish the proof for Theorem 4.1.3. □

Before proving Theorem 4.3.1, we have to introduce some lemmas. Based on the proof of the first part of Theorem 4.1.3, we deduce following lemma:

Lemma 4.3.1. *Let $(u(t), \dot{u}(t))$ be a global solution to equation (4.3) in S. If $V(x,t), u(t)$ satisfy (1), then for $n \geq 1, b \in (0, 1/2)$, $\alpha \in (b, \min(1-b, 1-\frac{2-\delta}{n}))$,*

$$\Omega_{\alpha,}\ \vec{u}(0) := s\text{-}\lim_{t\to\infty} F_c F_1 \int_0^t ds e^{-is\sqrt{H_0+1}} V(s)u(s) \quad (4.189)$$

exists in S for any $\vec{u}(0) \in S$, and

$$w\text{-}\lim_{t\to\infty} (1 - F_c F_1) \int_0^t ds e^{-is\sqrt{H_0+1}} V(s)u(s) = 0. \quad (4.190)$$

Proof. The proof of Lemma 4.3.1 follows from a similar argument for Theorem 1. (4.190) follows from that for each $\phi(x)\in L^2_x$,

$$\|(1-F_c(\frac{|x|}{t^\alpha}\le 1)F_1(t^{1/2-\epsilon}|P|>1))\phi(x)\|_{L^2_x}\to 0,\quad \text{as } t\to\infty. \tag{4.191}$$

□

Before we prove Theorem 4.3.1, we need following lemma:

Lemma 4.3.2 (Minimal and Maximal velocity bounds)**.** *For* $a>0, c\in(0,t]$, $e>1-b>\alpha>b>0, t\ge 1$, $j=1,\cdots,n$,

$$\|(F_2(\frac{x_j}{t^e}>1))F_1(t^bP_j>1/10)e^{ia\sqrt{H_0+1}}\langle x_j\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)}\lesssim_b \frac{1}{|t^e+\sqrt{a}|^\delta}, \tag{4.192}$$

$$\|(F_2(\frac{x_j}{t^e}>1))F_1(t^bP_j\le 1/10)e^{-ic\sqrt{H_0+1}}\langle x_j\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)}\lesssim_b \frac{1}{|t^e+\sqrt{c}|^\delta}, \tag{4.193}$$

$$\|(F_2(-\frac{x_j}{t^e}>1))F_1(-t^bP_j>1/10)e^{ia\sqrt{H_0+1}}\langle x_j\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)}\lesssim_b \frac{1}{|t^e+\sqrt{a}|^\delta}, \tag{4.194}$$

$$\|(F_2(-\frac{x_j}{t^e}>1))F_1(-t^bP_j\le 1/10)e^{-ic\sqrt{H_0+1}}\langle x_j\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)}\lesssim_b \frac{1}{|t^e+\sqrt{c}|^\delta}, \tag{4.195}$$

$$\begin{aligned}\|F_{2,t}(-x_j>t^e)\bar{F}_1(-t^bP_j\le 1/10)e^{-it\sqrt{H_0+1}}\bar{F}_c(\frac{|x|}{t^\alpha}>1)e^{it\sqrt{H_0+1}}e^{-ic\sqrt{H_0+1}}\langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)}\\ \lesssim_b \frac{1}{|t^e+\sqrt{c}|^\delta}.\end{aligned} \tag{4.196}$$

Remark 20. *When we use Lemma 4.3.2, we need $\delta>2$ in order to make it integrable in a or b when $|a|,|b|\ge 1$.*

Proof of Lemma 4.3.2. It is sufficient to check the case when $j=1$. Break the LHS of (4.192) into two pieces

$$\begin{aligned}(F_2(\frac{x_1}{t^e}>1))F_1(t^bP_1>1/10)e^{ia\sqrt{H_0+1}}\langle x_1\rangle^{-\delta}=&\\ (F_2(\frac{x_1}{t^e}>1))F_1(t^bP_1>1/10)e^{ia\sqrt{H_0+1}}\langle x_1\rangle^{-\delta}\chi(|x_1|\ge(t^e+\sqrt{a})/1000)+&\\ (F_2(\frac{x_1}{t^e}>1))F_1(t^bP_1>1/10)e^{ia\sqrt{H_0+1}}\langle x_1\rangle^{-\delta}\chi(|x_1|<(t^e+\sqrt{a})/1000)&\\ =: A_1+A_2.&\end{aligned} \tag{4.197}$$

For A_1,

$$\|A_1\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim \|(F_2(\frac{x_1}{t^e}>1))F_1(t^bP_1>1/10)e^{ia\sqrt{H_0+1}}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)}\frac{1}{|t^e+\sqrt{a}|^\delta}$$
$$\lesssim \frac{1}{|t^e+\sqrt{a}|^\delta}. \quad (4.198)$$

For A_2, since by using factor $(F_2(\frac{x_1}{t^e}>1))F_1(t^bq_1>1/10)$ and factor $\chi(|y_1|<(t^e+\sqrt{a})/1000)$,

$$e^{ix_1q_1}e^{ia\sqrt{|q|^2+1}}e^{-iq_1y_1} = \frac{1}{i(x_1+\frac{aq_1}{\sqrt{q^2+1}}-y_1)}\partial_{q_1}[e^{ix_1q_1}e^{ia\sqrt{q^2+1}}e^{-iq_1y_1}] \quad (4.199)$$

with

$$|x_1+\frac{aq_1}{\sqrt{q^2+1}}-y_1| \gtrsim t^e\chi(|a|\le t^{2e})+\sqrt{a}\chi(a>t^{2e}) \gtrsim |t^e+\sqrt{a}|, \quad (4.200)$$

we have

$$\|A_2\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_\epsilon \frac{1}{|t^e+\sqrt{a}|^\delta} \quad (4.201)$$

via taking integration by parts in q_1 for enough times. Thus, we get (4.192). Similarly, we get (4.193), (4.194) and (4.195). For (4.196),

$$\text{LHS of (4.196)} \le \text{LHS of (4.193) or LHS of (4.195)} + R \quad (4.202)$$

with

$$R := \|F_2(-x_j>t^e)\bar{F}_1(-t^bP_j\le 1/10)$$
$$e^{-it\sqrt{H_0+1}}F_c(\frac{|x|}{t^\alpha}\le 1)e^{it\sqrt{H_0+1}}e^{-ic\sqrt{H_0+1}}\langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)}. \quad (4.203)$$

Since

$$R \le \|F_2(-x_j>t^e)\bar{F}_1(-t^bP_j\le 1/10)e^{-it\sqrt{H_0+1}}F_c(\frac{|x|}{t^\alpha}\le 1)\langle x\rangle^{\delta}\|_{L^2_x\to L^2_x}$$
$$\times\|\langle x\rangle^{-\delta}e^{it\sqrt{H_0+1}}e^{-ic\sqrt{H_0+1}}\langle x\rangle^{-\delta}\|_{L^2_x\to L^2_x}$$
$$\lesssim_{N,b}\frac{1}{t^N}\frac{1}{\langle t-c\rangle^\delta}$$
$$(\text{choose } N \text{ sufficiently large}) \lesssim_{N,b}\frac{1}{\langle |t|+|c|\rangle^\delta}, \quad (4.204)$$

by using (4.193), (4.195), we get (4.196) and finish the proof. □

Proof of Theorem 4.3.1. First of all, we consider $u_+(t)$. Let

$$u_s(t) := \int_0^t ds \langle P\rangle^{-1} e^{-it\sqrt{H_0+1}} F_c(\frac{|x|}{t^\alpha} \le 1) F_1(t^b|P| > 1) \frac{e^{is\sqrt{H_0+1}}}{2i} V(s)u(s) \tag{4.205}$$

and

$$u_w(t) := \int_0^t ds \langle P\rangle^{-1} e^{-it\sqrt{H_0+1}} \bar{F}_{c,1} \frac{e^{is\sqrt{H_0+1}}}{2i} V(s)u(s) \tag{4.206}$$

where

$$\bar{F}_{c,1} := 1 - F_c(\frac{|x|}{t^\alpha} \le 1) F_1(t^b|P| > 1). \tag{4.207}$$

Then

$$u_+(t) = u_s(t) + u_w(t). \tag{4.208}$$

Based on Lemma 4.3.1, we know that

$$\| u_s(t) - e^{-it\sqrt{H_0+1}} u_s(\infty) \|_{H^1_x} \to 0, \text{ as } t \to \infty. \tag{4.209}$$

Then for $u_+(t)$, it is sufficient to show that $u_w(t)$ is equal to a sum of a localized part and a part which will go to 0 as $t \to \infty$. In the following context, we will prove such decomposition. Let

$$u_{j,+}(t) := \langle P\rangle^{-1} F_{2,t}(x_j > t^e) \langle P\rangle u_w(t) \tag{4.210}$$

and

$$u_{j,-}(t) := \langle P\rangle^{-1} F_{2,t}(-x_j > t^e) \langle P\rangle u_w(t) \tag{4.211}$$

where

$$F_{2,t}(x_j > t^e) := \left(\Pi_{l=1}^{j-1} \bar{F}_2(|x_l| \le t^e)\right) F_2(x_j > t^e), \tag{4.212}$$

and

$$F_{2,t}(-x_j > t^e) := \left(\Pi_{l=1}^{j-1} \bar{F}_2(|x_l| \le t^e)\right) F_2(-x_j > t^e). \tag{4.213}$$

Then

$$u_w(t) = \langle P\rangle^{-1} \left(\Pi_{l=1}^{n} \bar{F}_2(|x_l| \le t^e)\right) \langle P\rangle u_w(t) + \sum_{j=1}^{n} \left(u_{j,+}(t) + u_{j,-}(t)\right). \tag{4.214}$$

Set

$$u^{+}_{w,e,\alpha,b}(t) := \langle P\rangle^{-1} \left(\Pi_{l=1}^{n} \bar{F}_2(|x_l| \le t^e)\right) \langle P\rangle u_w(t). \tag{4.215}$$

In the following, we will show

$$\|u_{j,\ }(t)\|_{L^2_x} \to 0, \text{ as } t \to \infty. \tag{4.216}$$

Break $u_{j,\ }(t)$ into three pieces

$$\begin{aligned} u_{j,\ }(t) = \langle P\rangle^{\ 1} F_{2,t}(\ x_j > t^e) F_1(\ t^b P_j > 1/10) \langle P\rangle u_+(t) \\ \langle P\rangle^{\ 1} F_{2,t}(\ x_j > t^e) F_1(\ t^b P_j > 1/10) \langle P\rangle u_s(t) + \\ \langle P\rangle^{\ 1} F_{2,t}(\ x_j > t^e) \bar{F}_1(\ t^b P_j \quad 1/10) \langle P\rangle u_w(t) =: u_{j,\ ,1} + u_{j,\ ,2} + u_{j,\ ,r}. \end{aligned} \tag{4.217}$$

According Lemma 4.3.2, we have

$$\|u_{j,\ ,r}\|_{L^2_x(\mathbb{R}^n)} \to 0, \text{ as } t \to \infty. \tag{4.218}$$

For $u_{j,\ ,1} + u_{j,\ ,2}$, write it as

$$\begin{aligned} u_{j,\ ,1} + u_{j,\ ,2} = \langle P\rangle^{\ 1} \Big(F_{2,t}(\ x_j > t^e) F_1(\ t^b P_j > 1/10) \langle P\rangle e^{\ it\sqrt{H_0+1}} u_s \\ F_{2,t}(\ x_j > t^e) F_1(\ t^b P_j > 1/10) \langle P\rangle u_s(t) \Big) \\ \langle P\rangle^{\ 1} F_{2,t}(\ x_j > t^e) F_1(\ t^b P_j > 1/10) \langle P\rangle e^{\ it\sqrt{H_0+1}} (u_s \quad u_+(t)) \\ =: u_{j,\ ,1,1} + u_{j,\ ,1,2} \end{aligned} \tag{4.219}$$

where

$$u_s := \int_0^\infty ds \frac{e^{it\sqrt{H_0+1}}}{2i\sqrt{H_0+1}} e^{\ is\sqrt{H_0+1}} V(s)u(s). \tag{4.220}$$

For $u_{j,\ ,1,2}$,

$$\begin{aligned} u_{j,\ ,1,2} = \quad \langle P\rangle^{\ 1} F_{2,t}(\ x_j > t^e) F_1(\ t^b P_j > 1/10) \\ e^{\ it\sqrt{H_0+1}} \int_t^\infty ds \frac{e^{is\sqrt{H_0+1}}}{2i} V(s)u(s). \end{aligned} \tag{4.221}$$

Due to Lemma 4.3.2,

$$\|u_{j,\ ,1,2}(t)\|_{H^1(\mathbb{R}^n)} \to 0, \text{ as } t \to \infty. \tag{4.222}$$

For $u_{j,\ ,1,1}(t)$, due to (4.222), we have $u_{j,\ ,1,1}(t) \in H^1$, which means

$$\langle P\rangle^{\ 1} F_{2,t}(\ x_j > t^e) F_1(\ t^b P_j > 1/10) \langle P\rangle e^{\ it\sqrt{H_0+1}} u_s \in H^1. \tag{4.223}$$

If we can show that in L^2_x,

$$u_s = u_s(\infty), \quad \text{in the weak sense,} \tag{4.224}$$

then due to (4.223),

$$\langle P\rangle^{-1}F_{2,t}(-x_j > t^e)F_1(-t^bP_j > 1/10)\langle P\rangle e^{-it\sqrt{H_0+1}}u_s = \langle P\rangle^{-1}F_{2,t}(-x_j > t^e)F_1(-t^bP_j > 1/10)\langle P\rangle e^{-it\sqrt{H_0+1}}u_s(\infty) \tag{4.225}$$

and therefore by using Lemma 4.3.1,

$$\|u_{j,-,1,1}(t)\|_{L^2_x(\mathbb{R}^n)} \to 0, \text{ as } t \to \infty. \tag{4.226}$$

Now let us prove that in L^2_x,

$$u_s = u_s(\infty) \quad \text{in the weak sense.} \tag{4.227}$$

(4.227) is true due to (4.190). So we have

$$\|u_{j,-,1,1}(t)\|_{L^2_x(\mathbb{R}^n)} \to 0, \text{ as } t \to \infty. \tag{4.228}$$

Similarly, we get the same result for $u_-(t)$.

For $\dot{u}_+(t)$, let

$$\dot{u}_s(t) := -\int_0^t ds\, e^{-it\sqrt{H_0+1}}F_c(\frac{|x|}{t^\alpha} - 1)F_1(t^b|P| > 1)\frac{e^{is\sqrt{H_0+1}}}{2}V(s)u(s) \tag{4.229}$$

and

$$\dot{u}_w(t) := -\int_0^t ds\, e^{-it\sqrt{H_0+1}}\bar{F}_{c,1}\frac{e^{is\sqrt{H_0+1}}}{2}V(s)u(s). \tag{4.230}$$

Via a similar argument as what we did for $u_+(t)$, we get the same result for $\dot{u}_+(t)$ by setting

$$\dot{v}^+_{w,e,\alpha,b}(t) := \left(\Pi_{l=1}^n \bar{F}_2(|x_l| - t^e)\right)\dot{u}_w(t). \tag{4.231}$$

Similarly, we get the same result for $\dot{u}_-(t)$. We finish the proof. □

4.4 Applications

4.4.1 Estimates for free radiation

Let W^1_x denote the L^1_x Sobolev space. In this section, we show that if

$$\vec{u}(0)=\Omega_\alpha^\beta \vec{v}, \quad n\geq 3 \tag{4.232}$$

for some $\vec{v}\in L^2_x(\mathbb{R}^n)\times L^2_x(\mathbb{R}^n)$ satisfying

$$\langle P\rangle^{\frac{n+3}{2}}\vec{v}\in W^1_x(\mathbb{R}^n)\times L^1_x(\mathbb{R}^n), \tag{4.233}$$

then for $t\geq 1$,

$$\|\langle x\rangle^{-(n+1)/2}\vec{u}(t)\|_{L^2_x\times L^2_x}\lesssim_n \frac{1}{\langle t\rangle^{n/2-1}}\left(1+\|V(x,t)\|_{L^1_t L^2_x}\right)\|\langle P\rangle^{\frac{n+3}{2}}v\|_{W^1_x\times L^1_x}. \tag{4.234}$$

Here

$$\Omega_\alpha^\beta := (\Omega_\alpha^{\beta,*})^*. \tag{4.235}$$

Theorem 4.4.1. *When $n\geq 3$, let $u(t),\dot{u}(t)$ be as in (4.3) and $\vec{u}(0)$ be as in (4.232). If*

$$\sup_{t\in\mathbb{R}}\|\vec{u}(t)\|_S\lesssim_{\vec{u}(0)} 1 \tag{4.236}$$

and if $V(x,t)\in L^1_t L^2_x(\mathbb{R}^n\times\mathbb{R})$, $\langle P\rangle^{\frac{n+3}{2}}\vec{v}\in W^1_x(\mathbb{R}^n)\times L^1_x(\mathbb{R}^n)$, then for $t\geq 1$,

$$\|\langle x\rangle^{-(n+1)/2}\vec{u}(t)\|_{L^2_x\times L^2_x}\lesssim_n \frac{1}{\langle t\rangle^{n/2-1}}\left(1+\|V(x,t)\|_{L^1_t L^2_x}\right)\|\langle P\rangle^{\frac{n+3}{2}}v\|_{W^1_x\times L^1_x}. \tag{4.237}$$

Proof. Compute

$$\vec{u}(t)=U(t,0)\Omega_\alpha^\beta\vec{v}=s\text{-}\lim_{s\to\infty}U(t,s)U_0(s,0)\bar{F}_1(|P|\leq s^\beta)F_c(\frac{|x|}{s^\alpha}\leq 1)\vec{v} \tag{4.238}$$

$$=s\text{-}\lim_{s\to\infty}U(t,s)U_0(s,t)U_0(t,0)\bar{F}_1(|P|\leq s^\beta)F_c(\frac{|x|}{s^\alpha}\leq 1)\vec{v} \tag{4.239}$$

$$=s\text{-}\lim_{s\to\infty}U(t,t+s)U_0(t+s,t)U_0(t,0)\bar{F}_1(|P|\leq (t+s)^\beta)F_c(\frac{|x|}{(t+s)^\alpha}\leq 1)\vec{v} \tag{4.240}$$

By using Duhamel's formula to expand $U(t,s+t)U_0(t+s,t)$ in the expression of $\vec{u}(t)$, provided that $\vec{u}(t)$ exists in S,

$$\vec{u}(t) = s\text{-}\lim_{s\to\infty}\Big(U_0(t,0)\bar{F}_1(|P| - (t+s)^{\beta})F_c(\frac{|x|}{(t+s)^{\alpha}} - 1)\vec{v} + \int_0^s duU(t,t+u)V(t+u)U_0(u+t,0)\bar{F}_1(|P| - (t+s)^{\beta})F_c(\frac{|x|}{(t+s)^{\alpha}} - 1)\vec{v}\Big)$$
$$= U_0(t,0)\vec{v} + \int_0^{\infty} duU(t,t+u)V(t+u)U_0(u+t,0)\vec{v}. \quad (4.241)$$

Let

$$\vec{v} = \begin{pmatrix} v \\ \dot{v} \end{pmatrix}, \quad \vec{v}(t) = \begin{pmatrix} v(t) \\ \dot{v}(t) \end{pmatrix}. \quad (4.242)$$

Since using Lemma 4.2.1 and Hölder's inequality,

$$\|V(x,t+u)\cos((t+u)\sqrt{H_0+1})v\|_{L^2_x} \lesssim \frac{1}{\langle t+u\rangle^{n/2}}\|V(x,t)\|_{L^{\infty}_t L^2_x}\|\langle P\rangle^{\frac{n+3}{2}}\vec{v}\|_{L^1_x \cap L^1_x}, \quad (4.243)$$

similarly since

$$\|V(x,t+u)\frac{\sin((u+t)\sqrt{H_0+1})}{\sqrt{H_0+1}}v\|_{L^2_x} \lesssim \frac{1}{\langle t+u\rangle^{n/2}}\|V(x,t)\|_{L^{\infty}_t L^2_x}\|\langle P\rangle^{\frac{n+1}{2}}\vec{v}\|_{L^1_x \cap L^1_x}, \quad (4.244)$$

and since

$$\|\langle x\rangle^{-(n+1)/2}U_0(t,0)\vec{v}\|_{L^2_x \cap L^2_x} \lesssim \frac{1}{\langle t\rangle^{n/2}}\|\langle P\rangle^{\frac{n+3}{2}}\vec{v}\|_{W^1_x \cap L^1_x}, \quad (4.245)$$

we get

$$\|\langle x\rangle^{-(n+1)/2}\vec{u}(t)\|_{L^2_x \cap L^2_x} \lesssim \frac{1}{\langle t\rangle^{n/2}}\|\langle P\rangle^{\frac{n+3}{2}}v\|_{W^1_x \cap L^1_x} + \int_0^{\infty} du\|(0, V(x,t+u)v(u+s))^T\|_S$$
$$\lesssim \frac{1}{\langle t\rangle^{n/2}}\|\langle P\rangle^{\frac{n+3}{2}}v\|_{W^1_x \cap L^1_x} + \int_0^{\infty} du\frac{1}{\langle t+u\rangle^{n/2}}\|V(x,t)\|_{L^{\infty}_t L^2_x}\|\langle P\rangle^{\frac{n+3}{2}}\vec{v}\|_{L^1_x \cap L^1_x}$$
$$\lesssim_n \frac{1}{\langle t\rangle^{n/2-1}}\left(1+\|V(x,t)\|_{L^{\infty}_t L^2_x}\right)\|\langle P\rangle^{\frac{n+3}{2}}v\|_{W^1_x \cap L^1_x} \quad (4.246)$$

and finish the proof. □

Remark 21. *Indeed, such* $\frac{1}{\langle t\rangle^{n/2-1}}$ *can be improved if both of the potential and* $\vec{v}$ *are localized in space and if* $\vec{v}$ *has frequency away from* 0.

Discussion

The above result is a generalization of local decay estimates in the following sense: In the case of time independent linear interaction term that is also localized, the range of the wave operator above is equal to the range of the projection on the continuous spectral part of the Hamiltonian. In this case local decay holds for all localized initial data, and with rate of decay which is optimal. However, when the interaction term is time dependent and or nonlinear, there is no such decomposition. In this case the question arises as to what decay estimates hold for solutions which dipserse? That is solutions which asymptotically have no weakly localized part. The above estimate is in fact a decay estimate for such solutions.

4.4.2 Application to typical nonlinear examples

In this section, we show that for $a(x), b(x) \in L^{\infty}_{\delta,x}(\mathbb{R})$, $\delta > 2$, if there is a global solution $\vec{u}(t)$ in $\mathcal{S}$ with a uniform $\mathcal{S}$ bound for following Nonlinear KG equation in one space dimension

$$\begin{cases} (\quad + 1 + V(x))u = a(x)u^2 + b(x)u^3 \\ \vec{u}(0) := (u(x,0), \dot{u}(x,0)) = (u_0(x), \dot{u}_0(x)) \in \mathcal{S} \end{cases}, \quad (x,t) \in \mathbb{R}^1 \quad \mathbb{R}, \quad m > 0 \quad (4.247)$$

then the asymptotic behavior of the solution can be rewritten as the sum of a free part plus a weakly localized part:

Theorem 4.4.2. *If $a(x), b(x) \in L^{\infty}_{\delta,x}(\mathbb{R})$ for some $\delta > 2$ if there is a global solution $\vec{u}(t)$ to (4.247) in $\mathcal{S}$ satisfying*

$$C(\|\vec{u}(0)\|_{\mathcal{S}}) := \sup_{t \in \mathbb{R}} \|\vec{u}(t)\|_{\mathcal{S}} \lesssim_{\vec{u}(0)} 1, \quad (4.248)$$

where $V(x) \quad 0$ is a generic potential satisfying $V(x) \in L^2_{\sigma,x}$ for some $\sigma > 2$, then for α, b also satisfy

$$e > 1 \quad b > \alpha > b \quad 0, \quad (4.249)$$

there exist $u^1_{+,e,\alpha,b}, u^2_{+,e,\alpha,b} \in H^1_x, \dot{u}^1_{+,e,\alpha,b}, \dot{u}^2_{+,e,\alpha,b} \in L^2_x$ such that we have the following asymptotic decomposition

$$\lim_{t \to \infty} \|u(t) \quad \cos(t\sqrt{H_0+1})u^1_{+,e,\alpha,b} \quad \frac{\sin(t\sqrt{H_0+1})}{\sqrt{H_0+1}}\dot{u}^1_{+,e,\alpha,b} \quad u_{w,e,\alpha,b}(t)\|_{H^1_x} = 0 \quad (4.250)$$

and

$$\lim_{t\to\infty}\|\dot{u}(t)+\sin(t\sqrt{H_0+1})\sqrt{H_0+1}u^2_{+,e,\alpha,b}$$
$$-\cos(t\sqrt{H_0+1})\sqrt{H_0+1}\dot{u}^2_{+,e,\alpha,b}-v_{w,e,\alpha,b}(t)\|_{L^2_x}=0 \quad (4.251)$$

where $u_{w,e,\alpha,b}, v_{w,e,\alpha,b}$ *are the weakly localized parts of the solution, with the following property: It is weakly localized in the region* $|x|\leq t^e$*, in the following sense*

$$(\langle P\rangle u_{w,e,\alpha,b}(t), |x|\langle P\rangle u_{w,e,\alpha,b}(t))_{L^2_x} \lesssim_b t^e C(\|\vec{u}(0)\|_S)^2, \quad (4.252)$$

and

$$(v_{w,e,\alpha,b}(t), |x| v_{w,e,\alpha,b}(t))_{L^2_x} \lesssim_b t^e C(\|\vec{u}(0)\|_S)^2. \quad (4.253)$$

Proof. Due to (4.248), we have

$$\|a(x)u(x,t)\|_{L^\infty_t L^2_{\delta,x}} \leq \|a(x)\|_{L^\infty_{\delta,x}}\|u(x,t)\|_{L^\infty_t L^2_x} \leq C(\|\vec{u}(0)\|_S)\|a(x)\|_{L^\infty_{\delta,x}}, \quad (4.254)$$

$$\|a(x)u(x,t)^2\|_{L^\infty_t L^2_{\delta,x}} \leq \|a(x)\|_{L^\infty_{\delta,x}}\|u(x,t)\|_{L^\infty_t L^\infty_x}\|u(x,t)\|_{L^\infty_t L^2_x} \leq C(\|\vec{u}(0)\|_S)\|a(x)\|_{L^\infty_{\delta,x}}, \quad (4.255)$$

$$\|b(x)u(x,t)^2\|_{L^\infty_t L^2_{\delta,x}} \leq \|b(x)\|_{L^\infty_{\delta,x}}\|u(x,t)\|_{L^\infty_t L^\infty_x}\|u(x,t)\|_{L^\infty_t L^2_x} \leq C(\|\vec{u}(0)\|_S)^2\|a(x)\|_{L^\infty_{\delta,x}}, \quad (4.256)$$

and

$$\|b(x)u(x,t)^3\|_{L^\infty_t L^2_{\delta,x}} \leq \|b(x)\|_{L^\infty_{\delta,x}}\|u(x,t)\|^2_{L^\infty_t L^\infty_x}\|u(x,t)\|_{L^\infty_t L^2_x} \leq C(\|\vec{u}(0)\|_S)^2\|a(x)\|_{L^\infty_{\delta,x}}. \quad (4.257)$$

Since $V(x,t)=a(x)u+b(x)u^2 \in L^\infty_x \cap L^2_{\delta,x}(\mathbb{R})$ for some $\delta>2$, so by using Theorem 4.1.3, we get desired conclusion and finish the proof. □

When space dimension $n\geq 3$, if there is a global solution $u(t)$ in H^1, then when $N(u,x,t)=V(x,t)+\sum_{j=1}^{N}\lambda_j|u|^{p_j}$ for $1\leq p_j\leq \frac{n}{n-2}$, the channel wave operator exists:

Theorem 4.4.3. *Let* $(u(t),\dot{u}(t))$ *be as in Theorem 4.1.1. If* $V(x,t)\in L^\infty_t L^2_x(\mathbb{R}^n\times\mathbb{R})$ *and if* $N(u,x,t)=V(x,t)+\sum_{j=1}^{N}\lambda_j|u|^{p_j}$ *for* $\lambda_j>0, 1\leq p_j\leq\frac{n}{n-2}, V(x,t)$ *satisfying*

$$V(x,t)=\sum_{j=1}^{M}V_j(x-g_j(t)v_j,t), \quad V_j(x,t)\in L^\infty_t L^2_x, v_j\in\mathbb{R}^n, \text{ real functions } g_j(t), \quad (4.258)$$

and if

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} \lesssim \|\vec{u}(0)\|_S, \tag{4.259}$$

then the channel wave operator exists.

Proof. When $1 \le p_j \le \frac{n}{n-2}$,

$$\||u(t)|^p\|_{L_t^\infty L_x^2} \le C(\|u(t)\|_{L_x^2}, \|u(t)\|_{L_x^{\frac{2n}{n-2}}}) \le C(\|u(t)\|_{L_x^2}, \|u(t)\|_{H^1}). \tag{4.260}$$

$$\|V(x,t)\|_{L_t^\infty L_x^2} \le \sum_{j=1}^{M} \|V_j(x,t)\|_{L_t^\infty L_x^2} < \infty. \tag{4.261}$$

The assumptions of Theorem 4.1.1 is satisfied and we get the existence of channel wave operator. □

Chapter 5

Local decay estimates

5.1 Introduction

Local decay estimates are a-priory estimates on the solutions of dispersive equations. It states that the solution (for an initial condition associated with scattering) decay at least in an integrable rate in time, in every compact region of space, for a dense set of initial conditions. An equivalent statement is that the resolvent of the Hamiltonian of the dynamics, is bounded on properly weighted L^2 space.

Such estimates played a crucial role in scattering theory, as they imply the existence and completeness of the Möller Wave Operators. Moreover, there are many crucial applications of local decay and other propagation estimates, that go beyond the proof of asymptotic completeness(AC for short); for example in linear and non-linear time dependent resonance theory [84, 82]. Important consequences of Local decay include Strichartz estimates, and propagation of regularity for nonlinear Dispersive equations [95]. It is an estimate on the RATE of convergence to the asymptotic dynamics.

Of course, proving such estimates for an interacting system is difficult, as we do not have the method of stationary phase to apply directly. It turns out that it is possible to prove AC without using Local Decay. This was first shown by V. Enss in 1978 [21]. The Enss method applied to both two and three particle scattering in Quantum Mechanics, but the proof of AC for four or more particles, required the proof of Local Decay [75].

This was done using the Mourre Estimate combined with the Mourre's method of Differential Inequalities [62, 61, 64, 22], or the method of Propagation Estimates of Sigal-Soffer [79, 80, 42, 24].

Recently, B. Liu and the first author introduced a general approach to proving AC

[58, 57]. Their method applies to both linear and non linear dispersive equations. Their method requires localization assumption on the interaction terms. So their method requires radial symmetry assumption for nonlinear equations.

Later the authors improved Liu-S. method by constructing the free channel wave operator in a new way [91, 90, 89]. Without radial symmetry assumption and localization assumption on the interaction terms, the method initiated in [91, 90, 89], works for both linear and non linear dispersive equations, including time dependent interaction terms and Klein-Gordon type equations.

Neither method uses local decay. Yet, it is not clear how to directly prove local decay in such cases. In this work, we use the knowledge of AC based on this new approach [91] to derive local decay estimates. In this article we demonstrate this for the two body time independent Schrödinger equation, and then for a potential that is quasi-periodic in time (and localized in space).

Time dependent potentials were treated before. Most notable is the work of Rodnianski-Schlag [71], where L^p estimates are proved for time-dependent potentials with a smallness assumption on the size of the (time-dependent part) potentials. See also [31, 99, 87, 23] for recent new progress in this direction.

5.1.1 Main results

Let $H_0 := -\Delta_x, H := H_0 + V(x)$. We consider first the time-independent Schrödinger equation

$$\begin{cases} i\partial_t \psi(x,t) = H\psi(x,t) \\ \psi(x,0) = \psi_0 \in L^2_x(\mathbb{R}^3) \end{cases}, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \tag{5.1}$$

for $V(x)$ satisfying $\langle x\rangle^\delta V(x) \in L^\infty_x(\mathbb{R}^3)$ for some $\delta > 6$. Here $\langle\cdot\rangle : \mathbb{R}^3 \to \mathbb{R}, x \mapsto \sqrt{|x|^2+1}$ denotes Japanese brackets. For $m > 0$, let $F(k \geq m)$ denote a smooth characteristic function with

$$F(k \geq m) = \begin{cases} 1 & \text{when } k/m > 2 \\ 0 & \text{when } k/m < 1/2 \end{cases}, \tag{5.2}$$

and let $\bar{F}$ denote the complement of F. Sometimes we also use $F(k < c)$ or $F(k \geq c)$ to denote smooth characteristic functions. Throughout this paper, $A \lesssim_a B$ means $A \leq CB$ for some constant $C = C(a)$.

We prove that if $V(x)$ is localized in x, there is a local decay estimate

$$\int_{-\infty}^{\infty} dt \|\langle x\rangle^{-\eta} e^{-itH} P_c \psi_0\|^2_{L^2_x(\mathbb{R}^3)} \lesssim_\eta \|\psi_0\|^2_{L^2_x(\mathbb{R}^3)} \tag{5.3}$$

for any $\eta > 1$. Here P_c denotes the projection on the space of scattering states; it is equal to the projection on the continuous spectral subspace of H, due to AC. Similar result can be proved in 4 or higher space dimensions.

Assumption 5.1.1. *$\langle x\rangle^{\delta} V(x) \in L^1_x(\mathbb{R}^3)$ for some $\delta > 6$.*

Remark 22. *This condition is not optimal. It can be made optimal by a standard scaling argument. Since our focus is the time dependent potential case, we will not give the details.*

Theorem 5.1.1. *Assume $V(x)$ satisfies Assumption 5.1.1. Then if 0 is neither an eigenvalue nor a resonance for H, for $\psi \in L^2_x(\mathbb{R}^3)$, any $\eta > 1$, (5.3) is true.*

This method applies to systems with time-dependent potentials $V(x,t)$ which can be quasi-periodic in t. When the potential is time dependent, we rely on known results for proving global existence in L^2. Another method of proving global existence was developed in [87], where explicit conditions are given on the potential that imply a bound in L^p on the composition of the Full dynamics (forward) and the free dynamics(backward) in time.

If the Fourier transform of $V(x,t)$ in t is a finite measure, then by general principles based on Wiener's theorem, it can be written as a sum of almost periodic potential and a part that decays in time. The decaying part will be small eventually, so we focus on the quasi periodic case.

We illustrate the method for the case of quasi-periodic potential with N frequencies(N

2). To be precise, we consider the Schrödinger equation with a quasi-periodic time-dependent potential

$$\begin{cases} i\partial_t\psi(x,t) = (H_0 + V(x,t))\psi(x,t), \\ \psi(x,0) = \psi_0 \in L^2_x(\mathbb{R}^5) \end{cases}, (x,t) \in \mathbb{R}^5 \times \mathbb{R}, \tag{5.4}$$

where real-valued $V(x,t)$ has the form

$$V(x,t) = V_0(x) + \sum_{j=1}^{N_1} V_j(x)\sin(\omega_j t) + \sum_{j=N_1+1}^{N} V_j(x)\cos(\omega_j t), \quad \omega_j \in \mathbb{R} \setminus \{0\}. \tag{5.5}$$

Here ω_j are irrationally related to each other and $V_j(x)$ satisfy following assumption.

Assumption 5.1.2. *$\langle x\rangle^\delta V_j(x) \in L^\infty_x(\mathbb{R}^5)$ for some $\delta > 6$, $j = 0, 1, \cdots, N$. In this case, global existence of the solution in L^2 is known. See e.g. [67].*

A typical example of (5.5) is

$$V(x,t) = V_0(x) + V_1(x)\sin(t) + V_2(x)\sin(\sqrt{2}t). \tag{5.6}$$

We have shown in [91] that when $V(x,t) \in L^\infty_t L^2_x(\mathbb{R}^{n+1})$, $n \geq 3$, the free channel wave operator

$$\Omega_{\alpha,\pm} := s\text{-}\lim_{t\to\pm\infty} e^{itH_0} F_c\left(\frac{|x - 2tP|}{t^\alpha} \leq 1\right)U(t,0), \quad \text{on } L^2_x(\mathbb{R}^n) \tag{5.7}$$

exists for all $\alpha \in (0, 1-\frac{2}{n})$. Here $P := -i\nabla_x$, $U(t,0)$ denotes the solution operator to (6.1) and F_c, a smooth characteristic function.

Now let us introduce the assumption on the "time-dependent" bound states. For this, we need the notion of a Floquet system. For $\vec{s} \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_N (\mathbb{T}_j := \mathbb{R}/[0, 2\pi/|\omega_j|])$, let $U_{\vec{s}}(t,0)$ denote the solution operator to system

$$i\partial_t U_{\vec{s}}(t,0) = (-\Delta_x + V_0(x) + \sum_{j=1}^{N_1} V_j(x)\sin(\omega_j(t+s_j)) + \sum_{j=N_1+1}^{N} V_j(x)\cos(\omega_j(t+s_j)))U_{\vec{s}}(t,0). \tag{5.8}$$

Then

$$e^{itH_0}U_{\vec{s}}(t,0) = e^{itK_0}e^{-itK}, \quad \text{on } L^2_x(\mathbb{R}^n) \tag{5.9}$$

where $K_0 := H_0 + \sum_{j=1}^{N} P_{s_j}$, the free Floquet operator,

$$V_F(x,\vec{s}) := V_0(x) + \sum_{j=1}^{N_1} V_j(x)\sin(\omega_j s_j) + \sum_{j=N_1+1}^{N} V_j(x)\cos(\omega_j s_j), \tag{5.10}$$

the Floquet potential, and $K := K_0 + V_F(x, \vec{s})$, the perturbed Floquet operator. K is the hamiltonian of the Floquet system

$$i\partial_t \psi(x, \vec{s}, t) = K\psi_{x,\vec{s},t}, \quad (x, \vec{s}, t) \in \mathbb{R}^5 \times \mathbb{T}_1 \times \cdots \times \mathbb{T}_N \times \mathbb{R}. \tag{5.11}$$

Lemma 5.1.1. (5.9) *is true.*

Proof. It follows from the fact that

$$e^{-it\sum_{j=1}^{N} P_{s_j}} U_{\vec{s}}(t, 0) = e^{-itK}. \tag{5.12}$$

(5.12) is true since

$$\begin{aligned} i\partial_t[e^{-it\sum_{j=1}^{N} P_{s_j}} U_{\vec{s}}(t,0)] &= (H_0 + V_0(x) + \sum_{j=1}^{N} P_{s_j})e^{-it\sum_{j=1}^{N} P_{s_j}} U_{\vec{s}}(t,0) + \\ e^{-it\sum_{j=1}^{N} P_{s_j}} &\left(\sum_{j=1}^{N_1} V_j(x)\sin(\omega_j(t+s_j)) + \sum_{j=N_1+1}^{N} V_j(x)\cos(\omega_j(t+s_j))\right) U_{\vec{s}}(t,0) \\ &= (H_0 + V_0(x) + \sum_{j=1}^{N} P_{s_j})e^{-it\sum_{j=1}^{N} P_{s_j}} U_{\vec{s}}(t,0) + \left(\sum_{j=1}^{N_1} V_j(x)\sin(\omega_j s_j) + \right. \\ &\left. \sum_{j=N_1+1}^{N} V_j(x)\cos(\omega_j s_j)\right) e^{-it\sum_{j=1}^{N} P_{s_j}} U_{\vec{s}}(t,0). \end{aligned} \tag{5.13}$$

Here we use

$$e^{-it\sum_{j=1}^{N} P_{s_j}} f(s_l + t) e^{it\sum_{j=1}^{N} P_{s_j}} = f(s_l) \quad \text{for } l = 1, \cdots, N. \tag{5.14}$$

□

Let $P_{F,b}$ denote the projection on the space of all bound states of the Floquet operator K in $\mathcal{H}_F$ with

$$\mathcal{H}_F := L^2_{\vec{s}} L^2_x(\mathbb{R}^5 \times \mathbb{T}_1 \times \cdots \times \mathbb{T}_{N_1+N_2}). \tag{5.15}$$

Then for any $f(x) \in L^2_x(\mathbb{R}^5)$, $f(x) \in \mathcal{H}_F$. So $P_{F,b}f(x)$ is well-defined.

Assumption 5.1.3. *For all* $\eta \in [0, 3]$,

$$\sup_{\vec{s} \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_{N_1+N_2}} \|P_{F,b}\langle x\rangle^{\eta} f(x)\|_{L^2_x(\mathbb{R}^5)} \lesssim \|f(x)\|_{L^2_x(\mathbb{R}^5)}. \tag{5.16}$$

Remark 23. *See* (5.228) *for the relationship between* $P_{b,F}f(x)$ *and* $P_b(t)f(x)$. *See Lemma 5.3.8 for the dimension of the space of time-dependent eigenfunctions(bound states). So one can see Assumption 5.1.3 is equivalent to that all bound states are localized in space since for each time* t, *there are finitely many linearly independent bound states.*

Let

$$P_c(t) := s\text{-}\lim_{v\to\infty} U(t,t+v)\bar{F}_c(\frac{|x-2vP|}{v^{\alpha}}\leq 1)U(t+v,t) \quad \text{on } L^2_x(\mathbb{R}^5). \tag{5.17}$$

The existence of $P_c(t)$ follows from the unitarity of $U(t,v+t)$ on $L^2_x(\mathbb{R}^5)$ and the existence of

$$s\text{-}\lim_{v\to\infty} e^{ivH_0}\bar{F}_c(\frac{|x-2vP|}{v^{\alpha}}\leq 1)U(t+v,t), \quad \text{on } L^2_x(\mathbb{R}^5) \tag{5.18}$$

and

$$s\text{-}\lim_{v\to\infty} U(t,t+v)\bar{F}_c(\frac{|x-2vP|}{v^{\alpha}}\leq 1)e^{-ivH_0}, \quad \text{on } L^2_x(\mathbb{R}^5). \tag{5.19}$$

The existence of the above limits follows from [91]. Let

$$\Omega_{\alpha,+} := s\text{-}\lim_{t\to\infty} e^{itH_0}F_c(\frac{|x-2tP|}{t^{\alpha}}\leq 1)U(t,0) \quad \text{on } L^2_x(\mathbb{R}^n), n\geq 3 \tag{5.20}$$

and

$$\Omega_{+} := s\text{-}\lim_{t\to\infty} e^{itH_0}U(t,0)P_c(0) \quad \text{on } L^2_x(\mathbb{R}^n), n\geq 3. \tag{5.21}$$

Existence of $\Omega_{\alpha,+}$ for $\alpha\in(0,1-2/n)$ was proved in [91]. It implies the space of all scattering states is equal to the range of the channel wave operator $\Omega_{\alpha,+}$, which is equal to the range of Ω_+.

Remark 24. *Assumption 5.1.3 implies that the projection on the space of non-scattering states preserves certain localization in space*

$$\|\langle x\rangle^{\sigma}(1-P_c(t))f(x)\|_{L^2_x(\mathbb{R}^5)} \lesssim \|f(x)\|_{L^2_x(\mathbb{R}^5)} \tag{5.22}$$

since by setting

$$t_j := t-[\frac{t|\omega_j|}{2\pi}]\frac{2\pi}{|\omega_j|}, \tag{5.23}$$

$$(1 - P_c(t))f(x) = (1 - s\text{-}\lim_{v \to \infty} e^{ivK} \bar{F}_c(\frac{|x - 2vP|}{v^\alpha} \leq 1)e^{-ivK})f(x) = [P_{F,b}f(x)]|_{\vec{s}=\vec{t}}. \tag{5.24}$$

It is a reasonable assumption since the interaction term is localized in space. In time-independent cases, it is well-known that if 0 *is neither an eigenvalue nor a resonance, all bound states are localized in space and in* 5 *or higher space dimensions, there are no thresholds. See [50].*

Remark 25. *One can derive some localization and smoothness of all time-dependent bound states by the argument of Lemma 5.3.8. But for simplicity we add assumption 5.1.3.*

When ψ_0 is a scattering state, that is, when

$$\psi_0 = \Omega_{\alpha,+}\phi, \quad \text{for some } \phi \in L^2_x(\mathbb{R}^5) \tag{5.25}$$

we prove that

$$\int dt \| \langle x \rangle^{-\eta} U(t,0)\Omega_{\alpha,+}\phi\|^2_{L^2_x(\mathbb{R}^5)} \lesssim \|\phi\|^2_{L^2_x(\mathbb{R}^5)} \tag{5.26}$$

for all $\eta > 5/2$.

Theorem 5.1.2. *If* $V(x,t)$ *has the form of* (5.5) *and satisfies Assumption 5.1.2 and Assumption 5.1.3, then* (5.26) *is true.*

Strichartz estimates directly follow by using local decay estimates. Strichartz estimates say that

$$\|U(t,0)P_c(0)f\|_{L^q_t L^r_x(\mathbb{R}^{n+1})} \leq C_q \|f\|_{L^2_x(\mathbb{R}^n)} \tag{5.27}$$

for $2 \leq r,q \leq \infty$, $\frac{n}{r} + \frac{2}{q} = \frac{n}{2}$, and $(q,r,n) \neq (2,\infty,2)$.

Theorem 5.1.3. *If* $V(x,t)$ *has the form of* (5.5) *and satisfies Assumption 5.1.2 and Assumption 5.1.3, then Strichartz estimates(See* (5.27)*) are valid for all admissible* (q,r,n) *with* $n = 5$.

5.1.2 Outline of the proof

The proof is based on the following three steps: First, we prove AC by showing the existence and completeness of the Möller wave operators. These operators satisfy an intertwining property, that we use to reduce the estimate to a free flow. Since one can not commute the localization weight through the wave operators, in the second step we split the solution by decomposing the space to incoming and outgoing parts. This allows us to write the solution as a sum of quantities

$$\psi(t) = \psi_f(t) + C\psi(t) \tag{5.28}$$

for some $\psi_f(t)$ satisfying local decay estimate. We refer to $C\psi(t)$ as the perturbation term.

In the third step, it is shown that one can invert the coefficient of the perturbation term by proving its compactness. The compactness argument is based on propagation estimates for the FREE flow $e^{\ itH_0}$.

Time-independent cases

Let us begin with the proof for time-independent cases. We split $e^{\ itH}P_c\psi$ into four pieces, using incoming/outgoing decomposition(see (5.58) and (5.59) for the definition of incoming/outgoing projections)

$$\begin{aligned} e^{\ itH}P_c\psi &= P^+e^{\ itH}P_c\psi + P\ \ e^{\ itH}P_c\psi \\ &= P^+e^{\ itH_0}\Omega_+P_c\psi + P^+(1\quad \Omega_+)e^{\ itH}P_c\psi + P\ \ e^{\ itH_0}\Omega\ \ P_c\psi + P\ \ (1\quad \Omega\ \ )e^{\ itH}P_c\psi \\ &= P^+e^{\ itH_0}\Omega_+P_c\psi + P\ \ e^{\ itH_0}\Omega\ \ P_c\psi + Ce^{\ itH}P_c\psi \end{aligned} \tag{5.29}$$

where

$$C_1 := P^+(1\quad \Omega_+), \tag{5.30}$$

$$C_2 := P\ \ (1\quad \Omega\ \ ), \tag{5.31}$$

$$C := C_1 + C_2 \tag{5.32}$$

and $\Omega\ \ $ are the conjugate wave operators

$$\Omega\ \ := s\text{-}\lim_{t\ !\ \ 1} e^{itH_0}e^{\ itH}P_c \quad \text{on } L^2_x(\mathbb{R}^3). \tag{5.33}$$

Here we also use the intertwining property

$$\Omega\ \ e^{\ itH} = e^{\ itH_0}\Omega\ \ \qquad \text{on } L^2_x(\mathbb{R}^5). \tag{5.34}$$

Remark 26. *The above compactness of C_1, C_2 is similar in some respects to the compactness estimates used by Enss [21] and Davies [15]. But there are differences: Since we already know that AC holds, we prove the compactness for the adjoint of the wave operator. Moreover, we do not need to localize the hamiltonian away from the thresholds 0 and ∞. That would not be possible in the time dependent case. Instead, we use the local smoothing estimates of the free flow to deal with the high energy part. In 3,4 dimensions we assume that 0 is a regular point of the spectrum of H. In the time dependent case, we consider only dimensions 5 or higher, so, we do not need a regularity assumption.*

Here we start by using Ω_-^*. Indeed the existence of the free channel wave operators $\Omega_{\alpha,-}^*$ implies the existence of Ω_-^* and they are equal to each other, see [91]. C is compact on L^2_x(for compactness, see Lemma 5.2.3).

Based on (5.63) in Lemma 5.2.1,

$$\int dt \|\langle x\rangle^{-\sigma} P^- e^{-itH_0}\Omega_-^* P_c\psi\|^2_{L^2_x(\mathbb{R}^3)} \lesssim_\sigma \|\psi\|^2_{L^2_x(\mathbb{R}^3)} \tag{5.35}$$

for all $\sigma > 1$.

Lemma 5.1.2. *C can be expressed by*

$$C = C_r + C_m \tag{5.36}$$

with C_r, C_m satisfying

$$\|C_r\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)} \leq 1/100, \tag{5.37}$$

$$\|\langle x\rangle^{-\eta}(1-C_r)^{-1}C_r\langle x\rangle^{\sigma}\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)} \lesssim_{\sigma,\eta} 1, \quad \eta > 1, \sigma \in (1, 101/100), \tag{5.38}$$

and

$$\int dt \|C_m e^{-itH}P_c\psi\|^2_{L^2_x(\mathbb{R}^3)} \lesssim \|\psi\|^2_{L^2_x(\mathbb{R}^3)}. \tag{5.39}$$

Let

$$\psi_f(t) := P^+ e^{-itH_0}\Omega_+^* P_c\psi + P^- e^{-itH_0}\Omega_-^* P_c\psi. \tag{5.40}$$

Using (5.36),

$$e^{-itH}P_c\psi = \psi_f(t) + Ce^{-itH}P_c\psi = \psi_f(t) + (C_r + C_m)e^{-itH}P_c\psi. \tag{5.41}$$

Based on (5.37), $(1-C_r)^{-1}$ exists on $L^2_x(\mathbb{R}^3)$ and one could rewrite $e^{-itH}P_c\psi$ as

$$e^{-itH}P_c\psi = (1-C_r)^{-1}\psi_f(t) + (1-C_r)^{-1}C_m e^{-itH}P_c\psi. \tag{5.42}$$

By using (5.38), (5.39), (5.35), using that

$$(1-C_r)^{-1} = 1 + (1-C_r)^{-1}C_r \tag{5.43}$$

and taking $\sigma = 1001/1000$, one has that

$$\begin{aligned}\int dt\|\langle x\rangle^{-\eta}e^{-itH}P_c\psi\|^2_{L^2_x(\mathbb{R}^3)} \lesssim & \int dt\|\langle x\rangle^{-\eta}\psi_f(t)\|^2_{L^2_x(\mathbb{R}^3)}+\\ & \|\langle x\rangle^{-\eta}(1-C_r)^{-1}C_r\langle x\rangle^{\sigma}\|^2_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\int dt\|\langle x\rangle^{-\sigma}\psi_f(t)\|^2_{L^2_x(\mathbb{R}^3)}+\\ & \|\langle x\rangle^{-\eta}(1-C_r)^{-1}\|^2_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\int dt\|C_m e^{-itH}P_c\psi\|^2_{L^2_x(\mathbb{R}^3)} \lesssim_\eta \|\psi\|^2_{L^2_x(\mathbb{R}^3)}\end{aligned} \tag{5.44}$$

and we finish the proof.

Time-dependent cases

By AC the subspace of scattering states is identified by the range of the wave operators $\Omega_\pm$.

For $U(t,0)\Omega_+\phi$, using incoming/outgoing decomposition, we split $U(t,0)\Omega_+\phi$ into four pieces:

$$U(t,0)\Omega_+\phi = P^+e^{-itH_0}\phi + P^-e^{-itH_0}\Omega_-^*\Omega_+\phi + \tag{5.45}$$

$$P^+(1-\Omega_{t,+})U(t,0)\Omega_+\phi + P^-(1-\Omega_{t,-})U(t,0)\Omega_+\phi \tag{5.46}$$

$$=:\psi_1(t)+\psi_2(t)+C_1(t)U(t,0)\Omega_+\phi + C_2(t)U(t,0)\Omega_+\phi \tag{5.47}$$

with

$$P_c(t) = s\text{-}\lim_{s\to\infty} U(t,t+s)F_c\left(\frac{|x-2sP|}{s^\alpha}\leq 1\right)U(t+s,t), \quad \text{on } L^2_x(\mathbb{R}^5), \tag{5.48}$$

$$P^+\Omega_{t,+}^* := s\text{-}\lim_{a\to\infty} P^+e^{iaH_0}U(t+a,t)P_c(t) \quad \text{on } L^2_x(\mathbb{R}^5), \tag{5.49}$$

$$P^-\Omega_{t,-}^* := s\text{-}\lim_{a\to-\infty} e^{iaH_0}U(t+a,t)P_c(t) \quad \text{on } L^2_x(\mathbb{R}^5) \tag{5.50}$$

and

$$C_1(t) := P^+(1-\Omega_{t,+}^*), \quad C_2(t) := P^-(1-\Omega_{t,-}^*). \tag{5.51}$$

It is not clear here whether $\Omega_{t,+}$ and $\Omega_{t,-}$ exist with $P_c(t)$ defined only in one direction($t \to \infty$), but with P_-, $P_-\Omega_{t,-}$ exist on $L^2_x(\mathbb{R}^5)$. Here we also use following time-dependent intertwining property

$$\Omega_{t,-} U(t,0) = e^{-itH_0}\Omega_- \qquad \text{on } L^2_x(\mathbb{R}^5). \tag{5.52}$$

Lemma 5.1.3. *If V satisfies Assumption 5.1.2 and Assumption 5.1.3, then*

$$C_j(t) = C_{jm}(t) + C_{jr}(t), \quad j = 1, 2 \tag{5.53}$$

for some operators $C_{jm}(s,u)$ and $C_{jr}(s,u)$ satisfying

$$\sup_{t\in\mathbb{R}} \|C_{jr}(t)\|_{L^2_x \to L^2_x} \le 1/1000, \tag{5.54}$$

$$\sup_{t\in\mathbb{R}} \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} C_{jr}(t)\langle x\rangle^{\sigma}\|_{L^2_x \to L^2_x} \lesssim_\sigma 1 \tag{5.55}$$

for all $\eta > 5/2, \sigma \in (1, 101/100)$ and

$$\sup_{t\in\mathbb{R}} \|C_{jm}U(t,0)\Omega_+\phi(x)\|_{L^2_{x,t}} \lesssim \|\phi\|_{L^2_x(\mathbb{R}^5)}. \tag{5.56}$$

Then according to a similar argument as what we did for the time-independent system, using Lemma 5.1.3 instead of Lemma 5.1.2, we get (5.26) and finish the proof.

5.2 Time-independent Problems

In this section, we introduce the notion of incoming/outgoing waves, prove the compactness of the operator C and then prove Theorem 5.1.1.

5.2.1 Incoming/outgoing waves

The incoming/outgoing wave decompositions are similar to the ones initiated by Mourre [62]. The dilation generator A is defined by

$$A := \frac{1}{2}(x \cdot P + P \cdot x). \tag{5.57}$$

Definition 1 (Incoming/outgoing waves)**.** *The projection on outgoing waves is defined by [86]:*

$$P^+ := (\tanh(\frac{A - M}{R}) + 1)/2 \tag{5.58}$$

for some sufficiently large $R, M > 0$ such that Lemma 5.2.1 holds. The projection on incoming waves is defined by

$$P^- := 1 - P^+. \tag{5.59}$$

They enjoy the following properties (when the energy is both away from 0 and ∞, such estimates are proved in [42]).

Lemma 5.2.1. *When $R \geq R_0$ for some sufficiently large R_0, the incoming and outgoing waves satisfy:*

1. **High Energy Estimate**

   For all $\delta > 1, t \geq 0, c > 0, N \geq 1$, when the space dimension $n \geq 1$,

   $$\|P^{\pm} F(|P| > c) e^{\mp itH_0} \langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} \lesssim_{c,n} \frac{1}{\langle t\rangle^{\delta}}. \tag{5.60}$$

2. **Pointwise Smoothing Estimate**

   For $\delta > 0, t \geq v > 0, c > 0, l \in [0, \delta)$, when the space dimension $n \geq 1$,

   $$\|P^{\pm} F(|P| > c) e^{\mp itH_0} |P|^l \langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} \lesssim_{c,n,v,l} \frac{1}{\langle t\rangle^{\delta}}. \tag{5.61}$$

3. **Time Smoothing Estimate**

   For $\delta > 2, c > 0$, when the space dimension $n \geq 1$,

   $$\int_0^\infty dt t^2 \|P^{\pm} F(|P| > c) e^{\mp itH_0} |P|^2 \langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} \lesssim_{c,n} 1. \tag{5.62}$$

4. **Microlocal Decay**

   There exists some $\delta = \delta(n) > 1$ such that when $n \geq 3$,

   $$\|\langle x\rangle^{-\delta} P^{\pm} e^{\mp itH_0}\|_{L^2_x(\mathbb{R}^n) \to L^2_{x,t}(\mathbb{R}^{n+1})} \lesssim_{c,n} 1. \tag{5.63}$$

   In particular, when $n = 3$, δ can be any positive number which is greater than 1.

5. **Near Threshold Estimate**

   For $\delta > 1$ when $t \geq 1$, ($\epsilon \in (0, 1/2)$)

   $$\|P^{\pm} F(|P| > \frac{1}{\langle t\rangle^{1/2-\epsilon}}) e^{\mp itH_0} \langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} \lesssim_{n,\epsilon} \frac{1}{t^{(1/2-\epsilon)\delta}}. \tag{5.64}$$

In particular, when $n \geq 5$, one has ($\epsilon \in (0, n/4)$)

$$\|P^- e^{-itH_0}\langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_{n,\epsilon} \frac{1}{\langle t\rangle^{n/4-\epsilon}} \in L^1_t(\mathbb{R}) \tag{5.65}$$

since the L^2 volume of $F(|P| \leq \frac{1}{\langle t\rangle^{1/2-\epsilon}})f$ in frequency space is controlled by $\frac{1}{\langle t\rangle^{n/4-n\epsilon/2}}\|f\|_{L^1_x}$ up to some constant.

Proof. The estimates are for the free hamiltonian dynamics, which on $\mathbb{R}^n$ can be written explicitly by Fourier transform. It can then be estimated by standard Stationary phase methods. Some of these estimates may be cumbersome to derive in such a way. But they also follow from the propagation estimates of [79, 42]. Alternatively, one can follow the arguments of [42] combined with the analytic construction of propagation observables(PROBs for short) of [86], to get a shorter and direct proof. Most of these estimates will follow from direct computation of the resulting PRES applied to the PROBs: $P^-(\frac{A-M}{R})$, $\sum_{j=1}^{N} x_j P^-(\frac{A-M}{R})x_j$, $(A-M)P^-(\frac{A-M}{R})$. These PROBs will lead to estimates which hold for all energies, including $0, \infty$. By further localizing the energy(frequency) of the initial data away from the thresholds $0, \infty$ we can then use similar operators as above but with $A(t) \equiv A - bt$ instead of A, the usual dilation operator on $\mathbb{R}^n$. Here we take $b < 2E_m$, E_m is the lowest energy in the support of the initial data.

We put the proof in the appendix.

□

Lemma 5.2.2. *For all $a \in [0, \pi R/2)$, $n \geq 1$, $f \in H^a_x(\mathbb{R}^n)$($H^a_x$ denotes L^2_x Sobolev space),*

$$\||P|^a P^- f\|_{L^2_x(\mathbb{R}^n)} \lesssim_{\pi/2-a/R, a/R} \||P|^a f\|_{L^2_x(\mathbb{R}^n)} \tag{5.66}$$

and

$$\||x|^a P^- f\|_{L^2_x(\mathbb{R}^n)} \lesssim_{\pi/2-a/R, a/R} \||x|^a f\|_{L^2_x(\mathbb{R}^n)}. \tag{5.67}$$

Proof. Let

$$g(k) := (\tanh(k)+1)/2, \quad k \in \mathbb{R}. \tag{5.68}$$

Compute $(|P|^a P^+ f - P^+|P|^a)f(x)$

$$
\begin{aligned}
(|P|^a P^+ - P^+|P|^a)f(x) &= c\int dw\hat{g}(w)(|P|^a e^{iw(\frac{A-M}{R})} - e^{iw(\frac{A-M}{R})}|P|^a)f(x)\\
&= c\int dw\hat{g}(w)e^{-i\frac{wM}{R}}(\frac{1}{e^{aw/R}}-1)e^{iwA/R}|P|^a f(x). \qquad (5.69)
\end{aligned}
$$

Since

$$
\hat{g}(w) = \frac{c_1}{\sinh(\pi w/2)} + c_2\delta(w) \quad \text{for some constants } c_1, c_2, \qquad (5.70)
$$

one has

$$
\begin{aligned}
\|(|P|^a P^+ - P^+|P|^a)f(x)\|_{L^2_x(\mathbb{R}^n)} &\leq 2cc_1\int_{|w|\geq 1} dw\left|\frac{1}{\sinh(\pi w/2)}\right| e^{a|w|/R}\||P|^a f\|_{L^2_x(\mathbb{R}^n)} +\\
&\quad cc_1\int_{|w|<1} dw|\hat{g}(w)|\left|\left(\frac{1}{e^{aw/R}}-1\right)\right|\||P|^a f\|_{L^2_x(\mathbb{R}^n)}\\
&\lesssim_{a/R} \int_{|w|\geq 1} dw\left|\frac{1}{\sinh(\pi w/2)}\right| e^{a|w|/R}\||P|^a f\|_{L^2_x(\mathbb{R}^n)} + \int_{|w|<1} dw|w\hat{g}(w)|\||P|^a f\|_{L^2_x(\mathbb{R}^n)}\\
&\lesssim_{\pi/2-a/R,a/R} \||P|^a f\|_{L^2_x(\mathbb{R}^n)} \qquad (5.71)
\end{aligned}
$$

when $a < R\pi/2$. In addition, $\|P^+\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \leq 1$. So we have

$$
\||P|^a P^+ f(x)\|_{L^2_x(\mathbb{R}^n)} \lesssim_{\pi/2-a/R,a/R} \||P|^a f(x)\|_{L^2_x(\mathbb{R}^n)} \qquad (5.72)
$$

which also implies

$$
\||P|^a P^- f(x)\|_{L^2_x(\mathbb{R}^n)} \lesssim_{\pi/2-a/R,a/R} \||P|^a f(x)\|_{L^2_x(\mathbb{R}^n)}. \qquad (5.73)
$$

Similarly, one has (5.67). We finish the proof.

□

5.2.2 Compactness of C

We prove the compactness of C in this subsection.

Lemma 5.2.3. *If $V(x)$ satisfies assumption (5.1.1), then C is a compact operator on $L^2_x(\mathbb{R}^3)$.*

In order to prove Lemma 5.2.3, it suffices to show the compactness of C_1 and C_2. Recall that

$$
C_1 = P^+(1-\Omega_+) \qquad (5.74)
$$

and

$$C_2 = P_-(1 - \Omega_-). \tag{5.75}$$

The proof requires the following lemma.

Lemma 5.2.4 (Representation formula for Ω_-)**.** *Assume* 0 *is neither an eigenvalue nor a resonance for* H *and assume* $\langle x\rangle^{2\sigma}V(x) \in L^\infty_x(\mathbb{R}^3)$ *for some* $\sigma > 1$*. For* $g(x)$ *satisfying* $\langle x\rangle^{\sigma}g(x) \in L^2_x(\mathbb{R}^3)$*,*

$$\Omega_- g = (1 + V(x)\frac{1}{H_0})^{-1}g - i\int_0^{\infty} dtH_0e^{itH_0}\Omega_- V(x)\frac{e^{-itH_0}}{H_0}(1 + V(x)\frac{1}{H_0})^{-1}g. \tag{5.76}$$

Remark 27. *In* 5 *or higher space dimensions, we don't need this lemma since there are no zero-frequency issues.*

Proof. Let $L^2_{\sigma,x}(\mathbb{R}^3)$ denote the weighted L^2_x space

$$L^2_{\sigma,x}(\mathbb{R}^3) := \{f : \langle x\rangle^{-\sigma}f \in L^2_x(\mathbb{R}^3)\}. \tag{5.77}$$

$(1 + V(x)\frac{1}{H_0})^{-1} : L^2_{\sigma,x}(\mathbb{R}^3) \to L^2_{\sigma,x}(\mathbb{R}^3)$, is bounded for $\sigma > 1$ since 0 is neither an eigenvalue nor a resonance for H.(Here 1 comes from resolvent estimates, see Lemma 22.2 in [50] for example) So the integrand of (5.76) is well-defined. Now we prove the validity of (5.76). Using Duhamel's formula and employing integration by parts, one has

$$\begin{aligned}
\Omega_+ =&\Omega_+\Omega_+ + \Omega_+(1 - \Omega_+) && (5.78)\\
=&1 + (-i)\int_0^{\infty} dte^{itH_0}\Omega_+V(x)e^{-itH_0} && (5.79)\\
=&1 + e^{itH_0}\Omega_+V(x)\frac{e^{-itH_0}}{H_0}\Big|_{t=0}^{t=\infty} - i\int_0^{\infty} dtH_0e^{itH_0}\Omega_+V(x)\frac{e^{-itH_0}}{H_0} && (5.80)\\
=&1 - \Omega_+V(x)\frac{1}{H_0} - i\int_0^{\infty} dtH_0e^{itH_0}\Omega_+V(x)\frac{e^{-itH_0}}{H_0} && (5.81)
\end{aligned}$$

which implies

$$\Omega_+(1 + V(x)\frac{1}{H_0})f = f - i\int_0^{\infty} dtH_0e^{itH_0}\Omega_+V(x)\frac{e^{-itH_0}}{H_0}f. \tag{5.82}$$

Taking $f = (1 + V(x)\frac{1}{H_0})^{-1}g$ and plugging it into (5.82), one gets (5.76) for Ω_+. Similarly, one gets (5.76) for Ω_-. We finish the proof. □

Proposition 5.2.1. *If $\langle x\rangle^{\delta}V(x) \in L^\infty_x(\mathbb{R}^3)$ for some $\delta > 4$, then C_1 and C_2 are compact operators.*

Proof. Let

$$C_1(v) := iP^+\int_v^\infty du e^{iuH_0}\Omega_+ V(x)e^{-iuH_0}. \tag{5.83}$$

Break $C_1(v)$ into two pieces

$$C_1(v) = iP^+\int_v^\infty du F(|P| > 1/\langle u\rangle^{1/2-\epsilon})e^{iuH_0}\Omega_+V(x)e^{-iuH_0}+$$
$$iP^+\int_v^\infty du\bar{F}(|P|\le 1/\langle u\rangle^{1/2-\epsilon})e^{iuH_0}\Omega_+V(x)e^{-iuH_0} =: C_{1,l}(v)+C_{1,s}(v) \tag{5.84}$$

for some $\epsilon > 0$ satisfying $(1/2-\epsilon)(\delta-\epsilon) > 2$. Due to Lemma 5.2.1(Pointwise Smoothing Estimate and Near Threshold Estimate), using Duhamel formula to expand Ω_+,

$$\|C_{1,l}(v)\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)} \lesssim_v \int_v^\infty \frac{du}{\langle u\rangle^{(1/2-\epsilon)(\delta-\epsilon)}}\|\langle x\rangle^{\delta-\epsilon}\langle P\rangle^{-\epsilon}V(x)\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}+$$
$$\int_v^\infty du\int_0^\infty ds\frac{du}{\langle u+s\rangle^{(1/2-\epsilon)(\delta-\epsilon)}}\|\langle x\rangle^{\delta-\epsilon}\langle P\rangle^{-\epsilon}V(x)\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\quad\|V(x)\|_{L^\infty_x}$$
$$\lesssim \|\langle x\rangle^{\delta-\epsilon}\langle P\rangle^{-\epsilon}V(x)\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}(1+\|V(x)\|_{L^\infty_x}) \tag{5.85}$$

for all $v > 0$ and

$$\|C_{1,l}(v)\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)} \to 0 \quad \text{as } v\to\infty. \tag{5.86}$$

Since $\langle x\rangle^{-\epsilon}\langle P\rangle^{-\epsilon}$ is compact on $L^2_x(\mathbb{R}^n)$ for all $n\ge 1$, $C_{1,l}(v)$ is compact on $L^2_x(\mathbb{R}^3)$. Here we want to emphasize that when the space frequency is slightly away from the origin, by using Lemma 5.2.1, we could get enough decay in u when $V(x)$ is well-localized. It will be the same in time-dependent cases. For compactness and integrability of u, all we have to check is the part with small space frequency.

For $C_{1,s}(v)$, using Lemma 5.2.4, one has

$$C_{1,s}(v) = iP^+\int_v^\infty du\bar{F}(|P|\le 1/\langle u\rangle^{1/2-\epsilon})e^{iuH_0}\Omega_+V(x)e^{-iuH_0}$$
$$= iP^+\int_v^\infty du\bar{F}(|P|\le 1/\langle u\rangle^{1/2-\epsilon})e^{iuH_0}(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0}+$$
$$P^+\int_v^\infty du\bar{F}(|P|\le 1/\langle u\rangle^{1/2-\epsilon})\int_0^\infty du_1H_0e^{i(u+u_1)H_0}\Omega_+V(x)\frac{e^{-iu_1H_0}}{H_0}(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0}$$
$$=: C_{1,s,1}(v)+C_{1,s,2}(v). \tag{5.87}$$

For $C_{1,s,1}(v)$, one has

$$\begin{aligned}\|C_{1,s,1}(v)\|_{L^2_x\to L^2_x} &\lesssim \int_v^\infty du\|\bar{F}(|P|-1/\langle u\rangle^{1/2-\epsilon})\|_{L^1_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\|(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0}\|_{L^2_x\to L^1_x}\\ &\lesssim \int_v^\infty du\frac{1}{\langle u\rangle^{3/4-3/2\epsilon}}\|(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0}\|_{L^2_x\to L^1_x}\\ &\lesssim \int_v^\infty du\frac{1}{\langle u\rangle^{3/4-3/2\epsilon}}\|(1+V(x)\frac{1}{H_0})^{-1}V(x)\|_{L^6_x\to L^1_x}\|e^{-iuH_0}\|_{L^2_x\to L^6_x}\lesssim_\epsilon 1 \end{aligned} \quad (5.88)$$

by using Strichartz estimates, choosing $\epsilon\in(0,1/6)$ and using that for some $\sigma>1$, $\langle x\rangle^\delta V(x)\in L^\infty_x$ with $\delta>4$,

$$\begin{aligned}&\|(1+V(x)\frac{1}{H_0})^{-1}V(x)\|_{L^6_x\to L^1_x}\le\|V(x)\|_{L^6_x\to L^1_x}+\\ &\|V(x)\frac{1}{H_0}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^1_x}\|\langle x\rangle^{\sigma}(1+V(x)\frac{1}{H_0})^{-1}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x}\|\langle x\rangle^{\sigma}V(x)\|_{L^6_x\to L^2_x}\lesssim 1.\end{aligned} \quad (5.89)$$

For $C_{1,s,2}(v)$, based on the proof of Proposition 5.2.1 for the part with high frequency cut-off(the part with $F(|P|>1/\langle u\rangle^{1/2-\epsilon})$), it suffices to check

$$\begin{aligned}\tilde{C}_{1,s,2}(v):=P^+\int_v^\infty du\bar{F}(|P|-1/\langle u+u_1\rangle^{1/2-\epsilon})\int_0^\infty du_1\\ H_0e^{i(u+u_1)H_0}\Omega_+V(x)\frac{e^{-iu_1H_0}}{H_0}(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0}.\end{aligned} \quad (5.90)$$

Here we mean

$$\|C_{1,s,2}(v)-\tilde{C}_{1,s,2}(v)\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\lesssim 1 \quad \text{for all } v>0 \quad (5.91)$$

and

$$\|C_{1,s,2}(v)-\tilde{C}_{1,s,2}(v)\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\to 0 \quad (5.92)$$

as $v\to\infty$ by using a similar argument as what we did for $C_{1,l}(v)$ in the proof of Proposition 5.2.1. Since

$$\begin{aligned}\|\langle x\rangle^{-\sigma}\frac{e^{-iu_1H_0}}{H_0}(1+V(x)\frac{1}{H_0})^{-1}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x}\lesssim\|\langle x\rangle^{-\sigma}\frac{e^{-iu_1H_0}}{H_0}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x}\\ \|\langle x\rangle^{\sigma}(1+V(x)\frac{1}{H_0})^{-1}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x}\lesssim\frac{1}{\langle u_1\rangle^{1/2}}\end{aligned} \quad (5.93)$$

for $\sigma > 1$, one has

$$\begin{aligned}
&\int_0^\infty du \int_0^\infty du_1 \|\bar{F}(|P| \geq 1/\langle u+u_1\rangle^{1/2-\epsilon}) H_0 e^{i(u+u_1)H_0}(1+V(x)\frac{1}{H_0})^{-1} \\
&\qquad V(x)\frac{e^{-iu_1H_0}}{H_0}(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0}\|_{L^2_x\to L^2_x} \\
&\lesssim \int_0^\infty du \int_0^\infty du_1 \|H_0\bar{F}(|P| \geq 1/\langle u+u_1\rangle^{1/2-\epsilon})\|_{L^1_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\|(1+V(x)\frac{1}{H_0})^{-1}V(x)\langle x\rangle^{\sigma}\|_{L^2_x\to L^1_x} \\
&\quad \|\langle x\rangle^{-\sigma}\frac{e^{-iu_1H_0}}{H_0}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x}\|\langle x\rangle^{\sigma}(1+V(x)\frac{1}{H_0})^{-1}V(x)\|_{L^6_x\to L^2_x}\|e^{-iuH_0}\|_{L^2_x\to L^6_x} \\
&\quad \lesssim \int_0^\infty du \int_0^\infty du_1 \frac{1}{\langle u+u_1\rangle^{7/4-7/2\epsilon}} \frac{1}{\langle u_1\rangle^{1/2}}\|e^{-iuH_0}\|_{L^2_x\to L^6_x} \lesssim \|e^{-iuH_0}\|_{L^2_x\to L^2_uL^6_x} \\
&\lesssim 1 \qquad (5.94)
\end{aligned}$$

for some $\sigma > 1$ and sufficiently small $\epsilon > 0$, and

$$\begin{aligned}
&\int_0^\infty du \int_0^\infty du_1 \int_0^\infty du_2 \|\bar{F}(|P| \geq 1/\langle u+u_1+u_2\rangle^{1/2-\epsilon}) H_0^2 e^{i(u+u_1+u_2)H_0}\Omega_+ V(x)\frac{e^{-iu_2H_0}}{H_0} \\
&\qquad (1+V(x)\frac{1}{H_0})^{-1}V(x)\frac{e^{-iu_1H_0}}{H_0}(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0}\|_{L^2_x\to L^2_x} \\
&\quad \lesssim \int_0^\infty du \int_0^\infty du_1 \int_0^\infty du_2 \|H_0^2\bar{F}(|P| \geq 1/\langle u+u_1+u_2\rangle^{1/2-\epsilon})\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)} \\
&\qquad \|V(x)\frac{e^{-iu_2H_0}}{H_0}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x}\|\langle x\rangle^{\sigma}(1+V(x)\frac{1}{H_0})^{-1}V(x)\langle x\rangle^{\sigma}\|_{L^2_x\to L^2_x} \\
&\quad \|\langle x\rangle^{-\sigma}\frac{e^{-iu_1H_0}}{H_0}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x}\|\langle x\rangle^{\sigma}(1+V(x)\frac{1}{H_0})^{-1}V(x)\|_{L^6_x\to L^2_x}\|e^{-iuH_0}\|_{L^2_x\to L^6_x} \\
&\lesssim \int_0^\infty du \int_0^\infty du_1 \int_0^\infty du_2 \frac{1}{\langle u+u_1+u_2\rangle^{2-4\epsilon}} \frac{1}{\langle u_2\rangle^{1/2}}\frac{1}{\langle u_1\rangle^{1/2}}\|e^{-iuH_0}\|_{L^2_x\to L^6_x} \lesssim \|e^{-iuH_0}\|_{L^2_x\to L^2_uL^6_x} \\
&\lesssim 1. \qquad (5.95)
\end{aligned}$$

Here we use

$$\begin{aligned}
&\|(1+V(x)\frac{1}{H_0})^{-1}V(x)\langle x\rangle^{\sigma}\|_{L^2_x\to L^1_x} \leq \|V(x)\langle x\rangle^{\sigma}\|_{L^2_x\to L^1_x}+\|V(x)\frac{1}{H_0}(1+V(x)\frac{1}{H_0})^{-1}V(x)\langle x\rangle^{\sigma}\|_{L^2_x\to L^1_x} \\
&\lesssim \|\langle x\rangle^{\sigma}V(x)\|_{L^2_x}+\|V(x)\langle x\rangle^{\sigma}\|_{L^2_x\to L^1_x}\|\langle x\rangle^{-\sigma}\frac{1}{H_0}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x}\|\langle x\rangle^{\sigma}(1+V(x)\frac{1}{H_0})^{-1}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x} \\
&\qquad \|\langle x\rangle^{2\sigma}V(x)\|_{L^\infty_x} \lesssim 1 \qquad (5.96)
\end{aligned}$$

and

$$\|\langle x\rangle^{\sigma}(1+V(x)\frac{1}{H_0})^{-1}V(x)\|_{L^6_x\to L^2_x} \leq \|\langle x\rangle^{\sigma}(1+V(x)\frac{1}{H_0})^{-1}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x}\|\langle x\rangle^{\sigma}V(x)\|_{L^3_x} \lesssim 1 \qquad (5.97)$$

for some $\sigma > 1$. Using Lemma 5.2.4, (5.94) and (5.95) imply

$$\|\tilde{C}_{1,s,2}(v)\|_{L^2_x \to L^2_x} \lesssim \int_0^\infty du \|\bar{F}(|P| \ \ 1/\langle u+u_1\rangle^{1/2-\epsilon})\int_0^\infty du_1$$
$$H_0 e^{i(u+u_1)H_0}\Omega_+ V(x)\frac{e^{-iu_1H_0}}{H_0}(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0}\|_{L^2_x \to L^2_x} \lesssim 1 \quad (5.98)$$

with

$$\|\tilde{C}_{1,s,2}(v)\|_{L^2_x \to L^2_x} \to 0 \quad \text{as } v \to \infty. \quad (5.99)$$

Hence,

$$\|C_{1,s,2}(v)\|_{L^2_x \to L^2_x} \lesssim 1 \quad (5.100)$$

with

$$\|C_{1,s,2}(v)\|_{L^2_x \to L^2_x} \to 0 \quad \text{as } v \to \infty. \quad (5.101)$$

Using (5.88) and (5.100), one has

$$\|C_{1,s}(v)\|_{L^2_x \to L^2_x} \lesssim 1. \quad (5.102)$$

According to (5.102) and (5.85), one has

$$\|C_1(v)\|_{L^2_x \to L^2_x} \lesssim 1 \quad (5.103)$$

with

$$\|C_1(v)\|_{L^2_x \to L^2_x} \to 0 \quad (5.104)$$

as $v \to \infty$. Next we prove that $C_1(v)$ is compact on $L^2_x(\mathbb{R}^3)$. Since $\langle P\rangle^{-\epsilon}\langle x\rangle^{-\epsilon}$ is compact on $L^2_x(\mathbb{R}^3)$ for any $\epsilon > 0$, according to (5.61) of Lemma 5.2.1, one has

$$C_{1,1}(v, M) := iP^+\int_v^M du e^{iuH_0}P_cV(x)e^{-iuH_0} \quad (5.105)$$

and

$$C_{1,2}(v, M) := iP^+\int_v^M du\int_0^\infty ds e^{i(u+s)H_0}V(x)e^{-isH}P_cV(x)e^{-iuH_0} \quad (5.106)$$

are compact on $L^2_x(\mathbb{R}^3)$ for any $M > v$. So

$$C_1(v, M) := iP^+\int_v^M du e^{iuH_0}\Omega_+V(x)\ {}^{-iuH_0} \quad (5.107)$$

is compact on $L^2_x(\mathbb{R}^3)$ by using Duhamel's formula to expand Ω_+. Then $C_1(v)$ is compact since

$$\lim_{M\to\infty} \|C_1(v) - C_1(v,M)\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)} = 0. \tag{5.108}$$

Since

$$\lim_{v\downarrow 0} \|P^+(1-\Omega_+) - C_1(v)\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)} = 0, \tag{5.109}$$

$C_1 = P^+(1-\Omega_+)$ is a compact operator on $L^2_x(\mathbb{R}^3)$. Similarly, C_2 is a compact operator on $L^2_x(\mathbb{R}^3)$. We finish the proof.

□

Remark 28. *Based on the estimate for $C_{1,l}(v)$, it suffices to check the part with low frequency on support of P^- when $|u| \ll 1$. It is the same as the time-dependent cases.*

Therefore, we conclude the compactness of C.

Proof of Lemma 5.2.3. It follows from proposition 5.2.1. □

5.2.3 Decomposition of operator C

In this part, we prove Lemma 5.1.2.

Lemma 5.2.5. *If $V(x)$ satisfies Assumption 5.1.1, then for any $\epsilon \in (0,1)$,*

$$\|C_j\langle x\rangle^{1-\epsilon}\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)} \lesssim_\epsilon 1, \quad j=1,2 \tag{5.110}$$

and for $\eta \in (1, 101/100)$, $\epsilon' \in (0, 1/10)$,

$$\|\langle x\rangle^{-(1-\epsilon')} C_j \langle x\rangle^{\eta}\|_{L^2_x\to L^2_x} \lesssim 1. \tag{5.111}$$

Proof. For $C_1\langle x\rangle^{1-\epsilon}$, based on Lemma 5.2.1, it suffices to control

$$C_{11}\langle x\rangle^{1-\epsilon} := i\int_0^\infty dt\bar{F}(|P| \leq \frac{1}{\langle t\rangle^{1/2-\epsilon/100}})e^{itH_0}\Omega_+ V(x)e^{-itH_0}\langle x\rangle^{1-\epsilon} \tag{5.112}$$

and

$$C_{12}\langle x\rangle^{1-\epsilon} := i\int_0^1 dtF(|P|>1)e^{itH_0}\Omega_+ V(x)e^{-itH_0}\langle x\rangle^{1-\epsilon}. \tag{5.113}$$

For $C_{11}\langle x\rangle^{1-\epsilon}$, based on the argument from (5.87) to (5.98), using Lemma 5.2.4, the leading term of $C_{11}\langle x\rangle^{1-\epsilon}$ is

$$\bar{C}_{11} := iP^+\int_0^\infty du\bar{F}(|P|\leq 1/\langle u\rangle^{1/2-\epsilon/100})e^{iuH_0}(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0}\langle x\rangle^{1-\epsilon}. \tag{5.114}$$

For $\bar{C}_{11}$, the leading term is

$$\tilde{C}_{11} := iP^+\int_0^\infty du\bar{F}(|P|\leq 1/\langle u\rangle^{1/2-\epsilon/100})e^{iuH_0}(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0} \bar{F}(|P|\leq 1/\langle u\rangle^{1/2-\epsilon/25})\langle x\rangle^{1-\epsilon} \tag{5.115}$$

since

$$\|P^+\int_0^\infty du\bar{F}(|P|\leq 1/\langle u\rangle^{1/2-\epsilon/100})e^{iuH_0}(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0} F(|P|> 1/\langle u\rangle^{1/2-\epsilon/25})\langle x\rangle^{1-\epsilon}\|_{L^2_x\to L^2_x}\lesssim 1 \tag{5.116}$$

by using the integral over u(the total phase of u is of order $\frac{1}{\langle u\rangle^{1-2\epsilon/25}}$ and $\frac{1}{\langle u\rangle^{1-2\epsilon/25}}\gtrsim\frac{1}{\langle u\rangle}$). For $\tilde{C}_{11}$, one has that for some $\sigma>1$,

$$\begin{aligned}\|\tilde{C}_{11}\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\lesssim&\int_0^\infty du\frac{1}{\langle u\rangle^{3/4-3\epsilon/200}}\\ &\times\|(1+V(x)\frac{1}{H_0})^{-1}V(x)e^{-iuH_0}\bar{F}(|P|\leq 1/\langle u\rangle^{1/2-\epsilon/25})\langle x\rangle^{1-\epsilon}\|_{L^2_x\to L^1_x}\\ \lesssim&\int_0^\infty du\frac{1}{\langle u\rangle^{3/4-3\epsilon/200}}\times(\|\langle x\rangle^\sigma V(x)\|_{L^\infty_x\to L^1_x}+\|V(x)\frac{1}{H_0}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^1_x}\\ &\times\|\langle x\rangle^\delta(1+V(x)\frac{1}{H_0})^{-1}\langle x\rangle^{-\delta}\|_{L^2_x\to L^2_x}\|\langle x\rangle^\sigma V(x)\|_{L^\infty_x\to L^2_x})\times\frac{1}{\langle u\rangle^{3/4-3\epsilon/50}}u^{1/2-\epsilon/3}\\ \lesssim_\epsilon&\|\langle x\rangle^\sigma V(x)\|_{L^\infty_x\to L^1_x}+\|V(x)\frac{1}{H_0}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^1_x}\|\langle x\rangle^\delta(1+V(x)\frac{1}{H_0})^{-1}\langle x\rangle^{-\delta}\|_{L^2_x\to L^2_x}\|\langle x\rangle^\sigma V(x)\|_{L^\infty_x\to L^2_x}\\ \lesssim_\epsilon& 1.\end{aligned} \tag{5.117}$$

Here we use

$$(1+V(x)\frac{1}{H_0})^{-1}=1-V(x)\frac{1}{H_0}(1+V(x)\frac{1}{H_0})^{-1}. \tag{5.118}$$

Hence, if $V(x)$ satisfies assumption 5.1.1,

$$\|C_{11}\langle x\rangle^{1-\epsilon}\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\lesssim_\epsilon 1. \tag{5.119}$$

For $C_{12}\langle x\rangle^{1-\epsilon}$, it is sufficient to control $C_{12}x_j^2$ for $j=1,2,3$ since C_{12} is bounded on $L^2_x(\mathbb{R}^3)$. Compute $C_{12}x_j^2$

$$C_{12}x_j^2 = i\int_0^\infty dt F(|P|>1)e^{itH_0}\Omega_+ V(x)(x_j - 2tP_j)^2 e^{-itH_0}. \tag{5.120}$$

According to (5.60) and (5.62) in Lemma 5.2.1, one has

$$\|C_{12}x_j^2\|_{L^2_x\to L^2_x}\lesssim 1. \tag{5.121}$$

So

$$\|C_{12}\langle x\rangle^2\|_{L^2_x\to L^2_x}\lesssim 1. \tag{5.122}$$

Based on (5.119) and (5.122), one has

$$\|C_1\langle x\rangle^{1-\epsilon}\|_{L^2_x\to L^2_x}\lesssim_\epsilon 1. \tag{5.123}$$

Similarly, one has the same result for $C_2\langle x\rangle^{1-\epsilon}$. For $\langle x\rangle^{-(1-\epsilon')}C_j\langle x\rangle^{\eta}$, it helps us to gain more decay by using the boundedness of $\langle x\rangle^{-(1-\epsilon')}P^-\langle x\rangle^{1-\epsilon'}$ on $L^2_x(\mathbb{R}^3)$ when we have 0 frequency issue. In other words,

$$\begin{aligned}&\|\langle x\rangle^{-(1-\epsilon')}P^+e^{iuH_0}F(|P|\le\frac{1}{\langle u\rangle^{1/2-\epsilon/100}})\|_{L^1_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\lesssim\|\langle x\rangle^{-(1-\epsilon')}P^+\langle x\rangle^{1-\epsilon'}\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\\ &\times\|\langle x\rangle^{-(1-\epsilon')}\|_{L^4_x}\times\|e^{iuH_0}F(|P|\le\frac{1}{\langle u\rangle^{1/2-\epsilon/100}})\|_{L^1_x\to L^4_x}\lesssim\frac{1}{\langle u\rangle^{(1/2-\epsilon/100)3\times 3/4}}\lesssim\frac{1}{\langle u\rangle}\end{aligned} \tag{5.124}$$

by choosing $\epsilon>0$ small enough and ϵ' closed to 0. Then

$$\|\langle x\rangle^{-(1-\epsilon')}\tilde{C}_{11}\langle x\rangle^{\epsilon+\eta-1}\|_{L^2_x\to L^2_x}\lesssim 1+\int_0^\infty du\frac{1}{\langle u\rangle}\times\frac{1}{\langle u\rangle^{3/4-3\epsilon/50}}u^{(1/2+\epsilon/25)\eta}\lesssim 1 \tag{5.125}$$

for $\eta\in(1,11/10)$. We finish the proof.

□

The wave operator is defined by

$$\Omega_- := s\text{-}\lim_{t\to-\infty} e^{itH}e^{-itH_0}, \quad \text{on } L^2_x(\mathbb{R}^3). \tag{5.126}$$

Lemma 5.2.6. *If $V(x)$ satisfies Assumption 5.1.1, then for any $B>0$, $\sigma<2$, $M>m>0$,*

$$\|\langle x\rangle^\sigma\Omega_+F_{m,M}(H_0)\chi(|x|\le B)\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)}\lesssim_{B,m,M}1. \tag{5.127}$$

Here $F_{m,M}(k):=F(k>m)\bar{F}(k\le M)$.

Proof. Let $\psi \in L^2_x(\mathbb{R}^3)$. By using Duhamel expansion, one has

$$\begin{aligned}\langle x\rangle^\sigma \Omega_+ F_{m,M}(H_0)\chi(|x|\le B)\psi &= \langle x\rangle^\sigma \bar F(H_0\le 1)F_{m,M}(H_0)\chi(|x|\le B)\psi+\\ &\quad i\langle x\rangle^\sigma \bar F(H_0\le 1)\int_0^\infty dt e^{itH}V(x)e^{-itH_0}F_{m,M}(H_0)\chi(|x|\le B)\psi+\\ &\quad \langle x\rangle^\sigma F(H_0>1)\Omega_+F_{m,M}(H_0)\chi(|x|\le B)\psi =: \psi_1+\psi_2+\psi_3. \quad (5.128)\end{aligned}$$

For ψ_1, one has

$$\|\psi_1\|_{L^2_x(\mathbb{R}^3)} \lesssim_{m,M} \langle B\rangle^\sigma \|\psi\|_{L^2_x(\mathbb{R}^3)} \quad (5.129)$$

since

$$\|\langle x\rangle^\sigma F_{m,M}(H_0)\langle x\rangle^{-\sigma}\|_{L^2_x(\mathbb{R}^3)\to L^2_x(\mathbb{R}^3)} \lesssim_{m,M} 1. \quad (5.130)$$

For ψ_2, due to (5.130), one has that

$$\begin{aligned}\|\psi_2\|_{L^2_x(\mathbb{R}^3)} &\lesssim \int_0^\infty dt\|\langle x\rangle^\sigma \bar F(H_0\le 1)e^{itH}V(x)e^{-itH_0}F_{m,M}(H_0)\chi(|x|\le B)\psi\|_{L^2_x(\mathbb{R}^3)}\\ &\lesssim \int_0^\infty dt\langle t\rangle^\sigma\|\langle x\rangle^\sigma V(x)e^{-itH_0}F_{m,M}(H_0)\chi(|x|\le B)\psi\|_{L^2_x}\\ &\lesssim_m \int_0^\infty dt\langle t\rangle^\sigma\|\langle x\rangle^{2\sigma+5/4}V(x)\|_{L^\infty_x(\mathbb{R}^3)}\frac{1}{\langle t\rangle^{\sigma+5/4}}\|\langle x\rangle^{\sigma+5/4}\chi(|x|\le B)\psi\|_{L^1_x\cap L^2_x}\\ &\lesssim_{m,B} \|\psi\|_{L^2_x} \quad (5.131)\end{aligned}$$

where we use

$$\|\langle x\rangle^\sigma \bar F(H_0\le 1)e^{itH}\langle x\rangle^{-\sigma}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim \langle t\rangle^\sigma \quad (5.132)$$

and

$$\|\langle x\rangle^{-\sigma-5/4}F_{m,M}(H_0)e^{-itH_0}\langle x\rangle^{-\sigma-5/4}\|_{L^2_x\to L^2_x} \lesssim_m \frac{1}{\langle t\rangle^{\sigma+5/4}}. \quad (5.133)$$

For ψ_3, using Duhamel's formula and intertwining property, one has

$$\begin{aligned}\psi_3 &= \langle x\rangle^\sigma F(H_0>1)\Omega_+F_{m,M}(H_0)\chi(|x|\le B)\psi\\ &= \langle x\rangle^\sigma F(H_0>1)F_{m,M}(H_0)\chi(|x|\le B)\psi+\langle x\rangle^\sigma F(H_0>1)(1-\Omega_+)\Omega_+F_{m,M}(H_0)\chi(|x|\le B)\psi\\ &= \langle x\rangle^\sigma F(H_0>1)F_{m,M}(H_0)\chi(|x|\le B)\psi+\\ &\quad i\int_0^\infty dt\langle x\rangle^\sigma F(H_0>1)e^{itH_0}V(x)\Omega_+e^{-itH_0}F_{m,M}(H_0)\chi(|x|\le B)\psi\\ &=: \psi_{31}+\psi_{32}. \quad (5.134)\end{aligned}$$

According to (5.130), one has

$$\|\psi_{31}\|_{L^2_x(\mathbb{R}^3)} \lesssim \langle B\rangle^{\sigma}\|\psi\|_{L^2_x(\mathbb{R}^3)}. \tag{5.135}$$

For ψ_{32},

$$\begin{aligned}\psi_{32} &= \langle x\rangle^{\sigma}F(H_0>1)\frac{e^{itH_0}}{H_0}V(x)\Omega_+e^{-itH_0}F_{m,M}(H_0)\chi(|x|-B)\psi|_{t=0}^{t=\infty}\\ &-i\int_0^{\infty}dt\langle x\rangle^{\sigma}F(H_0>1)\frac{e^{itH_0}}{H_0}V(x)\Omega_+H_0e^{-itH_0}F_{m,M}(H_0)\chi(|x|-B)\psi =: \psi_{321}+\psi_{322}.\end{aligned} \tag{5.136}$$

For ψ_{321}, when $t=\infty$,

$$\langle x\rangle^{\sigma}F(H_0>1)\frac{e^{itH_0}}{H_0}V(x)\Omega_+e^{-itH_0}F_{m,M}(H_0)\chi(|x|-B)\psi=0 \tag{5.137}$$

since

$$\begin{aligned}&\|\langle x\rangle^{\sigma}F(H_0>1)\frac{e^{itH_0}}{H_0}V(x)\Omega_+e^{-itH_0}F_{m,M}(H_0)\chi(|x|-B)\psi\|_{L^2_x(\mathbb{R}^5)}\\ \le{}&\|\langle x\rangle^{\sigma}F(H_0>1)\frac{e^{itH_0}}{H_0}\langle x\rangle^{-\sigma}\|_{L^2_x\to L^2_x}\big(\|\langle x\rangle^{\sigma}V(x)e^{-itH_0}F_{m,M}(H_0)\chi(|x|-B)\psi\|_{L^2_x(\mathbb{R}^5)}+\\ &\|\langle x\rangle^{\sigma}V(x)(\Omega_+-1)e^{-itH_0}F_{m,M}(H_0)\chi(|x|-B)\psi\|_{L^2_x(\mathbb{R}^5)}\big)\\ &\lesssim_{m,B}\langle t\rangle^{\sigma}\left(\frac{1}{\langle t\rangle^{\sigma+1/2}}+\frac{1}{\langle t\rangle^{\sigma+1/2}}\right)\to 0\end{aligned} \tag{5.138}$$

as $t\to\infty$. Here in (5.138), we use

$$\|\langle x\rangle^{-3}e^{-itH_0}F_{m,M}(H_0)\chi(|x|-B)\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}\lesssim_{m,B}\frac{1}{\langle t\rangle^{\sigma+1/2}}\quad\text{for }\sigma\in(0,2), \tag{5.139}$$

and

$$\begin{aligned}&\|(\Omega_+-1)e^{-itH_0}F_{m,M}(H_0)\chi(|x|-B)\psi\|_{L^2_x(\mathbb{R}^5)}\lesssim\\ &\int_0^{\infty}ds\|e^{isH}V(x)e^{-i(s+t)H_0}F_{m,M}(H_0)\chi(|x|-B)\psi\|_{L^2_x(\mathbb{R}^5)}\\ &\lesssim\int_0^{\infty}ds\|\langle x\rangle^{4}V(x)\|_{L^{\infty}_x}\|\langle x\rangle^{-4}e^{-itH_0}F_{m,M}(H_0)\chi(|x|-B)\psi\|_{L^2_x(\mathbb{R}^5)}\\ &\lesssim_{m,B}\int_0^{\infty}ds\|\langle x\rangle^{4}V(x)\|_{L^{\infty}_x}\frac{1}{\langle s+t\rangle^{\sigma+3/2}}\|\psi\|_{L^2_x(\mathbb{R}^5)}\\ &\lesssim_{m,B}\frac{1}{\langle t\rangle^{\sigma+1/2}}\|\langle x\rangle^{4}V(x)\|_{L^{\infty}_x}\|\psi\|_{L^2_x(\mathbb{R}^5)}\end{aligned} \tag{5.140}$$

by using Duhamel's formula to expand Ω_+. So one has

$$\|\psi_{321}\|_{L^2_x} = \|\langle x\rangle^{\sigma} F(H_0 > 1)\frac{1}{H_0}V(x)\Omega_+ F_{m,M}(H_0)\chi(|x| \quad B)\psi\|_{L^2_x}$$
$$\|\langle x\rangle^{\sigma} F(H_0 > 1)\frac{1}{H_0}V(x)\|_{L^2_x \to L^2_x}\|\psi\|_{L^2_x} \lesssim \|\psi\|_{L^2_x}. \quad (5.141)$$

For ψ_{322}, using Duhamel formula to expand Ω_+, one has

$$\|\psi_{322}\|_{L^2_x(\mathbb{R}^3)} \lesssim \int_0^\infty dt\langle t\rangle^{\sigma}\|\langle x\rangle^{2\sigma+1+\epsilon}V(x)\|_{L^2_x \to L^2_x} \frac{1}{\langle t\rangle^{\sigma+1+\epsilon}}\|\langle x\rangle^{2+\sigma+\epsilon}\chi(|x| \quad B)\psi\|_{L^2_x}$$
$$\lesssim_{\epsilon,\sigma,B} \|\psi\|_{L^2_x} \quad (5.142)$$

for some $\epsilon > 0$. Here we use

$$\|\langle x\rangle^{-(\sigma+\epsilon+1)}e^{-itH_0}F_{m,M}(H_0)\langle x\rangle^{-2-\sigma-\epsilon}\|_{L^2_x \to L^2_x} \lesssim_{\epsilon,\sigma} \frac{1}{\langle t\rangle^{\sigma+1+\epsilon}} \quad (5.143)$$

and

$$\|\int_0^\infty ds\langle x\rangle^{-(\sigma+\epsilon+1)}e^{isH}V(x)e^{-i(t+s)H_0}F_{m,M}(H_0)\langle x\rangle^{-2-\sigma-\epsilon}\|_{L^2_x \to L^2_x} \lesssim_{\epsilon,\sigma} \frac{1}{\langle t\rangle^{\sigma+1+\epsilon}}. \quad (5.144)$$

Based on (5.135), (5.141) and (5.142), one has

$$\|\psi_3\|_{L^2_x} \lesssim_{\epsilon,\sigma,B} \|\psi\|_{L^2_x}. \quad (5.145)$$

Based on (5.129), (5.131) and (5.145), one gets (5.127) and finish the proof.

□

Now it is time to prove Lemma 5.1.2

Proof of Lemma 5.1.2. By the compactness of C, C can be approximated by finite rank operators

$$\|C - CP_b - \sum_{l=1}^{N} c_l(\bar{\phi}_l(x), \cdot)_{L^2_x(\mathbb{R}^3)}\psi_l(x)\|_{L^2_x \to L^2_x} \leq \frac{1}{1000}, \quad j = 1, 2 \quad (5.146)$$

with $c_l > 0$,

$$\begin{cases} \|\psi_l(x)\|_{L^2_x(\mathbb{R}^3)} = 1, \\ \bar{\phi}_l(x) = P_c\bar{\phi}_l(x) \\ \|\bar{\phi}_l(x)\|_{L^2_x(\mathbb{R}^3)} = 1 \end{cases}. \quad (5.147)$$

Here $P_b := 1 - P_c$ is the projection on the space of bound states of H. By AC, for each $\bar{\phi}_l(x)$, there exists $\tilde{\phi}_l(x) \in L^2_x(\mathbb{R}^3)$ such that

$$\|\tilde{\phi}_l(x)\|_{L^2_x(\mathbb{R}^3)} = \|\bar{\phi}_l(x)\|_{L^2_x(\mathbb{R}^3)} = 1 \tag{5.148}$$

and

$$\bar{\phi}_l(x) = \Omega_+ \tilde{\phi}_l(x). \tag{5.149}$$

We rewrite $\bar{\phi}_l(x)$ as

$$\bar{\phi}_l(x) = \Omega_+(1 - F_{m_l,M_{l2}}(H_0)\chi(|x| < M_{l1}))\tilde{\phi}_l(x) + \tag{5.150}$$

$$\Omega_+ F_{m_l,M_{l2}}(H_0)\chi(|x| < M_{l1})\tilde{\phi}_l(x) \tag{5.151}$$

for some sufficiently large $M_{l1} > 0$, sufficiently large M_{l2} and sufficiently small m_l such that

$$\|(1 - F_{m_l,M_{l2}}(H_0)\chi(|x| < M_{l1}))\tilde{\phi}_l(x)\|_{L^2_x(\mathbb{R}^3)} \leq \frac{1}{1000\langle c_l\rangle N} \tag{5.152}$$

which implies

$$c_l\|(\Omega_+(1 - F_{m_l,M_{l2}}(H_0)\chi(|x| < M_{l1}))\tilde{\phi}_l(x), \cdot)_{L^2_x(\mathbb{R}^3)}\psi_l(x)\|_{L^2_x \to L^2_x} \leq \frac{1}{1000N}. \tag{5.153}$$

Take

$$\phi_l(x) = F_{m_l,M_{l2}}(H_0)\chi(|x| < M_{l1})\tilde{\phi}_l(x), \tag{5.154}$$

and one has

$$\|C - CP_b - \sum_{l=1}^{N} c_l(\Omega_+\phi_l(x), \cdot)_{L^2_x(\mathbb{R}^3)}\psi_l(x)\|_{L^2_x \to L^2_x} \tag{5.155}$$

$$\leq \|C - CP_b - \sum_{l=1}^{N} c_l(\bar{\phi}_l(x), \cdot)_{L^2_x(\mathbb{R}^3)}\psi_l(x)\|_{L^2_x \to L^2_x} + \tag{5.156}$$

$$\sum_{l=1}^{N} c_l\|(\Omega_+(1 - F_{m_l,M_{l2}}(H_0)\chi(|x| < M_{l1}))\tilde{\phi}_l(x), \cdot)_{L^2_x(\mathbb{R}^3)}\psi_l(x)\|_{L^2_x \to L^2_x} \tag{5.157}$$

$$\leq \frac{1}{1000} + \sum_{l=1}^{N} \frac{1}{1000N} \tag{5.158}$$

$$\leq \frac{1}{500}. \tag{5.159}$$

Take

$$C_m := CP_b + \sum_{l=1}^{N} c_l(\Omega_+\phi_l(x), \cdot)_{L^2_x(\mathbb{R}^3)}\psi_l(x) \tag{5.160}$$

and

$$C_r := C - C_m. \tag{5.161}$$

Then we get (5.37). (5.39) follows by writing

$$(\Omega_+\phi_l(x), e^{-itH}P_c\psi)_{L^2_x(\mathbb{R}^3)} \tag{5.162}$$

$$=(F_{m_l,M_{l2}}(H_0)\chi(|x|<M_{l1})\tilde{\phi}_l(x), e^{-itH_0}\Omega_+\psi)_{L^2_x(\mathbb{R}^3)} \tag{5.163}$$

via using intertwining property

$$e^{-itH_0}\Omega_+ = \Omega_+ e^{-itH}P_c \quad \text{on } L^2_x(\mathbb{R}^n). \tag{5.164}$$

Using Strichartz estimates for the free flow e^{-itH_0}, one has

$$(F_{m_l,M_{l2}}(H_0)\chi(|x|<M_{l1})\tilde{\phi}_l(x), e^{-itH_0}\Omega_+\psi)_{L^2_x(\mathbb{R}^3)} \in L^2_t(\mathbb{R}). \tag{5.165}$$

In addition, $CP_b e^{-itH}P_c\psi = 0$. Hence, one gets (5.39). For (5.38), according to Lemma 5.2.5, write $\langle x\rangle^{-\eta}(1-C_r)^{-1}C_r\langle x\rangle^{\sigma}$ as

$$\langle x\rangle^{-\eta}(1-C_r)^{-1}C_r\langle x\rangle^{\sigma} = \langle x\rangle^{-\eta}(1-C_r)^{-1}C_1\langle x\rangle^{\sigma} + \langle x\rangle^{-\eta}(1-C_r)^{-1}C_2\langle x\rangle^{\sigma} \tag{5.166}$$

$$-\langle x\rangle^{-\eta}(1-C_r)^{-1}C_m\langle x\rangle^{\sigma} \tag{5.167}$$

$$= \langle x\rangle^{-\eta}(1-C_r)^{-1}\langle x\rangle^{(1-\epsilon')}\langle x\rangle^{-(1-\epsilon')}C_1\langle x\rangle^{\sigma} + \tag{5.168}$$

$$\langle x\rangle^{-\eta}(1-C_r)^{-1}\langle x\rangle^{(1-\epsilon')}\langle x\rangle^{-(1-\epsilon')}C_2\langle x\rangle^{\sigma} \tag{5.169}$$

$$-\langle x\rangle^{-\eta}(1-C_r)^{-1}C_m\langle x\rangle^{\sigma}. \tag{5.170}$$

Using (5.110) in Lemma 5.2.5, one has that for $\sigma \in (1, 101/100)$,

$$\|\langle x\rangle^{-\eta}(1-C_r)^{-1}\langle x\rangle^{(1-\epsilon')}\langle x\rangle^{-(1-\epsilon')}C_1\langle x\rangle^{\sigma}\|_{L^2_x\to L^2_x} \lesssim$$
$$\|\langle x\rangle^{-\eta}(1-C_r)^{-1}\langle x\rangle^{(1-\epsilon')}\|_{L^2_x\to L^2_x}\|\langle x\rangle^{-(1-\epsilon')}C_1\langle x\rangle^{\sigma}\|_{L^2_x\to L^2_x} \lesssim 1. \tag{5.171}$$

Similarly, we have

$$\|\langle x\rangle^{-\eta}(1-C_r)^{-1}C_2\langle x\rangle^{\sigma}\|_{L^2_x\to L^2_x} \lesssim 1. \tag{5.172}$$

We also have

$$\|\langle x\rangle^{-\eta}(1-C_r)^{-1}C_m\langle x\rangle^{\sigma}\|_{L^2_x\to L^2_x} \lesssim 1 \tag{5.173}$$

by using Lemma 5.2.6 and using the fact that all bound states are localized in space. We finish the proof.

□

5.2.4 Proof of Theorem 5.1.1

Proof for Theorem 5.1.1. According to the outline of the proof for Theorem 5.1.1, it follows from Lemma 5.1.2. □

5.3 Time-dependent Problems

In this section, we prove Lemma 5.1.3 to get local decay estimates for time dependent potentials.

5.3.1 Lemmas for compactness

We need following lemmas for compactness.

Lemma 5.3.1. *If V satisfies assumption 5.1.2, then*

$$\sup_{t\in\mathbb{R}} \|C_j(t)\langle x\rangle^{1/4-0}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \lesssim_0 1, \quad j=1,2 \tag{5.174}$$

and for all $a\in[0,9/5]$,

$$\sup_{t\in\mathbb{R}} \||P|^{a-\frac{1}{2}+2-0}C_j(t)\frac{1}{|P|^{a+0}}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \lesssim 1, \quad j=1,2. \tag{5.175}$$

Here 0 means $\lim_{\epsilon\downarrow 0}\epsilon$.

Proof. Break $C_1(t)$ into three pieces

$$C_1(t)=\int_0^\infty dvP^+e^{ivH_0}V(x,v+t)\bar{F}(|P|\leq 1)U(t+v,t)+$$
$$\int_1^\infty dvP^+e^{ivH_0}V(x,v+t)F(|P|>1)U(t+v,t)+$$
$$\int_0^1 dvP^+e^{ivH_0}V(x,v+t)F(|P|>1)U(t+v,t)=:C_{11}(t)+C_{12}(t)+C_{13}(t). \tag{5.176}$$

For $C_{11}(t)$, by using

$$|(U(t+v,t)f(x),\langle x\rangle^{a-0}U(t+v,t)f(x))_{L^2_x}| \lesssim |v|^{a-0}\|\langle P\rangle f\|^2_{L^2_x}+\|\langle x\rangle^{a/2-0/2}f\|^2_{L^2_x} \tag{5.177}$$

for $a \in [0,1]$, using duality, for $\delta > 5/2$,

$$\|C_{11}(t)\langle x\rangle^{1/4-0}\|_{L^2_x \to L^2_x} \lesssim \int_0^{\infty} dv \|P^+ e^{ivH_0} V(x,v+t)\bar{F}(|P| \leq 1) U(t+v,t)\langle x\rangle^{1/4-0}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}$$
$$\lesssim \int_0^{\infty} dv |v|^{1/4-0} \|P^+ e^{ivH_0} V(x,t+v)\bar{F}(|P| \leq 1)\langle P\rangle\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} +$$
$$\int_0^{\infty} dv \|P^+ e^{ivH_0} V(x,t+v)\bar{F}(|P| \leq 1)\langle x\rangle^{1/4-0}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}$$
$$\text{(Use Lemma 5.2.1)} \lesssim_0 \|\langle x\rangle^{\delta} V(x,t)\|_{L^{\infty}_{x,t}}. \quad (5.178)$$

Here we use (5.177) since we want to get $|v|^{1/4-0}$ when we control $\langle P\rangle$. When $|v| \geq 1$, for high frequency part, $|v|^{1/4-0}$ will help to take the integral of v. Here we do not have low frequency issue since for $P^+ e^{ivH_0} F(|P| \leq 1/\langle v\rangle^{1/2-\epsilon}) V(x,t+v)$, we get $1/\langle v\rangle^{5/4(1/2-\epsilon)}$ using the $L^2(\mathbb{R}^5)$ volume of the frequency and choosing $\epsilon > 0$ small enough; then $1/\langle v\rangle^{5/4(1/2-\epsilon)-1/5} \in L^1_v(\mathbb{R})$, see Lemma 5.2.1(Near Threshold Estimate).

For $C_{12}(t)$, integrating by parts in v variable by setting

$$U(t+v,t) = \frac{1}{iH_0}\partial_v[e^{-ivH_0}][e^{ivH_0}U(t+v,t)], \quad (5.179)$$

one has

$$C_{12}(t) = P^+ e^{ivH_0} V(x,v+t) F(|P|>1) \frac{1}{iH_0} U(t+v,t)|_{v=1}^{v=\infty} +$$
$$\int_1^{\infty} dv P^+ H_0 e^{ivH_0} V(x,v+t) F(|P|>1) \frac{1}{H_0} U(t+v,t)$$
$$\int_1^{\infty} dv P^+ e^{ivH_0} V(x,v+t) F(|P|>1) \frac{1}{H_0} V(x,v+t) U(t+v,t) +$$
$$\int_1^{\infty} dv P^+ e^{ivH_0} \partial_v[V(x,v+t)] F(|P|>1) \frac{1}{iH_0} U(t+v,t). \quad (5.180)$$

Using (5.177), similarly, one has that for $\delta > 5/2$,

$$\|C_{12}(t)\langle x\rangle^{1/4-0}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \lesssim_0 \sum_{j=0}^{1} \|\langle x\rangle^{\delta} \partial_t^j [V(x,t)]\|_{L^{\infty}_{x,t}} \lesssim \sum_{j=0}^{N} \|\langle x\rangle^{\delta} V_j(x)]\|_{L^{\infty}_{x,t}}. \quad (5.181)$$

For $C_{13}(t)$, by using a similar argument as we did for C_{12} in the proof of Lemma 5.2.5, one has

$$\|C_{13}(t)\langle x\rangle^2\|_{L^2_x \to L^2_x} \lesssim 1. \quad (5.182)$$

So one concludes

$$\sup_{t\in\mathbb{R}} \|C_1(t)\langle x\rangle^{1/4-0}\|_{L^2_x(\mathbb{R}^5)\to \mathbb{R}^5} \lesssim_0 1. \quad (5.183)$$

Similarly, one has

$$\sup_{t\in\mathbb{R}}\|C_2(t)\langle x\rangle^{1/4-\epsilon_0}\|_{L^2_x(\mathbb{R}^5)\to\mathbb{R}^5}\lesssim_{\epsilon_0} 1. \tag{5.184}$$

For (5.175), it suffices to check

$$\sup_{t\in\mathbb{R}}\||P|^{a-1/2+2\epsilon_0}P\int_0^\infty du F(|P|\leq\frac{1}{\langle u\rangle^{1/2-\epsilon_0}})e^{iuH_0}V(x,u+t)U(t+u,t)\frac{1}{|P|^{a+\epsilon_0}}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}. \tag{5.185}$$

Let

$$A(a):=|P|^{a-1/2+2\epsilon_0}P\int_0^\infty du F(|P|\leq\frac{1}{\langle u\rangle^{1/2-\epsilon_0}})e^{iuH_0}V(x,u+t)U(t+u,t)\frac{1}{|P|^{a+\epsilon_0}}. \tag{5.186}$$

Break $A(a)$ into two parts

$$\begin{aligned}A(a)=&|P|^{a-1/2+2\epsilon_0}P\int_0^\infty du F(|P|\leq\frac{1}{\langle u\rangle^{1/2-\epsilon_0}})e^{iuH_0}V(x,u+t)U(t+u,t)\frac{1}{|P|^{a+\epsilon_0}}\\ &\bar{F}(|P|>\frac{1}{\langle u\rangle^{1/2}})+|P|^{a-1/2+2\epsilon_0}P\int_0^\infty du F(|P|\leq\frac{1}{\langle u\rangle^{1/2-\epsilon_0}})e^{iuH_0}V(x,u+t)U(t+u,t)\\ &\frac{1}{|P|^{a+\epsilon_0}}F(|P|\leq\frac{1}{\langle u\rangle^{1/2}})=:A_1(a)+A_2(a).\end{aligned} \tag{5.187}$$

For $A_1(a)$, using Lemma 5.2.2, one has

$$\begin{aligned}\|A_1(a)\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}\lesssim&\int_0^\infty du\||P|^{a-1/2+2\epsilon_0}F(|P|\leq\frac{1}{\langle u\rangle^{1/2-\epsilon_0}})\|_{L^1_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}\times\langle u\rangle^{1/2(a+\epsilon_0)}\\ \lesssim&\frac{1}{\langle u\rangle^{1/2a-1/4+\epsilon_0}}\times\frac{1}{\langle u\rangle^{5/4-\epsilon_0}}\times\langle u\rangle^{1/2(a+\epsilon_0)}\lesssim 1.\end{aligned} \tag{5.188}$$

For $A_2(a)$, since for $a\in[0,9/5]$,

$$\begin{aligned}&\|U(t+u,t)\frac{1}{|P|^{a+\epsilon_0}}F(|P|\leq\frac{1}{\langle u\rangle^{1/2}})\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)+L^\infty_x(\mathbb{R}^5)}\\ \leq&\|e^{-iuH_0}\frac{1}{|P|^{a+\epsilon_0}}F(|P|\leq\frac{1}{\langle u\rangle^{1/2}})\|_{L^2_x(\mathbb{R}^5)\to L^\infty_x(\mathbb{R}^5)}+\\ &\|\int_0^u dvU(t+u,t+v)V(x,t+v)e^{-ivH_0}\frac{1}{|P|^{a+\epsilon_0}}F(|P|\leq\frac{1}{\langle u\rangle^{1/2}})\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}\\ \lesssim&1+\frac{|u|}{\langle u\rangle^{5/4-a/2-\epsilon_0/2}}\lesssim 1+\frac{1}{\langle u\rangle^{1/4-a/2-\epsilon_0/2}},\end{aligned} \tag{5.189}$$

one has

$$\begin{aligned}\|A_2(a)\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}\lesssim&\int_0^\infty du\||P|^{a-1/2+2\epsilon_0}F(|P|\leq\frac{1}{\langle u\rangle^{1/2-\epsilon_0}})\|_{L^1_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}\times\\ &\|V(x,t+u)\|_{L^\infty_x+L^2_x\to L^1_x}(1+\frac{1}{\langle u\rangle^{1/4-a/2-\epsilon_0/2}})\\ \lesssim&\int_0^\infty du\frac{1}{\langle u\rangle^{1/2a-1/4+\epsilon_0}}\times\frac{1}{\langle u\rangle^{5/4-\epsilon_0}}(1+\frac{1}{\langle u\rangle^{1/4-a/2-\epsilon_0/2}})\lesssim 1.\end{aligned} \tag{5.190}$$

Based on (5.188) and (5.190), one gets (5.175) and finish the proof. $\square$

Lemma 5.3.2. *If $V(x,t)$ satisfies assumption 5.1.2, then for all $\epsilon > 0$, given $M_1 > 1$, there exists $M_0 > 0$ such that when $M \geq M_0$,*

$$\sup_{t\in\mathbb{R}} \left\| \int_M^\infty dv P^+ e^{ivH_0} V(x,v+t) U(t+v,t) \right\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \leq \epsilon, \tag{5.191}$$

$$\sup_{t\in\mathbb{R}} \left\| \int_0^{1/M} dv P^+ e^{ivH_0} V(x,v+t) U(t+v,t) \right\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \leq \epsilon, \tag{5.192}$$

$$\sup_{t\in\mathbb{R}} \left\| \int_{1/M}^M dv P^+ \bar{F}(H_0 \leq 1/M) e^{ivH_0} V(x,t+v) U(t+v,t) \right\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \leq \epsilon \tag{5.193}$$

and

$$\sup_{t\in\mathbb{R}} \left\| \int_{1/M}^M dv P^+ e^{ivH_0} V(x,t+v) U(t+v,t) \bar{F}(H_0 > M^{100} M_1^2) F(|x| \leq M_1) \right\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \leq \epsilon \tag{5.194}$$

Proof. (5.191) follows from Lemma 5.2.1. (5.192) follows from the fact that

$$\sup_{t\in\mathbb{R}} \left\| \int_0^{1/M} dv P^+ e^{ivH_0} V(x,v+t) U(t+v,t) \right\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \leq \frac{1}{M} \sup_{t\in\mathbb{R}} \|V(x,t)\|_{L^\infty_x(\mathbb{R}^5)}. \tag{5.195}$$

(5.193) follows from

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left\| \int_{1/M}^M dv P^+ \bar{F}(H_0 \leq 1/M) e^{ivH_0} V(x,t+v) U(t+v,t) \right\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \\
&\leq \sup_{t\in\mathbb{R}} \int_{1/M}^M dv \|\bar{F}(q^2 \leq 1/M)\|_{L^2_q(\mathbb{R}^5)} \|V(x,t+v) U(t+v,t)\|_{L^2_x(\mathbb{R}^5)\to \mathcal{F}L^\infty_x(\mathbb{R}^5)} \\
&\lesssim \sup_{t\in\mathbb{R}} \int_{1/M}^M dv \|\bar{F}(q^2 \leq 1/M)\|_{L^2_q(\mathbb{R}^5)} \|V(x,t+v) U(t+v,t)\|_{L^2_x(\mathbb{R}^5)\to L^1_x(\mathbb{R}^5)} \\
&\lesssim \sup_{t\in\mathbb{R}} \int_{1/M}^M dv \frac{1}{M^{5/4}} \|V(x,v)\|_{L^\infty_v L^2_x(\mathbb{R}^{5+1})} \lesssim \frac{1}{M^{1/4}} \sup_{t\in\mathbb{R}} \|V(x,t)\|_{L^\infty_t L^2_x(\mathbb{R}^{5+1})}.
\end{aligned} \tag{5.196}$$

Here $\mathcal{F}L^\infty_x(\mathbb{R}^5)$ denotes

$$\mathcal{F}L^\infty_x(\mathbb{R}^5) := \{f(x) : \text{ the Fourier transform of } f \text{ is in } L^\infty\}. \tag{5.197}$$

For (5.194), it follows by using Duhamel's formula to expand $U(v+t,t)$ from the right hand side

$$\begin{aligned}
U(t+v,t)\bar{F}(H_0 > M^{100}M_1^2)F(|x| \leq M_1) = e^{-ivH_0} \bar{F}(H_0 > M^{100}M_1^2) F(|x| \leq M_1) + \\
(-i)\int_0^v dv_1 U(t+v,t+v_1) V(x,t+v_1) e^{-iv_1H_0} \bar{F}(H_0 > M^{100}M_1^2) F(|x| \leq M_1),
\end{aligned} \tag{5.198}$$

using that when $v \in [1/M, M]$ and $v_1 \in [1/M^{10}, M]$,

$$\|\langle x\rangle^{-1} e^{-ivH_0}\bar{F}(H_0 > M^{100}M_1^2)F(|x| \le M_1)\|_{L_x^2(\mathbb{R}^5)\to L_x^2(\mathbb{R}^5)} \lesssim \frac{1}{M^{90}} \tag{5.199}$$

and using that when $v_1 \in [0, 1/M^{10}]$,

$$\int_0^{1/M^{10}} dv_1 \|U(t+v,t+v_1)V(x,t+v_1)e^{-iv_1H_0}\bar{F}(H_0 > M^{100}M_1^2)F(|x| \le M_1)\|_{L_x^2(\mathbb{R}^5)\to L_x^2(\mathbb{R}^5)}$$
$$\lesssim \frac{1}{M^{10}} \sup_{v_1 \in \mathbb{R}} \|V(x,v_1)\|_{L^\infty_{x,v_1}}. \tag{5.200}$$

Similarly, we have the same result for P_-. We finish the proof. □

Lemma 5.3.3. *If $V(x,t)$ satisfies assumption 5.1.2, then*

$$\lim_{M\to\infty} \sup_{t\in\mathbb{R}} \|(1 - F(|x| \le M)F(|P| \le M))P_-(\Omega_{t,-} - 1)\|_{L_x^2(\mathbb{R}^5)\to L_x^2(\mathbb{R}^5)} = 0. \tag{5.201}$$

Proof. Let

$$C_-(t,M) := \int_{\frac{1}{M^{1/100}}}^{M^{1/100}} ds e^{isH_0}V(x,s+t)U(s+t,t). \tag{5.202}$$

Based on Lemma 5.3.2, it suffices to check the validity of

$$\lim_{M\to\infty} \sup_{t\in\mathbb{R}} \|(1 - F(|x| \le M)F(|P| \le M))P_- C_-(t,M)\|_{L_x^2(\mathbb{R}^5)\to L_x^2(\mathbb{R}^5)} = 0. \tag{5.203}$$

On one hand,

$$\begin{aligned}
&\|F(|x| > M)P_- C_-(t,M)\|_{L_x^2(\mathbb{R}^5)\to L_x^2(\mathbb{R}^5)} \\
\le& \|F(|x| > M)P_- \int_{\frac{1}{M^{1/100}}}^{M^{1/100}} dsF(|P| > \sqrt{M})e^{isH_0}V(x,s+t)U(s+t,t)\|_{L_x^2(\mathbb{R}^5)\to L_x^2(\mathbb{R}^5)} + \\
&\|F(|x| > M)P_- \int_{\frac{1}{M^{1/100}}}^{M^{1/100}} ds\bar{F}(|P| \le \sqrt{M})e^{isH_0}V(x,s+t)U(s+t,t)\|_{L_x^2(\mathbb{R}^5)\to L_x^2(\mathbb{R}^5)} \\
&=: A_1(M,t) + A_2(M,t),
\end{aligned} \tag{5.204}$$

$$\lim_{M\to\infty} \sup_{t\in\mathbb{R}} A_1(M,t) = 0 \tag{5.205}$$

and

$$\lim_{M\to\infty} \sup_{t\in\mathbb{R}} A_2(M,t) = 0 \tag{5.206}$$

by using (5.61) in Lemma 5.2.1 and (5.67) in Lemma 5.2.2, respectively.

On the other hand, similarly,

$$\lim_{M\to\infty}\sup_{t\in\mathbb{R}}\|F(|P|>M)P\ C\ (t,M)\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}=0. \tag{5.207}$$

Thus,

$$\lim_{M\to\infty}\sup_{t\in\mathbb{R}}\sup\|(1-F(|x|\le M)F(|P|\le M))P\ C\ (t,M)\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}$$
$$\le\lim_{M\to\infty}\sup_{t\in\mathbb{R}}\sup\|F(|x|>M)P\ C\ (t,M)\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}+$$
$$\lim_{M\to\infty}\sup_{t\in\mathbb{R}}\sup\|F(|x|\le M)F(|P|>M)P\ C\ (t,M)\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}=0. \tag{5.208}$$

We finish the proof.

□

5.3.2 Properties of the time-dependent scattering projection

In this section, we derive properties of the projection on the space of scattering states for time-dependent problems. In this case, the space of scattering states may vary if we start at a different time. One can see it from the expression of such a projection

$$P_c(t)=\Omega_{t,+}\Omega_{t,+}. \tag{5.209}$$

Here the time-dependence comes from the definition of wave operator and its adjoint

$$\Omega_{t,+}=s\text{-}\lim_{u\to\infty}U(t,t+u)e^{-iuH_0}\quad\text{on } L^2_x(\mathbb{R}^5), \tag{5.210}$$

$$\Omega_{t,+}=\Omega_{\alpha,t,+}:=s\text{-}\lim_{u\to\infty}e^{iuH_0}F_c\left(\frac{|x-2uP|}{u^\alpha}\ge 1\right)U(u+t,t)\quad\text{on } L^2_x(\mathbb{R}^5) \tag{5.211}$$

for $\alpha\in(0,1-2/5)$.

Existence of time-dependent wave operators and Floquet wave operators

The existence of $\Omega_{t,+}$ follows from Cook's method.

Lemma 5.3.4. *If $V(x,t)\in L^1_tL^2_x(\mathbb{R}^{5+1})$, for $\alpha\in(0,1-2/5)$, $\Omega_{\alpha,t,+}$ exists. So $\Omega_{t,+}$ exists.*

Proof. It follows from the technique initiated in [91]. □

We change such time dependence to a time independent problem via Floquet wave operators.

Lemma 5.3.5. *For* $s_j = t - \frac{2\pi}{|\omega_j|}[\frac{|\omega_j| t}{2\pi}]$, $j = 1, \cdots, N$,

$$\Omega_{t,+} = \Omega_{K,+}(\vec{s}) := s\text{-}\lim_{a\to\infty} e^{iaH_0} U_{\vec{s}}(a,0) P_c(t), \quad \text{on } L^2_x(\mathbb{R}^5) \tag{5.212}$$

where $U_{\vec{s}}(t,0)$ *satisfies*

$$i\partial_a[U_{\vec{s}}(a,0)] = (-\Delta_x + V_0(x) + \sum_{j=1}^{N_1} V_j(x)\sin(a+\omega_j s_j) + \sum_{j=N_1+1}^{N} V_j(x)\cos(a+\omega_j s_j)) U_{\vec{s}}(a,0). \tag{5.213}$$

Proof. It follows from a similar argument as what we did in the proof of Lemma 5.9. □

Recall that $\mathcal{H}_F = L^2_{\vec{s}} L^2_x(\mathbb{R}^5 \times \mathbb{T}_1 \times \cdots \times \mathbb{T}_N)$. The Floquet wave operators are defined by

$$\Omega_{K,+} := s\text{-}\lim_{t\to\infty} e^{itK_0} e^{-itK} P_c(K) \quad \text{on } \mathcal{H}_F \tag{5.214}$$

and

$$\Omega_{K,+} := s\text{-}\lim_{t\to\infty} e^{itK} e^{-itK_0} \quad \text{on } \mathcal{H}_F \tag{5.215}$$

with $P_c(K)$, the projection on the continuous spectrum of K in $\mathcal{H}_F$ space. The existence of $\Omega_{K,+}$ follows from Cook's method.

Lemma 5.3.6. *If* $V_F(x,\vec{s}) \in L^\infty_{\vec{s}} L^2_x(\mathbb{R}^5 \times \mathbb{T}_1 \times \cdots \times \mathbb{T}_N)$, *for* $\alpha \in (0, 1-2/5)$, $\Omega_{K,\alpha,+}$

$$\Omega_{K,\alpha,+} := s\text{-}\lim_{t\to\infty} e^{itK_0} F_c(\frac{|x-2tP|}{t^\alpha} \leq 1) e^{-itK} \quad \text{on } \mathcal{H}_F \tag{5.216}$$

exists.

Proof. It follows from the technique initiated in [91]. □

Once we get the existence of the free channel wave operators $\Omega_{t,\alpha,+}, \Omega_{K,\alpha,+}$, one has the existence of $\Omega_{t,+}, \Omega_{K,+}$ via the construction of the projection on the space of corresponding scattering states.

Scattering Projection and properties of time-dependent bound states

Based on the existence of free channel wave operators, we define Scattering Projections on their range . In time-dependent section, we always assume that $V(x,t)$ satisfies assumption 5.1.2.

Definition 2 (Projection on the space of time-dependent scattering states)**.** *Projection on the space of time-dependent scattering states at time t $P_c(t)$ can be equivalently defined by*

$$P_c(t) = s\text{-}\lim_{u\to\infty} U(t,t+u)F_c\left(\frac{|x-2uP|}{u^\alpha}\le 1\right)U(t+u,t) \quad \textit{on } L^2_x(\mathbb{R}^5). \tag{5.217}$$

Let

$$P_c(\vec{s}) := s\text{-}\lim_{u\to\infty} U_{\vec{s}}(0,u)F_c\left(\frac{|x-2uP|}{u^\alpha}\le 1\right)U_{\vec{s}}(u,0) \quad \text{on } L^2_x(\mathbb{R}^5) \tag{5.218}$$

and

$$P_{F,c} := s\text{-}\lim_{u\to\infty} e^{iuK}F_c\left(\frac{|x-2uP|}{u^\alpha}\le 1\right)e^{-iuK} \quad \text{on } \mathcal{H}_F. \tag{5.219}$$

Lemma 5.3.7. *The limits defining $P_c(t), P_c(\vec{s})$ and $P_{F,c}$ exist.*

Proof. Their existence follows from the existence of free channel wave operators and their adjoint. ☐

Taking $s_j = t - \frac{2\pi}{|\omega_j|}[\frac{|\omega_j|t}{2\pi}]$, $j=1,\dots,N$,

$$P_c(\vec{s}) = P_c(t). \tag{5.220}$$

In addition,

$$P_c(\vec{s}')f(x) = [P_{F,c}f(x)]|_{\vec{s}=\vec{s}'}. \tag{5.221}$$

$P_{F,b}$(mentioned in the introduction), the projection on the space of all bound states of K in $\mathcal{H}_F$, is defined by

$$P_{F,b} = 1 - P_{F,c}. \tag{5.222}$$

Definition 3 (Projection on the space of all bound states)**.** *Projection on the space of all bound states at time t, $P_b(t)$, is defined by*

$$P_b(t) := 1 - P_c(t). \tag{5.223}$$

$P_b(t)$ exists on $L^2_x(\mathbb{R}^5)$ since $P_c(t)$ exists on $L^2_x(\mathbb{R}^5)$. Now we give a characterization of the space of bound states at time t. Let $\mathcal{B}(t)$ denote the space of all bound states at time t and

$$\mathcal{B}_F := \{[\phi(x,\vec{s})] : \phi(x,\vec{s}) \text{ is a bound state of Floquet operator } K \text{ on } \mathcal{H}_F\} \tag{5.224}$$

with

$$[\phi(x,\vec{s})] = \{f(x,\vec{s}) : f(x,\vec{s}) = e^{-i(\sum_{j=1}^{N} n_j\omega_j s_j)}\phi(x,\vec{s}) \quad \text{for some } (n_1,\ldots,n_N) \in \mathbb{Z}^N\}. \tag{5.225}$$

Let $P_{b,[\phi(x,\vec{s})]}$ denote the projection on $[\phi(x,\vec{s})]$ in $\mathcal{H}_F$. Then

$$P_{b,[\phi(x,\vec{s})]}f(x) = \sum_{(n_1,\ldots,n_N)\in\mathbb{Z}^N} (e^{-i(\sum_{j=1}^{N} n_j\omega_j s_j)}\phi(x,\vec{s}), f(x))_{\mathcal{H}_F} e^{-i(\sum_{j=1}^{N} n_j\omega_j s_j)}\phi(x,\vec{s}) \tag{5.226}$$

$$= \sum_{(n_1,\ldots,n_N)\in\mathbb{Z}^N} (\phi_{-n}(x), f(x))_{L^2_x} e^{-i(\sum_{j=1}^{N} n_j\omega_j s_j)}\phi(x,\vec{s}) \tag{5.227}$$

$$=(\phi(x,\vec{s}), f(x))_{L^2_x}\phi(x,\vec{s}) \tag{5.228}$$

where we write $\phi(x,\vec{s})$ as

$$\phi(x,\vec{s}) = \sum_{(n_1,\ldots,n_N)\in\mathbb{Z}^N} \phi_n(x) e^{i(\sum_{j=1}^{N} n_j\omega_j s_j)}. \tag{5.229}$$

Lemma 5.3.8. *For all $t \in \mathbb{R}$, $dim(\mathcal{B}(t)) = dim(\mathcal{B}_F) < \infty$.*

Proof. Given a normalized $\psi(t) \in \mathcal{B}(t)$ ($\|\psi(t)\|_{L^2_x} = 1$), using incoming and outgoing decomposition, one has

$$\psi(t) =P^+\psi(t) + P^-\psi(t) \tag{5.230}$$

$$=P^+(1-\Omega_{t,+})\psi(t) + P^-(1-\Omega_{t,-})\psi(t). \tag{5.231}$$

Based on Lemma 5.3.3, one has that

$$\psi(t) = F(|x| \leq M)F(|P| \leq M)[P^+(1-\Omega_{t,+}) + P^-(1-\Omega_{t,-})]\psi(t) + C_{M,r}\psi(t) \tag{5.232}$$

with

$$\lim_{M\to\infty} \|C_{M,r}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} = 0. \tag{5.233}$$

Choose $M > 0$ large enough such that

$$\|C_{M,r}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} < 1/100. \tag{5.234}$$

$F(|x| \le M)F(|P| \le M)$ is a time-independent compact operator which can be approximated by a finite rank operator. Choose $N_0 > 0$ such that

$$\|F(|x| \le M)F(|P| \le M) - \sum_{j=1}^{N_0}(\phi_j(x), \cdot)_{L^2_x(\mathbb{R}^5)}\psi_j(x)\|_{L^2_x(\mathbb{R}^5 \to L^2_x(\mathbb{R}^5))} < 1/100. \tag{5.235}$$

If $dim(B(t)) = \infty$, we can always choose $\psi(t) \in B(t)$ with $\|\psi(t)\|_{L^2_x(\mathbb{R}^5)} = 1$ such that

$$(\phi_j(x), \psi(t))_{L^2_x(\mathbb{R}^5)} = 0 \quad \text{for all } j = 1, \dots, N_0. \tag{5.236}$$

So

$$\|\psi(t)\|_{L^2_x(\mathbb{R}^5)} \le 1/100 + 1/100 < 1. \tag{5.237}$$

Contradiction. So for each $t \in \mathbb{R}$, $dim(B(t)) < \infty$. Since taking $s_j = t - \frac{2\pi}{|\omega_j|}[\frac{|\omega_j|t}{2\pi}]$, $j = 1, \dots, N$,

$$P_b(\vec{s}) = P_b(t), \quad P_b(\vec{s}')f(x) = [P_{F,c}f(x)]|\vec{s} = \vec{s}', \tag{5.238}$$

we have $dim(B_F) < \infty$ and $dim(B(t)) = dim(B_F)$ for all $t \in \mathbb{R}$.

□

Now we prove the continuity of $P_b(\vec{s})$ in $\vec{s}$. Let

$$\Omega_+(\vec{s}) := s\text{-}\lim_{t\to\infty} U_{\vec{s}}(0,t)e^{-itH_0} \quad \text{on } L^2_x(\mathbb{R}^5), \tag{5.239}$$

and

$$\Omega_+(\vec{s})^* := s\text{-}\lim_{t\to\infty} e^{itH_0}U_{\vec{s}}(t,0)P_c(\vec{s}) \quad \text{on } L^2_x(\mathbb{R}^5). \tag{5.240}$$

Lemma 5.3.9. *Given $c, M > 0$, for all $n \ge 1$,*

$$\lim_{\vec{s}'\to\vec{s}}\|(\Omega_+(\vec{s}')^* - \Omega_+(\vec{s})^*)F(|P| > c)\chi(|x| \le M)\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} = 0 \tag{5.241}$$

and

$$\lim_{\vec{s}'\to\vec{s}}\|(\Omega_+(\vec{s}') - \Omega_+(\vec{s}))\chi(|x| \le M)\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} = 0. \tag{5.242}$$

Proof. Using Duhamel's formula, one has

$$\begin{aligned}(\Omega_+(\vec{s}') - \Omega_+(\vec{s}))F(|P| > c)\chi(|x| \leq M) = \\ i\int_0^{\infty} dt U_{\vec{s}}(0,t)V(x, t + \sum_{j=1}^{N} s_j)e^{-itH_0}F(|P| > c)\chi(|x| \leq M) \\ - i\int_0^{\infty} dt U_{\vec{s}'}(0,t)V(x, t + \sum_{j=1}^{N} s_j')e^{-itH_0}F(|P| > c)\chi(|x| \leq M).\end{aligned} \tag{5.243}$$

For any $\epsilon > 0$, by using

$$\|\langle x\rangle^{-3}e^{-itH_0}F(|P| > c)\chi(|x| \leq M)\|_{L_x^2(\mathbb{R}^n)\to L_x^2(\mathbb{R}^n)} \lesssim_{n,M,c} \frac{1}{\langle t\rangle^2}, \tag{5.244}$$

there exists $b = b(\epsilon, c, M, n) > 0$ such that

$$\sup_{\eta\in\mathbb{R}^N} \|\int_b^{\infty} dt U_\eta(0,t)V(x, t+\sum_{j=1}^{N}\eta_j)e^{-itH_0}F(|P| > c)\chi(|x| \leq M)\|_{L_x^2(\mathbb{R}^n)\to L_x^2(\mathbb{R}^n)} < \epsilon/100. \tag{5.245}$$

So

$$\begin{aligned}\|\int_b^{\infty} dt U_{\vec{s}}(0,t)V(x, t + \sum_{j=1}^{N} s_j)e^{-itH_0}F(|P| > c)\chi(|x| \leq M) \\ - \int_b^{\infty} dt U_{\vec{s}'}(0,t)V(x, t + \sum_{j=1}^{N} s_j')e^{-itH_0}F(|P| > c)\chi(|x| \leq M)\|_{L_x^2(\mathbb{R}^n)\to L_x^2(\mathbb{R}^n)} < \epsilon/50.\end{aligned} \tag{5.246}$$

For

$$R_b(\vec{\eta}) := \int_0^b dt U_\eta(0,t)V(x, t + \sum_{j=1}^{N}\eta_j)e^{-itH_0}F(|P| > c)\chi(|x| \leq M), \tag{5.247}$$

use the continuity of $V(x, t + \sum_{j=1}^{N}\eta_j)$ in η and

$$\lim_{\vec{s}'\to\vec{s}} \sup_{t\in[0,b]} \|U_{\vec{s}}(0,t) - U_{\vec{s}'}(0,t)\|_{L_x^2(\mathbb{R}^n)\to L_x^2(\mathbb{R}^n)} = 0. \tag{5.248}$$

One has that there exists $\delta = \delta(n, \epsilon, c, M, n) > 0$ such that when $|\vec{s}' - \vec{s}| < \delta$,

$$\begin{aligned}\|\int_0^b dt U_{\vec{s}}(0,t)V(x, t + \sum_{j=1}^{N} s_j)e^{-itH_0}F(|P| > c)\chi(|x| \leq M) \\ - \int_0^b dt U_{\vec{s}'}(0,t)V(x, t + \sum_{j=1}^{N} s_j')e^{-itH_0}F(|P| > c)\chi(|x| \leq M)\|_{L_x^2(\mathbb{R}^n)\to L_x^2(\mathbb{R}^n)} < \epsilon/50.\end{aligned} \tag{5.249}$$

Based on (5.258) and (5.260), one has that when $|\vec{s}' - \vec{s}| < \delta$,

$$\|(\Omega_+(\vec{s}') - \Omega_+(\vec{s}))F(|P| > c)\chi(|x| \leq M)\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} < \epsilon/25. \tag{5.250}$$

Since

$$\lim_{c \downarrow 0} \|F(|P| > c)\chi(|x| \leq M)\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} = 0, \tag{5.251}$$

one has

$$\lim_{\vec{s}' \to \vec{s}} \|(\Omega_+(\vec{s}') - \Omega_+(\vec{s}))\chi(|x| \leq M)\|_{L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)} = 0. \tag{5.252}$$

We finish the proof. □

Lemma 5.3.10. *Given $\vec{s} \in \mathbb{R}^N$,*

$$s\text{-}\lim_{\vec{s}' \to \vec{s}} P_b(\vec{s}') - P_b(\vec{s}) = 0 \quad \text{on } L^2_x(\mathbb{R}^5). \tag{5.253}$$

Proof. First, we prove that for all $\vec{s} \in \mathbb{R}^N$,

$$s\text{-}\lim_{\vec{s}' \to \vec{s}} P_c(\vec{s}')P_c(\vec{s}) = P_c(\vec{s}) \quad \text{on } L^2_x(\mathbb{R}^5). \tag{5.254}$$

Given $f(x) \in L^2_x(\mathbb{R}^5)$, using AC, we can write $P_c(\vec{s})f(x)$ as

$$P_c(\vec{s})f(x) = \Omega_+(\vec{s})g_{\vec{s}}(x) \tag{5.255}$$

where

$$g_{\vec{s}}(x) := \Omega_+(\vec{s})f(x). \tag{5.256}$$

For any $\epsilon > 0$, there exists $M = M(\epsilon, g_{\vec{s}}(x)), c = c(\epsilon, g_{\vec{s}}(x)) > 0$ such that

$$\|(1 - F(|P| > c)\chi(|x| < M))g_{\vec{s}}(x)\|_{L^2_x(\mathbb{R}^5)} \leq \epsilon/100. \tag{5.257}$$

So

$$\begin{aligned}\|P_c(\vec{s}')\Omega_+(1 - F(|P| > c)\chi(|x| < M))g_{\vec{s}}(x) \\ - \Omega_+(1 - F(|P| > c)\chi(|x| < M))g_{\vec{s}}(x)\|_{L^2_x(\mathbb{R}^5)} \leq \epsilon/50.\end{aligned} \tag{5.258}$$

For

$$P_c(\vec{s}')\Omega_+(\vec{s})F(|P| > c)\chi(|x| < M)g_{\vec{s}}(x) - \Omega_+(\vec{s})F(|P| > c)\chi(|x| < M)g_{\vec{s}}(x), \tag{5.259}$$

using Lemma 5.3.9, one has that there exists $\delta = \delta(\epsilon, f, M) > 0$ such that when $|\vec{s}' - \vec{s}| < \delta$,

$$
\begin{aligned}
\|P_c(\vec{s}')\Omega_+(\vec{s})F(|P| > c)\chi(|x| < M)g_{\vec{s}}(x) - \Omega_+(\vec{s})F(|P| > c)\chi(|x| < M)g_{\vec{s}}(x)\|_{L^2_x(\mathbb{R}^5)} = \\
\| - P_b(\vec{s}')\Omega_+(\vec{s})F(|P| > c)\chi(|x| < M)g_{\vec{s}}(x)\|_{L^2_x(\mathbb{R}^5)} \\
= \|P_b(\vec{s}')(\Omega_+(\vec{s}')F(|P| > c)\chi(|x| < M)g_{\vec{s}}(x) - \Omega_+(\vec{s})F(|P| > c)\chi(|x| < M)g_{\vec{s}}(x))\|_{L^2_x(\mathbb{R}^5)} \\
\leq \|\Omega_+(\vec{s})F(|P| > c)\chi(|x| < M) - \Omega_+(\vec{s}')F(|P| > c)\chi(|x| < M)\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \|g_{\vec{s}}\|_{L^2_x(\mathbb{R}^5)} \\
\leq \|\Omega_+(\vec{s})F(|P| > c)\chi(|x| < M) - \Omega_+(\vec{s}')F(|P| > c)\chi(|x| < M)\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \|f(x)\|_{L^2_x(\mathbb{R}^5)} \\
\leq \epsilon/100. \quad (5.260)
\end{aligned}
$$

Thus, based on (5.258) and (5.260), one has that there exists $\delta = \delta(\epsilon, f, M) > 0$ such that when $|\vec{s}' - \vec{s}| < \delta$,

$$\|P_c(\vec{s}')\Omega_+(\vec{s})g_{\vec{s}}(x) - \Omega_+(\vec{s})g_{\vec{s}}(x)\|_{L^2_x(\mathbb{R}^5)} \leq \epsilon/2 < \epsilon. \quad (5.261)$$

Hence, we conclude (5.254).

(5.254) implies

$$s\text{-}\lim_{\vec{s}' \to \vec{s}} P_b(\vec{s}') - P_b(\vec{s}')P_b(\vec{s}) = 0 \quad \text{on } L^2_x(\mathbb{R}^5) \quad (5.262)$$

which implies

$$s\text{-}\lim_{\vec{s}' \to \vec{s}} P_b(\vec{s}) - P_b(\vec{s}')P_b(\vec{s}) = 0 \quad \text{on } L^2_x(\mathbb{R}^5) \quad (5.263)$$

by using Lemma 5.3.8 and Gram-Schmidt process. Thus, based on (5.262) and (5.263), one has

$$s\text{-}\lim_{\vec{s}' \to \vec{s}} P_b(\vec{s}') - P_b(\vec{s}) = 0 \quad \text{on } L^2_x(\mathbb{R}^5). \quad (5.264)$$

□

Lemma 5.3.11. *Based on assumption 5.1.3, for all* $\vec{s} \in \mathbb{R}^N$,

$$s\text{-}\lim_{\vec{s}' \to \vec{s}} P_c(\vec{s}) - P_c(\vec{s}') = 0 \quad \textit{on } L^2_x(\mathbb{R}^5). \quad (5.265)$$

In particular, for all $\eta \in [0, 3]$,

$$\|\langle x\rangle^{-\eta} P_c(\vec{s})\langle x\rangle^{\eta}\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \lesssim 1. \quad (5.266)$$

Proof. (5.265) follows from Lemma 5.3.10. (5.266) follows from (5.16). □

5.3.3 Proof of Theorem 5.1.2

We prove Theorem 5.1.2 by proving Lemma 5.1.3.

Proof of Lemma 5.1.3. For $C_1(t)$, due to Lemma 5.3.1 and Lemma 5.3.2, it suffices to consider the operator

$$\tilde{C}(t) := \int_{1/M}^{M} dv P^{+} F(H_0 > 1/M) e^{ivH_0} V(x, t+v) U(t+v, t) F(H_0 \leq M_2) F(|x| \leq M_1) \tag{5.267}$$

for some sufficiently large $M, M_1, M_2 \gg 1$. We can choose M, M_1, M_2 large enough such that

$$\sup_{t \in \mathbb{R}} \| \tilde{C}(t) - C_1(t) \|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \leq \frac{1}{10000000}. \tag{5.268}$$

Since $F(H_0 \leq M_2) F(|x| \leq M_1)$ is compact on $L^2_x(\mathbb{R}^5)$, $\tilde{C}(t)$ has the form of

$$\tilde{C}(t) := \tilde{C}_1(t) C_{cpt} \tag{5.269}$$

for a compact operator $C_{cpt} := F(H_0 \leq M_2) F(|x| \leq M_1)$ on $L^2_x(\mathbb{R}^5)$ by setting

$$\tilde{C}_1(t) := \int_{1/M}^{M} dv P^{+} F(H_0 > 1/M) e^{ivH_0} V(x, t+v) U(t+v, t). \tag{5.270}$$

And

$$\sup_{t \in \mathbb{R}} \| \tilde{C}_1(t) \|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \lesssim M, \tag{5.271}$$

since the integrand is bounded for V bounded. Due to compactness, there exists $N \in \mathbb{N}^{+}$, $\langle x \rangle^{100} \phi_k, \langle x \rangle^{100} \psi_k \in L^2_x(\mathbb{R}^5), \|\phi_k\|_{L^2_x} = 1, k = 1, \cdots, N$ such that

$$\| C_{cpt} - \sum_{k=0}^{N} (\phi_k(x), \cdot)_{L^2_x(\mathbb{R}^5)} \psi_k \|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \leq 1/5000M. \tag{5.272}$$

Here we want to emphasize that $\phi_k(x), \psi_k(x)$ are localized in space since $\langle x \rangle^{100} \phi_k, \langle x \rangle^{100} \psi_k \in L^2_x(\mathbb{R}^5)$. We are able to take $\phi_k(x), \psi_k(x)$ to be localized since $C_{cpt} := F(H_0 \leq M_2) F(|x| \leq M_1)$. Due to $U(t,0)\Omega_{+}\phi(x) = \Omega_{t,+} e^{-itH_0} \phi(x)$, one has

$$((1 - P_c(t))\phi_k(x), U(t,0)\Omega_{+}\phi(x))_{L^2_x(\mathbb{R}^5)} \psi_k = 0. \tag{5.273}$$

So it is sufficient to break $P_c(t)\phi_k(x)$. We claim that for any $\epsilon > 0$, there exists $M_{k,1}(\epsilon), M_{k,2}, M_{k,3}(\epsilon) \gg 1$ such that

$$\sup_{t \in \mathbb{R}} \| (1 - F_{k,\epsilon}(H_0, x)) \Omega_{t,+} \phi_k(x) \|_{L^2_x} \leq \frac{\epsilon}{1 + \|\psi_k\|_{L^2_x}} \tag{5.274}$$

where

$$F_{k,\epsilon}(H_0, x) := F(H_0 > \frac{1}{M_{k,3}})\bar{F}(H_0 < M_{k,2})\bar{F}(|x| < M_{k,1}). \tag{5.275}$$

We defer the proof of this claim to the end. Taking $\epsilon = \frac{1}{1000000NM}$,

$$C_{1m}(t) := \tilde{C}_1(t)\left(\sum_{k=1}^{N}((1 - P_c(t))\phi_k(x), \cdot)_{L^2_x(\mathbb{R}^5)}\psi_k + \right. \tag{5.276}$$

$$\left. \sum_{k=1}^{N}(\Omega_{t,+}^{*} F_{k,\epsilon}(H_0, x)\Omega_{t,+}^{*}\phi_k(x), \cdot)_{L^2_x(\mathbb{R}^5)}\psi_k\right) \tag{5.277}$$

and $C_{1r}(t) := C_1(t) - C_{1m}(t)$, one gets (5.54) and (5.56) when $j = 1$. We also have

$$\sup_{t\in\mathbb{R}} \|C_{1m}(t)\langle x\rangle^{\sigma}\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \lesssim 1 \quad \text{for any } \sigma \in [0, 2) \tag{5.278}$$

by using the assumed localization of bound states, if they exist, and a similar argument as we applied for time-independent cases, see Lemma 5.2.6. Similarly, one gets (5.54) and (5.56) when $j = 2$.

Now we prove (5.55). For (5.55), due to (5.278), it suffices to prove

$$\sup_{t\in\mathbb{R}} \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1}C_j(t)\langle x\rangle^{\sigma}\|_{L^2_x \to L^2_x} \lesssim_{\sigma,\eta} 1 \tag{5.279}$$

for all $\eta > 5/2, \sigma \in (1, 101/100)$, $j = 1, 2$. According to Lemma 5.2.1, it suffices to prove

$$\sup_{t\in\mathbb{R}} \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1}\tilde{C}_j(t)\langle x\rangle^{\sigma}\|_{L^2_x \to L^2_x} \lesssim_{\sigma,\eta} 1 \tag{5.280}$$

where

$$\tilde{C}_1(t) := \int_0^{\infty} dv P^+ F(|P| - \frac{1}{\langle v\rangle^{1/2-0}})e^{ivH_0}V(x, v+t)U(t+v, t) \tag{5.281}$$

and

$$\tilde{C}_2(t) := \int_{-\infty}^{0} dv P^- F(|P| - \frac{1}{\langle v\rangle^{1/2-0}})e^{ivH_0}V(x, v+t)U(t+v, t). \tag{5.282}$$

For $\tilde{C}_1(t)$, taking integration by parts by setting

$$U(t+v, t) = \frac{1}{iH_0}\partial_v[e^{-ivH_0}][e^{ivH_0}U(t+v, t)], \tag{5.283}$$

one has

$$
\begin{aligned}
\tilde{C}_1(t) = \int_0^\infty & dv P^+ F(|P| \le \frac{1}{\langle v\rangle^{1/2-0}}) e^{ivH_0} V(x,v+t)\bar{F}(|P| \le 1) U(t+v,t) + \\
& P^+ F(|P| \le \frac{1}{\langle v\rangle^{1/2-0}}) e^{ivH_0} V(x,v+t) F(|P|>1)\frac{1}{iH_0} U(t+v,t)|_{v=0}^{v=\infty} + \\
& \int_0^\infty dv P^+ H_0 F(|P| \le \frac{1}{\langle v\rangle^{1/2-0}}) e^{ivH_0} V(x,v+t) F(|P|>1)\frac{1}{H_0} U(t+v,t) \\
& \int_0^\infty dv P^+ F(|P| \le \frac{1}{\langle v\rangle^{1/2-0}}) e^{ivH_0} V(x,v+t) F(|P|>1)\frac{1}{H_0} V(x,v+t) U(t+v,t) + \\
& \int_0^\infty dv P^+ e^{ivH_0} \partial_v [F(|P| \le \frac{1}{\langle v\rangle^{1/2-0}}) V(x,v+t)] F(|P|>1)\frac{1}{iH_0} U(t+v,t). \quad (5.284)
\end{aligned}
$$

Using Lemma 5.2.2 and using

$$
|(U(t+v,t)f(x), \langle x\rangle^{a-0} U(t+v,t)f(x))_{L^2_x}| \lesssim |v|^{a-0} \|\langle P\rangle^2 f\|^2_{L^2_x} + \|\langle x\rangle^{a/2-0/2} f\|^2_{L^2_x} \quad (5.285)
$$

for $a \in [1,2]$, one has that for $\sigma \in (1, 101/100)$,

$$
\begin{aligned}
& \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} \tilde{C}_1(t) \langle x\rangle^{\sigma}\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \lesssim \\
& \int_0^\infty \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} P^+ F(|P| \le \frac{1}{\langle v\rangle^{1/2-0}})\|_{L^2_x(\mathbb{R}^5 \to L^2_x(\mathbb{R}^5)} \frac{1}{\langle v\rangle^{5/4}} \langle v\rangle^{101/100} \\
& \lesssim \sup_{v \ge 1} \langle v\rangle^{4/5} \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} P^+ F(|P| \le \frac{1}{\langle v\rangle^{1/2-0}})\|_{L^2_x(\mathbb{R}^5 \to L^2_x(\mathbb{R}^5)} \\
& \lesssim \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} \frac{1}{|P|^{9/5+0}}\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \\
& \quad \sup_{v \ge 1} \||P|^{9/5+0} P^+ F(|P| \le \frac{1}{\langle v\rangle^{1/2-0}})\|_{L^2_x(\mathbb{R}^5 \to L^2_x(\mathbb{R}^5)} \\
& \lesssim \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} \frac{1}{|P|^{9/5+0}}\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)}. \quad (5.286)
\end{aligned}
$$

Using (5.286), (5.175), (5.278) and Lemma 5.2.2, one has

$$
\begin{aligned}
& \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} \tilde{C}_1(t) \langle x\rangle^{\sigma}\|_{L^2_x \to L^2_x} \lesssim \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} \frac{1}{|P|^{9/5+0}}\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \\
& \lesssim \|\langle x\rangle^{-\eta} \frac{1}{|P|^{9/5+0}}\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} + \sum_{j=1}^{2} \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} C_j(t) \frac{1}{|P|^{9/5+0}}\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} + \\
& \sum_{j=1}^{2} \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} C_{jm}(t) \frac{1}{|P|^{9/5+0}}\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)} \\
& \lesssim 1 + \|\langle x\rangle^{-\eta}(1 - C_{1r}(t) - C_{2r}(t))^{-1} \frac{1}{|P|^{13/10+0}}\|_{L^2_x(\mathbb{R}^5) \to L^2_x(\mathbb{R}^5)}. \quad (5.287)
\end{aligned}
$$

Here we also use that for $\eta > 5/2$,

$$\|\langle x\rangle^{-\eta}\frac{1}{|P|^{9/5+0}}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \lesssim 1 \tag{5.288}$$

and

$$(1-C_{1r}(t)-C_{2r}(t))^{-1} = 1+(1-C_{1r}(t)-C_{2r}(t))^{-1}(C_{1r}(t)+C_{2r}(t)). \tag{5.289}$$

Keeping using (5.175), (5.278) and Lemma 5.2.2, one has

$$\begin{aligned}
&\|\langle x\rangle^{-\eta}(1-C_{1r}(t)-C_{2r}(t))^{-1}\tilde{C}_1(t)\langle x\rangle^{\sigma}\|_{L^2_x\to L^2_x} \lesssim \\
&\qquad 1+\|\langle x\rangle^{-\eta}(1-C_{1r}(t)-C_{2r}(t))^{-1}\frac{1}{|P|^{13/10+0}}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}\\
&\qquad \lesssim 1+\|\langle x\rangle^{-\eta}(1-C_{1r}(t)-C_{2r}(t))^{-1}\frac{1}{|P|^{4/5+0}}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}\\
&\qquad \lesssim 1+\|\langle x\rangle^{-\eta}(1-C_{1r}(t)-C_{2r}(t))^{-1}\frac{1}{|P|^{3/10+0}}\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}\\
&\lesssim 1+\|\langle x\rangle^{-\eta}(1-C_{1r}(t)-C_{2r}(t))^{-1}|P|^{1/5-0}\bar{F}(|P|\le 1)\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)}+\\
&\|\langle x\rangle^{-\eta}(1-C_{1r}(t)-C_{2r}(t))^{-1}\frac{1}{|P|^{3/10+0}}F(|P|>1)\|_{L^2_x(\mathbb{R}^5)\to L^2_x(\mathbb{R}^5)} \lesssim 1.
\end{aligned} \tag{5.290}$$

Similarly, one has

$$\|\langle x\rangle^{-\eta}(1-C_{1r}(t)-C_{2r}(t))^{-1}\tilde{C}_2(t)\langle x\rangle^{\sigma}\|_{L^2_x\to L^2_x} \lesssim 1. \tag{5.291}$$

And they are uniform in t. So one has (5.280), (5.279) and then (5.55).

Once we prove the claim (5.274), we can finish the proof. Now we prove (5.274) provided that $\langle x\rangle^{100}\phi_k(x)\in L^2_x(\mathbb{R}^5)$. It suffices to prove

$$\sup_{\vec{s}\in[0,T]^N}\|(1-F_{k,\epsilon}(H_0,x))\Omega_{K,+}(\vec{s})\phi_k(x)\|_{L^2_x} \le \frac{\epsilon}{1+\|\psi_k\|_{L^2_x}} \tag{5.292}$$

where

$$T:=\max_{j=1,\cdots,N}|\frac{2\pi}{\omega_j}|. \tag{5.293}$$

Given $\phi_k(x)$, first of all, for all $\vec{s}\in T^N$, there exists $M(\vec{s})$ such that

$$\|\chi(|x|>M(\vec{s}))\Omega_{K,+}(\vec{s})\phi_k(x)\|_{L^2_x} \le \frac{\epsilon}{100(1+\|\psi_k\|_{L^2_x})} \tag{5.294}$$

which means

$$\|\chi(|x|\le M(\vec{s}))\Omega_{K,+}(\vec{s})\phi_k(x)\|^2_{L^2_x} \ge \|P_c(\vec{s})\phi_k(x)\|^2_{L^2_x}-(\frac{\epsilon}{100(1+\|\psi_k\|_{L^2_x})})^2. \tag{5.295}$$

Using Lemma 5.3.9, one has

$$\lim_{\vec{t}\to\vec{0}}\|\chi(|x|\le M(\vec{s}))\Omega_{K,+}(\vec{s})\phi_k(x)-\chi(|x|\le M(\vec{s}))\Omega_{K,+}(\vec{s}+\vec{t})\phi_k(x)\|_{L^2_x(\mathbb{R}^5)}=0. \quad (5.296)$$

Based on Lemma 5.3.11, one has the continuity of $\|P_c(\vec{s})\phi_k(x)\|^2_{L^2_x}$ in $\vec{s}$. So using (5.296) and the continuity of $\|P_c(\vec{s})\phi_k(x)\|^2_{L^2_x}$, there exists $r(\vec{s})>0$ such that for all $\vec{s}'\in O(\vec{s}):=\{\vec{s}':|\vec{s}-\vec{s}'|<r(\vec{s})\}$,

$$\|\chi(|x|\le M(\vec{s}))\Omega_{K,+}(\vec{s}')\phi_k(x)\|_{L^2_x}-\|P_c(\vec{s}')\phi_k(x)\|^2_{L^2_x}\le\left(\frac{\epsilon}{90(1+\|\psi_k\|_{L^2_x})}\right)^2. \quad (5.297)$$

Then $\{O(\vec{s})\}_{\vec{s}\in[0,T]^N}$ is an open cover of $[0,T]^N$. Due to the compactness of $[0,T]^N$, there exists a finite subcover $\{O(\vec{s}(j))\}_{j=1}^{j=N}$ of $[0,T]^N$. Take $M_{k,1}:=10\max_{j\in\{1,\cdots,N\}}\{M(\vec{s}(j))\}$. One has that for all $s,u\in[0,10]$,

$$\begin{aligned}&\|F(|x|>M_{k,1})\Omega_{K,+}(\vec{s})\phi_k(x)\|_{L^2_x(\mathbb{R}^5)}\\ &\quad\le\|\chi(|x|>M(\vec{s}(c)))\Omega_{K,+}(\vec{s})\phi_k(x)\|_{L^2_x(\mathbb{R}^5)}\le\frac{\epsilon}{90(1+\|\psi_k\|_{L^2_x})}.\end{aligned} \quad (5.298)$$

with $\vec{s}(c)$ satisfying $\vec{s}\in O(\vec{s}(c))$. Using (5.296) and a similar compactness argument, one can find $M_{k,2},M_{k,3}\ge 1$ such that for all $\vec{s}\in[0,T]^N$,

$$\|F(H_0>M_{k,2})\bar{F}(|x|\le M_{k,1})\Omega_{K,+}(\vec{s})\phi_k(x)\|_{L^2_x(\mathbb{R}^5)}\le\frac{\epsilon}{90(1+\|\psi_k\|_{L^2_x})} \quad (5.299)$$

and

$$\|\bar{F}(H_0\le\frac{1}{M_{k,3}})\bar{F}(|x|\le M_{k,1})\Omega_{K,+}(\vec{s})\phi_k(x)\|_{L^2_x(\mathbb{R}^5)}\le\frac{\epsilon}{90(1+\|\psi_k\|_{L^2_x})}. \quad (5.300)$$

Then one conclude (5.274) and finish the proof of the claim. We finish the proof. □

5.3.4 An application: Strichartz estimates

The Strichartz estimates for the free flow(see [[93], [95]] for Strichartz estimates and [49] for end-point Strichartz estimates) state that for $2\le r,q\le\infty$, $\frac{n}{r}+\frac{2}{q}=\frac{n}{2}$, and $(q,r,n)\neq(2,\infty,2)$, the homogeneous Strichartz estimate holds

$$\|e^{-itH_0}f\|_{L^q_tL^r_x(\mathbb{R}^{n+1})}\lesssim_{n,q,r}\|f\|_{L^2_x(\mathbb{R}^n)}, \quad (5.301)$$

the dual homogeneous Strichartz estimate

$$\|\int_{\mathbb{R}}ds\,e^{-isH_0}F(s)\|_{L^2_x(\mathbb{R}^n)}\lesssim_{n,q,r}\|F\|_{L^{q'}_tL^2_x(\mathbb{R}^n)} \quad (5.302)$$

and the inhomogeneous Strichartz estimate

$$\Big\|\int_{s<t} ds\, e^{-i(t-s)H_0}F(s)\Big\|_{L_t^qL_x^r(\mathbb{R}^{n+1})} \lesssim_{n,q,r} \|F\|_{L_t^{q'}L_x^{r'}(\mathbb{R}^{n+1})} \tag{5.303}$$

where (r,r') and (q,q') are conjugate pairs. For a perturbed system, Strichartz estimates implies that

$$\|U(t,0)P_c(0)f\|_{L_t^qL_x^r(\mathbb{R}^{n+1})} \leq C_q\|f\|_{L_x^2(\mathbb{R}^n)} \tag{5.304}$$

for $2 \leq r,q \leq \infty$, $\frac{n}{r}+\frac{2}{q}=\frac{n}{2}$, and $(q,r,n)\neq(2,\infty,2)$. In this subsection, we show that our local decay estimates imply Strichartz estimates hold, by using the inhomogeneous Strichartz estimate of the free flow.

Proof of Theorem 5.1.3. It suffices to check the endpoint Strichartz estimates, that is, when $(q,r,n)=(2,10/3,5)$ in 5 space dimensions. Using Duhamel's formula,

$$U(t,0)P_c(0)\psi = e^{-itH_0}\psi + (-i)\int_0^t ds\, e^{-i(t-s)H_0}V(x,s)U(s,0)P_c(0)\psi \tag{5.305}$$

$$=: \psi_1(t)+\psi_2(t). \tag{5.306}$$

$\psi_1(t)$ enjoys Strichartz estimates of the free flow. For $\psi_2(t)$, based on the inhomogeneous Strichartz estimate of the free flow, and local decay estimates(take $\eta = 3-1/4 > 5/2$ in (5.26)), we have

$$\|\psi_2(t)\|_{L_t^2L_x^{10/3}(\mathbb{R}^{5+1})} \lesssim \|\chi(t\geq 0)V(x,t)U(t,0)P_c(0)\psi\|_{L_t^2L_x^{10/7}(\mathbb{R}^{5+1})} \tag{5.307}$$

$$\lesssim \|\langle x\rangle^{3-1/4}V(x,t)\|_{L_t^\infty L_x^5(\mathbb{R}^{5+1})}\|\langle x\rangle^{-3+1/4}U(t,0)P_c(0)\psi\|_{L_{x,t}^2(\mathbb{R}^{5+1})} \tag{5.308}$$

$$\lesssim \|\langle x\rangle^4V(x,t)\|_{L_{x,t}^\infty(\mathbb{R}^{5+1})}\|\psi\|_{L_x^2(\mathbb{R}^5)}. \tag{5.309}$$

Hence, we obtain the endpoint Strichartz estimates

$$\|\psi(t)\|_{L_t^2L_x^{10/3}(\mathbb{R}^{5+1})} \lesssim \|\langle x\rangle^4V(x,t)\|_{L_{x,t}^\infty(\mathbb{R}^{5+1})}\|\psi\|_{L_x^2(\mathbb{R}^5)}. \tag{5.310}$$

We finish the proof. □

5.4 Proof of Lemma 5.2.1

Full proof of Lemma 5.2.1. **Proof of** (5.60)**:** It suffices to check the case when $t > 1$. Let

$$F_{10^j}(|P|) := F(10^j < |P| < 10^{j+1}), \quad j = 1, 2, \dots. \tag{5.311}$$

Here $F_{2^j}(|P|)$ satisfies

$$\sum_{j=1}^{\infty} F_{10^j}(|P|) = F(|P| > 10). \tag{5.312}$$

It is known that for all $M > 1$, $c > 0$,

$$\|P^{\pm} \bar{F}(|P| \leq M)F(|P| > c)e^{-itH_0}\langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_{c,n,M} \frac{1}{\langle t\rangle^{\delta}}, \tag{5.313}$$

and

$$\|P^{\pm} F_{10}(|P|)e^{-itH_0}\langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{\langle t\rangle^{\delta}}, \tag{5.314}$$

see [42]. So it suffices to prove

$$\|\sum_{j=1}^{\infty} P^{\pm} F_{10^j}(|P|)e^{-itH_0}\langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{\langle t\rangle^{\delta}}. \tag{5.315}$$

Indeed, using dilation transformation, one has

$$\begin{aligned}\|P^{\pm} F_{10^j}(|P|)e^{-itH_0}\langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} &= \|P^{\pm} F_{10}(|P|)e^{-i10^{2j}tH_0}\langle x/10^j\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)}\\ &\leq \|P^{\pm} F_{10}(|P|)e^{-i10^{2j}tH_0}\langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \|\langle x\rangle^{\delta}\langle x/10^j\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)}\\ &\lesssim_n \frac{1}{\langle 10^{2j}t\rangle^{\delta}} \times 10^{j\delta} \lesssim_n \frac{1}{10^{j\delta}} \times \frac{1}{\langle t\rangle^{\delta}},\end{aligned} \tag{5.316}$$

which implies

$$\|\sum_{j=1}^{\infty} P^{\pm} F_{10^j}(|P|)e^{-itH_0}\langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_n \frac{1}{\langle t\rangle^{\delta}}. \tag{5.317}$$

We finish the proof for (5.60).

Proof of (5.61)**:** (5.61) follows from a similar argument since when $t \geq 1$,

$$\|P^{\pm} F_{10^j}(|P|)e^{-itH_0}|P|^l\langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_{l,n} \frac{1}{10^{j(\delta-1)}} \times \frac{1}{\langle t\rangle^{\delta}} \tag{5.318}$$

and when $t \in (v, 1)$, using dilation to replace H_0 with H_0/v implies that (5.61) holds.

Proof of (5.62)**:** Since

$$
\begin{aligned}
&\int_0^1 t^2 dt \| P^- F_{10^j}(|P|) e^{-itH_0} |P|^2 \langle x\rangle^{-\delta} \|_{L_x^2(\mathbb{R}^n) \to L_x^2(\mathbb{R}^n)} \\
&\leq \int_0^{1/10^{3j/4}} t^2 dt \| P^- F_{10^j}(|P|) e^{-itH_0} |P|^2 \langle x\rangle^{-\delta} \|_{L_x^2(\mathbb{R}^n) \to L_x^2(\mathbb{R}^n)} + \\
&\int_{1/10^{3j/4}}^1 t^2 dt \| P^- F_{10^j}(|P|) e^{-itH_0} |P|^2 \langle x\rangle^{-\delta} \|_{L_x^2(\mathbb{R}^n) \to L_x^2(\mathbb{R}^n)} =: A_{j,1} + A_{j,2}. \quad (5.319)
\end{aligned}
$$

For $A_{j,1}$, one has

$$
A_{j,1} \lesssim \int_0^{1/10^{3j/4}} dt t^2 \times 10^{2j} \lesssim 10^{-j/4} \quad (5.320)
$$

which is summable over j. For $A_{j,2}$, using dilation to replace $|P|$ with $10^j|P|$, one has

$$
\begin{aligned}
A_{j,2} =& 10^{2j} \int_{1/10^{3j/4}}^1 t^2 dt \| P^- F_1(|P|) e^{-i10^{2j}tH_0} |P|^2 \langle x/10^j\rangle^{-\delta} \|_{L_x^2(\mathbb{R}^n) \to L_x^2(\mathbb{R}^n)} && (5.321)\\
\lesssim_n& 10^{2j} \int_{1/10^{3j/4}}^1 t^2 dt \times \frac{1}{\langle t10^{2j}\rangle^{\delta}} \times 10^{j\delta} && (5.322)\\
\lesssim_n& \frac{1}{10^{j(\delta-2)}} && (5.323)
\end{aligned}
$$

which is summable over j. Here we use

$$
\| P^- F_1(|P|) e^{-itH_0} |P|^2 \langle x\rangle^{-\delta} \|_{L_x^2(\mathbb{R}^n) \to L_x^2(\mathbb{R}^n)} \lesssim_n \frac{1}{\langle t\rangle^{\delta}}, \quad (5.324)
$$

see [42]. Then based on the estimates on $A_{j,1}$ and $A_{j,2}$, one gets (5.62).

Proof of (5.63)**:** It follows from Strichartz estimates, a commutator argument and scaling. Choose $f \in L_x^2(\mathbb{R}^n)$ and break $\langle x\rangle^{-\delta} P^+ e^{itH_0} f$ into two parts

$$
\begin{aligned}
\langle x\rangle^{-\delta} P^+ e^{itH_0} f =& P^+ \langle x\rangle^{-\delta} e^{itH_0} f + [\langle x\rangle^{-\delta}, P^+] e^{itH_0} f && (5.325)\\
=:& f_1(x,t) + f_2(x,t). && (5.326)
\end{aligned}
$$

For $f_1(x,t)$, by using Strichartz estimates, we have that when $n \geq 3$, there exists $\delta_0 = \delta_0(n) > 0$ such that when $\delta > \delta_0$,

$$
\| f_1(x,t) \|_{L_t^2 L_x^2(\mathbb{R}^{n+1})} = \| \langle x\rangle^{-\delta} e^{itH_0} f \|_{L_t^2 L_x^2(\mathbb{R}^{n+1})} \lesssim \| e^{itH_0} f \|_{L_t^2 L_x^q(\mathbb{R}^{n+1})} \quad (5.327)
$$

with q satisfying

$$
1 + \frac{n}{q} = \frac{n}{2}. \quad (5.328)
$$

For $f_2(x,t)$, use Fourier representation to express P^+. Let

$$F((A-M)/R) := P^+ = \frac{\tanh((A-M)/R)+1}{2}. \tag{5.329}$$

$$\begin{aligned} f_2(x,t) &= c\int dw\hat{F}(w)[\langle x\rangle^{-\delta}, e^{i(A-M)w/R}]e^{itH_0}f \\ &= c\int dw\hat{F}(w)\langle x\rangle^{-\delta}(e^{i(\frac{A-M}{R})w}\langle x\rangle^{\delta}e^{-i(\frac{A-M}{R})w}-\langle x\rangle^{\delta})e^{iAw}\langle x\rangle^{-\delta}f \\ &= c\int dw\int_0^w dw_1\hat{F}(w)\langle x\rangle^{-\delta}e^{i(\frac{A-M}{R})w_1}i[\frac{A-M}{R},\langle x\rangle^{\delta}]e^{i(\frac{A-M}{R})(w-w_1)}\langle x\rangle^{-\delta}e^{-itH_0}f. \end{aligned} \tag{5.330}$$

Since

$$|i[\frac{A-M}{R},\langle e^{w_1/R}x\rangle^{\delta}]| \lesssim_\delta \frac{1}{R}\langle e^{w_1/R}x\rangle^{\delta}, \tag{5.331}$$

and since

$$\hat{F}(w) = \frac{c_1}{\sinh(\pi w/2)} + c_2\delta(w) \quad \text{for some constants } c_1, c_2, \tag{5.332}$$

one has

$$\begin{aligned} &\|\int dw\int_0^w dw_1\hat{F}(w)\langle x\rangle^{-\delta}e^{i(\frac{A-M}{R})w_1}i[\frac{A-M}{R},\langle x\rangle^{\delta}]e^{i(\frac{A-M}{R})(w-w_1)}\langle x\rangle^{-\delta}e^{-itH_0}f\|_{L^2_{x,t}(\mathbb{R}^{n+1})} \\ &= \|\int dw\int_0^w dw_1\hat{F}(w)\langle x\rangle^{-\delta}i[\frac{A-M}{R},\langle e^{w_1/R}x\rangle^{\delta}]\langle e^{w/R}x\rangle^{-\delta}e^{nw/(2R)}[e^{-itH_0}f](e^{w/R}x)\|_{L^2_{x,t}(\mathbb{R}^{n+1})} \\ &\text{(Choose } R \text{ large enough)} \lesssim_\delta \frac{1}{R}\|\langle x\rangle^{-\delta}\|_{L^r_x(\mathbb{R}^n)}\|\int dwe^{nw/(2R)}|w\hat{F}(w)|\,|[e^{-itH_0}f](e^{w/R}x)|\|_{L^2_tL^q_x(\mathbb{R}^{n+1})} \\ &\lesssim_\delta \frac{1}{R}\|\langle x\rangle^{-\delta}\|_{L^r_x(\mathbb{R}^n)}\int dwe^{nw/(rR)}|w\hat{F}(w)|\|[e^{-itH_0}f](y)\|_{L^2_tL^q_y(\mathbb{R}^{n+1})} \lesssim_\delta \frac{1}{R}\|f(x)\|_{L^2_x(\mathbb{R}^5)} \end{aligned} \tag{5.333}$$

where

$$\frac{1}{r}+\frac{1}{q}=\frac{1}{2}, \quad \delta > \frac{n}{r}, \tag{5.334}$$

and we choose R_0 large enough such that $n/(rR_0) < \pi/20$ which implies

$$\int |w\hat{F}(w)|e^{nw/(rR)} \lesssim 1 \quad \text{for } R \geq R_0. \tag{5.335}$$

So we conclude that when $\delta > \frac{n}{r}, R_0 > 20n/r$,

$$\|\langle x\rangle^{-\delta}P^+e^{itH_0}f\|_{L^2_{x,t}(\mathbb{R}^{n+1})} \lesssim_\delta \|f(x)\|_{L^2_x(\mathbb{R}^5)}. \tag{5.336}$$

Similarly, one has that when $\delta > \frac{n}{r}, R_0 > 20n/r$,

$$\|\langle x\rangle^{-\delta} P^- e^{-itH_0} f\|_{L^2_{x,t}(\mathbb{R}^{n+1})} \lesssim_\delta \|f(x)\|_{L^2_x(\mathbb{R}^5)}. \tag{5.337}$$

For (5.64), since A commutes with P^-, using dilation transformation and (5.61),

$$\begin{aligned}
&\|P^- F(|P| > \frac{1}{\langle t\rangle^{1/2-\epsilon}}) e^{-itH_0} |P| \langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} = \\
&\quad \frac{1}{t^{1/2-\epsilon}} \|P^- F(|P| > 1) e^{-it^{2\epsilon}H_0} |P| \langle t^{1/2-\epsilon} x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \\
&\leq \frac{1}{t^{1/2-\epsilon}} \|P^- F(|P| > 1) e^{-it^{2\epsilon}H_0} |P| \chi(|x| \leq 1) \langle t^{1/2-\epsilon} x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} + \\
&\frac{1}{t^{1/2-\epsilon}} \|P^- F(|P| > 1) e^{-it^{2\epsilon}H_0} |P| \chi(|x| > 1) \langle x\rangle^{-1} \langle x\rangle \langle t^{1/2-\epsilon} x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \\
&\lesssim_{n,\epsilon,c,N} \frac{1}{\langle t\rangle^{N\epsilon}} + \frac{1}{t^{1/2-\epsilon}} \times \frac{1}{\langle t\rangle^{2\epsilon}} \times \frac{1}{t^{(1/2-\epsilon)\delta}} \lesssim_{n,\epsilon,c} \frac{1}{\langle t\rangle^{2\epsilon}} \times \frac{1}{t^{(1/2-\epsilon)(\delta+1)}}.
\end{aligned} \tag{5.338}$$

Proof of (5.64)**:** Using dilation transformation, one has

$$\begin{aligned}
&\|P^- F(|P| > \frac{1}{\langle t\rangle^{1/2-\epsilon}}) e^{-itH_0} \langle x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} = \\
&\quad \|P^- F(|P| > 1) e^{-it^{2\epsilon}H_0} \langle\langle t\rangle^{1/2-\epsilon} x\rangle^{-\delta}\|_{L^2_x(\mathbb{R}^n)\to L^2_x(\mathbb{R}^n)} \lesssim_{n,\epsilon} \frac{1}{\langle t\rangle^{(1/2-\epsilon)\delta}}.
\end{aligned} \tag{5.339}$$

Here we use the fact that when $|x| \geq 1$ on the right-hand side, $\langle\langle t\rangle^{1/2-\epsilon} x\rangle^{-\delta}$ gives $\frac{1}{\langle t\rangle^{(1/2-\epsilon)\delta}}$ decay and when $|x| < 1$, the right-hand side is well-localized and one could get the same decay at least by applying (5.60). □

Chapter 6

L^p boundedness of the Scattering Wave Operators of Schrödinger Dynamics with Time-dependent Potentials

6.1 Introduction

In this paper, we let $H_0 = -\Delta_x$, where $\Delta_x = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_n)^2$ is the Laplacian in $L^2(\mathbb{R}^n)$. Consider a time-dependent Schrödinger equation

$$i\partial_t \psi(t) = H(t)\psi(x) \tag{6.1}$$

with $H(t) := -\Delta_x + V(x,t)$ for some real-valued $V(x,t)$. The wave operator associated with a pair H_0, H of self-adjoint operators, and its conjugate $\Omega^{\pm *}$ are defined by

$$\Omega^{\pm} = s\text{-}\lim_{T \to \pm\infty} U(0,T)e^{-iH_0T}, \quad \text{on } L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \tag{6.2}$$

and

$$\Omega^{\pm *} = s\text{-}\lim_{T \to \pm\infty} e^{iTH_0}U(T,0)P_c, \quad \text{on } L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{6.3}$$

Here $U(T,0)$ denotes the dynamical group of the Schrödinger equation with a hamiltonian $H(T)$ and P_c denotes the projection on the space of the free scattering states of system (6.1).

Remark 29. *The existence of $U(T,0)$ in L^2 and H^1 is well-understood, see e.g. [67]. When $H = -\Delta_x + W(x)$ is time-independent, P_c denotes the projection on the continuous spectrum of $-\Delta_x + W(x)$. When H is time-dependent, it is well-known that the range of wave operator is contained in the space of all scattering states, see e.g. [68]. In [91], we give a characterization of this space. [91] proved that the space of all free scattering states is qual to the range of new free channel wave operator Ω_α*

$$\Omega_\alpha := s\text{-}\lim_{t \to \infty} e^{itH_0}F_c\left(\frac{|x-2tP|}{t^\alpha} \leq 1\right)U(t,0) \quad \textit{on } L^2(\mathbb{R}^n), \quad n \geq 3, \alpha \in (0, 1-2/n) \tag{6.4}$$

where F_c denotes a smooth characteristic function.

A smooth characteristic function of an interval I is a smooth approximation of the $\chi_I(x)$. This smooth function F maybe defined as

$$F(x \in I) \approx \int g(x-y)\chi_I(y)dy; \quad g(x) \approx c_\alpha e^{-\alpha x^2}; \quad F \leq 1.$$

Let $P := -i\nabla_x$. Let $\beta(t > M) := \beta(\frac{t}{M})$ with $\beta(\lambda) \in C^\infty(\mathbb{R})$, a smooth cut-off function satisfying $\beta(\lambda) = 0$ for $-\infty < \lambda < 1/2$ and $\beta(\lambda) = 1$ for $\lambda \geq 1$. $\bar{\beta}(t \leq M) := 1 - \beta(t > M)$. This paper is devoted to the study of L^p boundedness of the wave operator Ω (in the general case, with a high frequency cut-off).

To be precise, we proved that for some sufficiently large $M \gg 1$,

$$\|\Omega \beta(|P| > M)\|_{L^p_x(\mathbb{R}^3) \to L^p_x(\mathbb{R}^3)} \lesssim 1 \quad \text{all } 1 \leq p \leq \infty. \tag{6.5}$$

For Example: Mikhlin-type potentials (in t variable) $V(x,t)$ satisfying

$$\sup_{t \in \mathbb{R}} \frac{(1+|t|)^a}{a!} \sum_{l,j=0}^{2} \sum_{m,r=1}^{3} \left| \frac{\partial^a}{\partial t^a} \left[\partial^l_{\xi \cdot e_r} \partial^j_{\xi \cdot e_m} \hat{V}(\xi,t) \right] \right| \leq c^a \hat{V}_0(\xi), \text{ for all } a \in \mathbb{N}, \text{ some } c \geq 1 \tag{6.6}$$

with $\hat{V}_0(\xi) \in L^1_\xi(\mathbb{R}^3) \cap L^\infty_\xi(\mathbb{R}^3)$, $\{e_1, e_2, e_3\}$, a basis in $\mathbb{R}^3$. Using Duality, one has

$$\|\beta(|P| > M)\Omega\|_{L^p_x(\mathbb{R}^3) \to L^p_x(\mathbb{R}^3)} \lesssim 1 \quad \text{all } 1 \leq p \leq \infty. \tag{6.7}$$

Throughout this paper, we stick to $T \to +\infty$. The space dimension is 3. The same result can be extended to higher space dimensions.

Remark 30. *For low frequency part $\Omega \bar{\beta}(|P| \leq M)$, usually the technique is different. See [88] for a proof for low frequency part. However, the cancellation Lemmas we prove and use, hold for all frequencies. We demonstrate several new results which hold for all frequencies. This is in particular in the study of NLS type equations in L^p.*

We let $\Omega := \Omega_+$.

6.1.1 Background and previous works

The first general approach to the proof of these estimates was developed by Journé, Soffer, and Sogge [48]. They proved decay estimates for time-independent potentials, by using a time-dependent method which combined spectral and scattering theory with harmonic analysis. Their method involved splitting solutions into high- and low-energy parts, and using Kato's smoothing and the local energy decay on the corresponding pieces. Both parts relied on a **cancellation lemma**(CL), see Lemma 6.1.1. The Fourier transform of $f(x)$ in x variable in n-dimension is defined by

$$\hat{f}(k,t) := \mathcal{F}_x[f(x)](k,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ik\cdot x} f(x,t)d^n x, \tag{6.8}$$

and

$$f(x,t) = \mathcal{F}_k^{-1}[\hat{f}(k,t)](x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ik\cdot x} \hat{f}(k,t)d^n k. \tag{6.9}$$

Let

$$\mathcal{F}L^1_x(\mathbb{R}^n) := \{f(x) : \hat{f}(\xi) \in L^1_\xi(\mathbb{R}^n)\}. \tag{6.10}$$

CL says

Lemma 6.1.1 (**Cancellation lemma**). *If $V(x) \in \mathcal{F}L^1_x$, then the* time translated*(tT) potential*

$$\mathcal{K}_t(V(x)) := e^{iH_0 t}V(x)e^{-iH_0 t} : L^p \to L^p, \textit{ is bounded for all } 1 \le p \le \infty. \tag{6.11}$$

In [48], they also assumed that zero is neither an eigenvalue, nor a resonance, and, roughly $|V(x)| \le C|x|^{-4-n}$, $\hat{V} \in L^1$. Recall that a resonance is a distributional solution of $H\psi = 0$ so that $\psi \notin L^2$ but $(1+|x|^2)^{-\frac{\delta}{2}}\psi(x) \in L^2$ for any $\delta > 1/2$ but not for $\delta = 0$, see [**?**]. Their work was preceded by related results of Rauch [65], Jensen, Kato [**?**], and Jensen [45], [46], who established decay estimates on weighted L^2 space

$$\|\langle x\rangle^{-\delta} e^{-itH} f\|_{L^2(\mathbb{R}^n)} \le Ct^{-n/2}\|\langle x\rangle^{\delta'} f\|_{L^2(\mathbb{R}^n)} \tag{6.12}$$

for some sufficiently large δ and δ'. They also developed the small energy asymptotic expansions of the resolvent which are used in [48] to deal with low energy estimates.

Here

$$\langle x\rangle = \sqrt{|x|^2+1}.$$

After the work of [48], many works followed.

L^p estimates for wave operators were first introduced by Yajima [106]. He used a stationary method to prove the L^p boundedness of the wave operators, either when the Fourier transform of $\langle x\rangle^\delta V$ is small in some norm, or when $\partial^\alpha V/\partial x^\alpha$ decays rapidly for $|\alpha| \leq N$, some $N \in \mathbb{N}^+$. These assumptions on the potential are weaker than those in [48]. Yajima's theorem implies the dispersive bounds by using intertwining property of the wave operators [68]

$$\Omega^- e^{-itH_0} = e^{-itH}\Omega^- \qquad \text{on } L^2(\mathbb{R}^n). \tag{6.13}$$

However, such intertwining property does not hold when $H(t)$ is not periodic in t. One can see it since

$$\Omega^- e^{-itH_0} = s\text{-}\lim_{s\to-\infty} U(0,s)e^{-i(s+t)H_0} = U(0,t)\Omega^-(t) \tag{6.14}$$

where

$$\Omega^-(t) := s\text{-}\lim_{s\to-\infty} U_t(0,s)e^{-isH_0} \tag{6.15}$$

with $U_t(0,s)$ satisfying

$$i\partial_s U_t(s,0) = (H_0 + V(x,s+t))U_t(s,0). \tag{6.16}$$

See also [66].

See also Weder [100] for results of time-independent case in one dimension, $n = 1$, and Yajima [[2], [107], [108], [102]] for all $n \geq 1$.

For time-dependent potentials, the analogue of Kato's scattering result was proved by Howland [38]. When $V(x,t)$ decays in time (in the sense of integrability), wave operators were constructed by Howland [37] and Davies [14].

For potentials periodic in t, Soffer, Weinstein [82] presented a theory of resonances for a class of nonautonomous Hamiltonians; the existence/completeness of the wave operators follows in L^2.

Closely related to the boundedness of the wave operator on L^p, are L^p decay estimates for the free Schrödinger equation ($H(t) = H_0$) on $\mathbb{R}^n$:

$$\|e^{itH_0}f\|_{L^p} \leq C_p|t|^{-n(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^{p'}}, \quad p \geq 2, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{6.17}$$

They imply the Strichartz estimates [49]

$$\|e^{itH_0} f\|_{L^q_t L^r_x} \le C_q \|f\|_{L^2}, \quad 2 \le r, q \le \infty, \frac{n}{r} + \frac{2}{q} = \frac{n}{2}, \text{ and } (q,r,n) \ne (2,\infty,2). \quad (6.18)$$

The non-endpoint Strichartz estimates(when $q \ne 2$) were addressed in [28], [105] and of course the original work of Strichartz [?]. The more delicate endpoint cases are established by Keel and Tao [49].

Such decay estimates play a fundamental role in the theory of nonlinear dispersive equations, among other things. The extension of such estimates to inhomogeneous problems (either due to curvature, local potentials, or coherent structure such as solitons, vortices, etc.) then motivated the efforts to establish the L^p decay estimates for more general Hamiltonians.

Rodnianski and Schlag [71] proved L^p decay estimates for small time-dependent potentials which also satisfy the following condition

$$\sup_t \|V(t,\cdot)\|_{L^{3/2}(\mathbb{R}^3)} + \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \int \frac{|V(\hat{\tau},x)|}{|x-y|} d\tau dx < c_0, \text{ for some small constant } c_0 > 0 \quad (6.19)$$

where $V(\hat{\tau},x)$ denotes the Fourier transform in t variable. Their proof uses the representation of $U(t,0)$ as an infinite series of oscillatory integrals; they also established non-endpoint Strichartz estimates for large time-independent potentials with $\langle x \rangle^{-2-\epsilon}$ decay.

Goldberg proved in [29] dispersive estimates for almost-critical potentials and, in [31] Strichartz estimates for $L^{n/2}$ and thus scaling-critical potentials. See also [30]. Later, Beceanu [3] proved Strichartz estimates for time-dependent potentials by using Wiener theorem.

Now we go back to the existence of the wave operator. It is not hard to prove the existence of wave operator Ω by using Cook's method, if $V(x,t) \in L^\infty_t L^2_x \cap L^\infty_x(\mathbb{R}^{n+1})$, see [73]. However, it is difficult to prove the existence of the adjoint of the wave operator Ω. In fact, the existence of Ω is equivalent to asymptotic completeness of all scattering states. In this paper, based on the existence of Ω on L^2, a dense set of L^p, we first prove its L^p boundedness. Then one uses the theorem about extensions of continuous linear transformations (B.L.T., see [?], Theprem I.7) to extend the domain to the full L^p space by continuity. We obtain the L^p boundedness of Ω . L^p boundedness of Ω follows from duality. For applications to nonlinear Schrödinger equations(NLSs), we always assume that there is a global well-posedness of the solution in some Sobolev space first, then determine the potential and after that, study Ω via studying Ω .

6.1.2 New cancellation lemma, main result and applications

Let

$$\Omega_\epsilon := 1 + i\int_0^\infty dt e^{-\epsilon t}\Omega(t)e^{itH_0}V(x,t)e^{-itH_0}, \quad \Omega(t) := U(t,0)e^{-itH_0}. \tag{6.20}$$

We prove (6.5) by showing

$$\sup_{\epsilon\in(0,1)} \|\Omega_\epsilon \beta(|P| > M)\|_{L^p_x(\mathbb{R}^3)\to L^p_x(\mathbb{R}^3)} \lesssim 1 \quad \text{for all } 1 \le p \le \infty. \tag{6.21}$$

Using Duhamel's formula to expand $\Omega_\epsilon\beta(|P| > M)$ and iterating it, one gets

$$\Omega_\epsilon\beta(|P| > M) = \beta(|P| > M) + \sum_{k=1}^{\infty} I_\epsilon^{(k)}\beta(|P| > M) \tag{6.22}$$

where

$$I_\epsilon^{(k)} := i^k\int_0^\infty ds_1\int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k \mathcal{K}_{s_k}(V(x,s_k)) \cdots \mathcal{K}_{s_1}(V(x,s_1))e^{-\epsilon s_1}. \tag{6.23}$$

When $k = 1$,

$$I_\epsilon^{(1)} = i\int_0^\infty ds \mathcal{K}_s(V(x,s))e^{-\epsilon s}. \tag{6.24}$$

Based on the Duhamel expansion, one can see that in order to get L^p boundedness of $\Omega_\epsilon\beta(|P| > M)$ or L^p boundedness of Ω_ϵ, we have to prove the L^p boundedness of $I_\epsilon^{(1)}$. $I_\epsilon^{(1)}$ is the first non-trivial term in the expansion of Ω. L^p boundedness of $I_0^{(1)}$ for all $1 \le p \le \infty$ is close to the notion of **improved cancellation lemma**(ICL) in this note. ICL for V means that the integrated tT potential

$$\mathcal{IK}(V) := \int_0^\infty dt e^{itH_0}V(x,t)e^{-itH_0}e^{-0t} : L^p_x \to L^p_x, \text{ is bounded for all } 1 \le p \le \infty. \tag{6.25}$$

See section 6.2 for more details. Throughout the paper, we write $\mathcal{IK}$ to represent $\mathcal{IK}(V)$ for convenience.

Given a potential V, we prove the ICL for V first, and then use it to show

$$\|I_\epsilon^{(k)}\beta(|P|>M)\|_{L^p_x(\mathbb{R}^3)\to L^p_x(\mathbb{R}^3)} \le \frac{C^k}{M^{k-1}}, \quad k=1,2,\dots. \tag{6.26}$$

Then (6.21) follows right away by using (6.26).

Main result

We first consider a class of Mikhlin-type potentials (in t variable) $V(x,t)$ satisfying

$$\sup_{t\in\mathbb{R}} \frac{(1+|t|)^a}{a!}\sum_{l,j=0}^{2}\sum_{m,r=1}^{3}\left|\frac{\partial^a}{\partial t^a}\left[\partial^l_{\xi\, e_r}\partial^j_{\xi\, e_m}\hat{V}(\xi,t)\right]\right| \le c^a\hat{V}_0(\xi), \text{ all } a\in\mathbb{N}, \text{ some } c\ge 1, \tag{6.27}$$

for some $\hat{V}_0(\xi)\in L^1_\xi(\mathbb{R}^3)\cap L^\infty_\xi(\mathbb{R}^3)$. Here $\{e_1,e_2,e_3\}$ is a basis in $\mathbb{R}^3$.

Theorem 6.1.1. *If $V(x,t)$ satisfies condition (6.27), there exists $M=M(V(x,t))>0$ such that for all $1\le p\le\infty$,*

$$\Omega\beta(H_0>M^2) = s\text{-}\lim_{\epsilon\downarrow 0}\Omega_\epsilon\beta(H_0>M^2), \text{ on } L^p, \tag{6.28}$$

and $\beta(H_0>M^2)\Omega^*, \Omega\beta(H_0>M^2)$ are bounded on L^p.

For detailed proof, see section 6.4. Typical examples are

$$V(x,t)=V_0(x)+\frac{\sin(\ln\langle t\rangle)}{\langle t\rangle^\delta}V_1(x), \text{ for } \delta\ge 0, \tag{6.29}$$

and

$$V(x,t)=V_0(x)+V_1\left(x-\frac{\sin(\omega\ln\langle t\rangle)}{\langle t\rangle^\delta}v\right), \text{ for } \delta\ge 0, \tag{6.30}$$

see Corollary 6.4.7, Corollary 6.4.8.

Remark 31. *The first example above has a potential that decays arbitrarily slow in time, to a time independent potential. Since the decay in time is NOT in L^1, this case is not covered by the known results. See e.g. [83]. The second example is more involved: it corresponds to a charge transfer type hamiltonian, where the moving potential is a non linear path in time. Previous works required the path to be linear up to a fast decaying term. The case of general path, which however converges to an end point, was considered in [6]; the path was allowed to be a rough function of time. Yet, the method* *introduced by [6] did not apply to the charge transfer case, as a time independent part V_0 was not allowed. All previous works were focused on proving time decay estimates of the dynamics, but not L^p boundedness of wave operators or L^p decay. See e.g. [72], [29], [3].*

Remark 32. *When $\partial_t[V](t,\xi) \in L^1_t(0,\infty)$, it means asymptotic energy exists and is bounded. It may mean that $\delta > 0$ is optimal. But we show that our method can handle the case when $\delta = 0$, see (6.29). In this case, it is not known in general if the frequency support of the solution remains localized.*

We also consider the case of self similar potentials.

$$V(x,t) = V_1(g(t)x,t) + \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^{\infty} f_j(t) e^{ig_j(t)x \cdot a_j} \tag{6.31}$$

with

$$h(t) := \int d^n\xi |\hat{V}(\xi,t)| + \sum_{j=1}^{\infty} |f_j(t)| \in L^1_t[0,\infty), g(x), \text{a real function on } [0,\infty). \tag{6.32}$$

Theorem 6.1.2. *If $V(x,t)$ is as in (6.31) and satisfies condition (6.32), then*

$$\lim_{T\to\infty} \|U(0,T)e^{-iTH_0} - \Omega\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = 0 \tag{6.33}$$

and

$$\|\Omega\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \exp\left(\frac{\|h(t)\|_{L^1_t(0,\infty)}}{(2\pi)^{\frac{n}{2}}}\right) \tag{6.34}$$

for all $n \geq 3$.

A typical example is that when $\hat{V}_1(\xi)$ is a finite measure, for some $\alpha > 0$,

$$V(x,t) = \frac{\chi(|t| - c)\sin(\omega t)}{t^{1+\alpha}} V_1(\frac{x}{t}), \text{ for some } c > 0, \omega \in \mathbb{R}, \text{ in dimension } n \geq 3 \tag{6.35}$$

which can be used to study self-similar solutions for some NLS or other equations. For detailed proof, see section 6.5.

The results above imply the crucial L^p decay estimates and Strichartz estimates for the high frequency part in L^p space, by using operators $\Omega(0,T)\beta(|H_0| > M^2)$ with

$$\Omega(0,T) := U(0,T)e^{-iH_0T}. \tag{6.36}$$

In fact, L^p boundedness of $\Omega\beta(|H_0|>M^2)$ implies the L^p boundedness of $P_c\Omega(0,T)\beta(|H_0|>M^2)$. Here recall that P_c denotes the projection on the space of all free scattering states. This operator norm bound is independent on T, see Corollary 6.4.6, Corollary 6.5.1. This is used to pass the decay properties of e^{iH_0T} to $P_cU(T,0)\beta(H_0>M^2)$ by using

$$\|P_cU(0,T)\beta(|H_0|>M)\|_{L^p\to L^{p'}} \leq \sup_{T\in\mathbb{R}}\|P_c\Omega(0,T)\beta(H_0>M^2)\|_{L^{p'}\to L^{p'}}\|e^{itH_0}\|_{L^p\to L^{p'}} \tag{6.37}$$

for $p\in[1,2]$.

As another example, we can prove decay estimates for a charge transfer potential when the potential has the form of $V(x-\sqrt{1+|t|}v)$ satisfying

$$|||V(x)|||_p := \sum_{l=0}^{2}\sum_{m=1}^{3}\|(|\xi|+1)^3|\partial^l_{\xi\cdot e_m}\hat{V}(\xi)|\|_{L^1_\xi}+\|V(x)\|_{L^1_x\cap L^2_x}<\infty. \tag{6.38}$$

Theorem 6.1.3. *If $V(x,t)=V(x-\sqrt{1+|t|}v)$ satisfies assumption 6.38, then for a sufficiently large $M>0$,*

$$\sup_{T\in\mathbb{R}}|T|^{3/2}\|U(0,T)\beta(|P|>M)\|_{L^1_x\to L^\infty_x}<\infty. \tag{6.39}$$

We remark that this type of a potential problem is difficult, since the moving potential has no limit point, and furthermore it moves sub-linearly in time. Such type of motion may be observed in the motion of vortices for example. See section 6.5 for more details.

6.1.3 Applications to NLS

The method developed here can be applied to some NLS dynamics. Consider Hartree-type NLS

$$i\partial_t\psi(t)=(H_0+V(x,t))\psi(t)+N(|\psi(t)|)\psi(t),\quad \psi(0)=\psi_0\in L^2(\mathbb{R}^3) \tag{6.40}$$

with $N(\cdot): L^2_x\cap L^p_x\to L^2_x\cap \mathcal{F}L^1_x$ for some $2\leq p<6$, satisfying the following **advanced cancellation criterion(ACC)** and some other conditions: for some (q,r) satisfying

$$\frac{2}{q}+\frac{3}{r}=\frac{3}{2},\quad 2\leq q\leq\infty,\ 2\leq r<6, \tag{6.41}$$

1. (**ACC1**): For $\psi(t) \in L_t^q L_x^r(\mathbb{R}^3 \times [-T,T]) \cap C_t L_x^2(\mathbb{R}^3 \times [-T,T])$, some $k_1 > 1$,

$$\|(N(|\psi(t)|))\|_{L_t^{k_1} FL_x^1(\mathbb{R}^3 \times [-T,T])} \lesssim C(\|\psi(t)\|_{L_t^q L_x^r(\mathbb{R}^3 \times [-T,T]) \cap C_t L_x^2(\mathbb{R}^3 \times [-T,T])}). \tag{6.42}$$

2. (**ACC2**): For $\psi(t), \phi(t) \in L_t^q L_x^r(\mathbb{R}^3 \times [-T,T]) \cap C_t L_x^2(\mathbb{R}^3 \times [-T,T])$,

$$\int_{-T}^{T} dt \|N(|\psi(t)|) - N(|\phi(t)|)\|_{FL_x^1} \lesssim C(T)\|\psi(t) - \phi(t)\|_{L_t^q([-T,T])L_x^r \cap C_t L_x^2}$$
$$\times C(\|\psi(t)\|_{L_t^q([-T,T])L_x^r \cap C_t L_x^2}, \|\phi(t)\|_{L_t^q([-T,T])L_x^r \cap C_t L_x^2}) \tag{6.43}$$

with some constant $C(T)$ satisfying

$$C(T) \to 0, \text{ as } T \to 0. \tag{6.44}$$

A typical example: when

$$N(|\psi|) = \frac{1}{|x|^\delta} * |\psi|^2 \quad \text{for any } \delta \in (0, 3/2), \tag{6.45}$$

conditions 1 and 2 are satisfied by taking $r = 4, q = 8/3, k_1 = 4/3$.

3. (**Condition**):

$$\|N(|f(x)|)f(x)\|_{L_x^1} \lesssim \|f(x)\|_{L_x^r}^{q_0} \tag{6.46}$$

with

$$0 \le q_0 \le q. \tag{6.47}$$

Here the potential $V(x,t)$ satisfies following **advanced cancellation criterion** and another condition:

1. (**ACC3**): For all $1 \le p \le \infty$, any $a \in \mathbb{R}$, some $k_2 > 1$,

$$\|V(x, t+a)\|_{L_t^{k_2}([-T,T])FL_x^1} \le_T 1. \tag{6.48}$$

2. (**Condition**): for any $a, T \in \mathbb{R}$,

$$\|V(x,t)\|_{L_t^{q_1}([a,a+T])L_x^{r'}} \le_T 1 \tag{6.49}$$

with

$$\frac{1}{r'} + \frac{1}{r} = 1, \quad \frac{1}{q'} + \frac{1}{q} = 1, \quad q_1 \le q'. \tag{6.50}$$

Theorem 6.1.4. *If $V(x,t)$ satisfies conditions 1 and 2 and if N satisfies 1-2, then* (6.40) *has global wellposedness in L^2_x and in addition, if $\psi_0 \in L^1_x \cap L^2_x$ and N also satisfies 3, then for any $c>0$,*

$$\sup_{|t|\geq c} \|\psi(t)\|_{L^\infty_x} \lesssim_{\|\psi_0\|_{L^1_x\cap L^2_x}, c} 1. \tag{6.51}$$

Remark 33. *Here for global wellposedness, k_1 in* (1) *can be equal to* 1.

The proof for Theorem 6.1.4 relies on advanced CL by using advanced cancellation criterion. Based on such advanced CL for $N(|e^{-itH_0}\psi_0|)$, a new iteration scheme and standard contraction mapping argument, we get local wellposedness in L^2_x and local Strichartz estimate for solution $\psi(t)$. Based on such a result, we are able to build advanced CL for $N(|\psi(t)|)$, which helps to establish the L^∞_x boundedness for $\psi(t)$ when $|t| \geq 1$. Such upper bound is independent on $t \in (-\infty, -c] \cup [c, \infty)$ with given $c>0$. Typical examples are

$$N(|\psi(t)|) = -\lambda[\frac{1}{|x|^{3/2-\delta}} * |\psi(t)|^2](x), \text{ for } \delta \in (0, \frac{3}{2}), \lambda > 0 \tag{6.52}$$

and

$$N(|\psi(t)|) = -\lambda[\frac{e^{-c|x|}}{|x|^{3/2-\delta}} * |\psi(t)|^2](x), \text{ for } \delta \in (0, \frac{3}{2}), \lambda > 0, c > 0. \tag{6.53}$$

Here for (6.52), we have global wellposedness and for (6.53), global wellposedness and L^∞ boundedness when $|t| \geq c$ for any $c>0$.

In order to illustrate the theory, we also prove Theorem 6.1.4 by showing that how the method works in an example:

Theorem 6.1.5. *In*

$$\begin{cases} i\partial_t \psi(t) = H_0\psi(t) + [f * |\psi(t)|^2](x)\psi(t), \\ \psi(0) = \psi_0 \in L^2(\mathbb{R}^3) \end{cases}, \quad \text{with } f(x,t) \in C_t L^2_x, \tag{6.54}$$

(6.54) *has global wellposedness in L^2_x and in addition, if $\psi_0 \in L^1_x \cap L^2_x$, then for any $c>0$,*

$$\sup_{|t|\geq c} \|\psi(t)\|_{L^\infty_x} \lesssim_{\|\psi_0\|_{L^1_x\cap L^2_x}, c} 1. \tag{6.55}$$

When it comes to solutions in H^1_x for the following NLS

$$\begin{cases} i\partial_t\psi(x,t) = (-\Delta_x + N(|\psi(x,t)|))\psi(x,t) \\ \psi(x,0) = \psi_0(x) \in H^1_x(\mathbb{R}^3) \cap L^1_x(\mathbb{R}^3) \end{cases}, \quad \text{in 3 dimensions} \tag{6.56}$$

where H^1_x denotes the Sobolev space of order 1. We show the L^p boundedness of $e^{itH_0}U(t,0) - 1$(including $\Omega - 1$) on $L^{p_0}_x \cap H^1_x$ for any $p_0 \in (6, \infty]$.

For $\psi_0 \in H^1_x$ leading to a global solution with H^1_x uniformly bounded in t, if N satisfies

$$\begin{cases} N(\cdot) : H^1_x \to L^2_x, \text{ is bounded} \\ N_1(\cdot) : H^1_x \to L^2_x, \text{ is bounded} \\ N'(\cdot) : H^1_x \to L^3_x, \text{ is bounded} \end{cases} \tag{6.57}$$

where

$$N'(k) := \frac{d}{dk}[N(k)], \quad N_1(k) = \frac{N(k)}{|k|}, \tag{6.58}$$

then we have:

Theorem 6.1.6 (Existence of free channel wave operator in L^p_x)**.** *For any* $p \in [2, \infty], p_0 \in (6, \infty]$*, if* N *satisfies* (6.57) *and if*

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1_x} \leq C(\|\psi_0\|_{H^1_x}), \tag{6.59}$$

then

$$\|(e^{itH_0}U(t,0) - 1)\psi_0\|_{L^p_x} \leq C(\|\psi_0\|_{H^1_x \cap L^{p_0}_x}, \sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1_x}). \tag{6.60}$$

Furthermore, if we also have

$$\|N(|f(x)|)f(x)\|_{L^{p'}_x} \lesssim_{\|f(x)\|_{H^1_x}} 1, \quad \textit{for some } p \in (6, \infty] \tag{6.61}$$

then for $\psi_0 \in H^1_x \cap L^p_x$ *satisfying* (6.59)*, for* $p > 6$*,*

$$\Omega\ \psi_0 := \lim_{t\to\infty} e^{itH_0}U(t,0)\psi_0 \textit{ exists in } L^p_x \tag{6.62}$$

and

$$\|\Omega\ \psi_0\|_{L^p_x} \leq C(\|\psi_0\|_{H^1_x \cap L^p_x}, \sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1_x}). \tag{6.63}$$

Remark 34. *Here $p > 6$ makes $e^{itH_0} : L_x^{p'} \to L_x^p$, bounded with a bound $|t|^{3(\frac{1}{2}-\frac{1}{p'})}$ integrable on $\mathbb{R} \setminus (-1, 1)$. We will give a proof for the case when $p = \infty$ and the result for other $p \in (6, \infty)$ will follow in a similar way.*

In addition, if we only have $\psi_0 \in H_x^1$, we are able to prove the L^p boundedness of $e^{itH_0}U(t, 0) - 1$ for $2 \leq p < \infty$:

Theorem 6.1.7. *For any $p \in [2, \infty)$, if N satisfies (6.57) and if*

$$\sup_{t\in\mathbb{R}} \|\psi(t)\|_{H_x^1} \lesssim_{\|\psi_0\|_{H_x^1}} 1, \tag{6.64}$$

then $e^{itH_0}U(t, 0) - 1 : H_x^1 \to L_x^p$, is bounded uniformly in $t \in (-\infty, -1] \cup [1, \infty)$. In particular, if $\psi_0 \in L_x^\infty \cap H_x^1$, then $e^{itH_0}U(t, 0)\psi_0 \in L_x^\infty$.

The proof for Theorem 6.1.6 mainly relies on L^∞ boundedness of $e^{itH_0}U(t, 0) - 1$ on $H_x^1 \cap L_x^{p_0}$ since $e^{itH_0}U(t, 0) - 1$ is already bounded on H_x^1 and since L^p result can follow via interpolation inequality. And the L^∞ boundedness relies on the method of ItT potential(advanced CL). The proof for Theorem 6.1.7 mainly relies on the statement that if $\psi_0 \in H_x^1$, then $\psi(t) \in L_x^\infty + FL_x^{1+\epsilon}$ for any $\epsilon \in (0, 1)$.

Here are some examples: when

$$N(f) := |f|^3, \text{ or } -|f|^2 + |f|^3, \tag{6.65}$$

the assumption (6.57) is satisfied: When $N(f) = |f|^3$, that $\psi_0 \in H_x^1$ implies global wellposedness in H_x^1 due to energy conservation

$$E(\psi(t)) := \int d^n x(\frac{1}{2}|\nabla_x \psi(t)|^2 + \frac{1}{2}|\psi(t)|^4). \tag{6.66}$$

When $N(f) = -|f|^2 + |f|^3$, we have following lemma:

Lemma 6.1.2 ([98])**.** *If $\psi_0(x) \in H_x^1$, then with $N(f) = -|f|^2 + |f|^3$,*

$$\|\psi(t)\|_{S^1(I\times\mathbb{R}^3)} \lesssim C(|I|, \|\psi_0\|_{H_x^1}). \tag{6.67}$$

Here we say that a pair of exponents (q, r) is admissible if $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $2 \leq q, r \leq \infty$. And Strichartz norm $\|\cdot\|_{S^1(I\times\mathbb{R}^3)}$ is defined by

$$\|u\|_{S^1(I\times\mathbb{R}^3)} := \sup_{(q,r) \text{ is admissible}} (\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} + \|\nabla_x u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)}) \tag{6.68}$$

for all u *satisfying*

$$\sup_{(q,r)\text{ is admissible}} \left(\|u\|_{L^q_t L^r_x(I\times\mathbb{R}^3)} + \|\nabla_x u\|_{L^q_t L^r_x(I\times\mathbb{R}^3)}\right) < \infty. \tag{6.69}$$

Our method also has some other applications, e.g. the ionization problem for more general potentials [83]. Decay estimates of $\beta(|P|\geq M)P_cU(t,0)$ with rough potentials will be treated in a future publication. In Theorem 6.1.6, the solution is not always dispersive, due to the possible presence of solitons or other bound states.

6.1.4 Other result of this paper and outline of the proof of the main theorems

In section 6.2, we introduce basic properties of CL and improved CL. In section 6.3, we show how this new method works for time-independent potentials. We stick to 3 space dimensions and write $L^p_\xi(\mathbb{R}^3), L^p(\mathbb{R}^3)$ as L^p_ξ, L^p respectively for simplicity.

For some time-independent system with a potential $V(x)$, let $L_{\eta,l,j}(k,\hat{\xi},\epsilon)$ denote the Fourier transform of $\chi(|\xi|\geq 0)\,|\xi|\partial^j_{\xi\cdot e_l}[\hat{V}(\xi-\eta)]e^{-\frac{\epsilon}{|\xi|}}$ in $|\xi|$ variable for $l=1,2,3$, $j=0,1,2$, $\eta\in\mathbb{R}^3$, and

$$K_1(V(x),\eta) := \max_{l=1,2,3,j=0,1,2}\int_{S^2} d\sigma(\xi)\int_{-\infty}^{\infty} dk\,\sup_{\epsilon\geq 0}|L_{\eta,l,j}(k,\hat{\xi},\epsilon)|. \tag{6.70}$$

$$K_m(V) = \sup_{\eta\in\mathbb{R}^3}|K_1(V(x),\eta)|. \tag{6.71}$$

Theorem 6.1.8. *If*

$$\hat{V}_a(\xi) := \sum_{j,l=0}^{2}\sum_{r,m=1}^{3}|\partial^j_{\xi\cdot e_r}\partial^l_{\xi\cdot e_m}\hat{V}(\xi)| \in L^1_\xi \text{ and } K_m(V(x)) := \sup_{\eta\in\mathbb{R}^3}|K_1(V(x),\eta)| < \infty, \tag{6.72}$$

then there exists $M=M(V(x,t))>0$ *such that*

$$\Omega\beta(H_0>M^2) = s\text{-}\lim_{\epsilon\downarrow 0}\Omega_\epsilon\beta(H_0>M^2), \text{ exists on } L^p, 1\leq p\leq\infty \tag{6.73}$$

and $\beta(H_0>M^2)\Omega^*, \Omega\beta(H_0>M^2): L^p\to L^p$ *are bounded.*

Here the assumption $K_m(V(x))<\infty$ can be realized for example, if $\langle|P_\xi|\rangle^2[\hat{V}](\xi)\in L^\infty_\xi(\mathbb{R}^3)(P_\xi := -i\nabla_\xi)$:

Proposition 6.1.1. *If* $\langle|P_\xi|\rangle^2[\hat{V}](\xi) \in L^1_\xi(\mathbb{R}^3)$ *and*

$$\|\langle|P_\xi|\rangle^4[\hat{V}](\xi)\|_{K(\mathbb{R}^3)} < \infty \tag{6.74}$$

where $(P_\xi)_j := -i\partial_{\xi_j}$ *and* $\|\cdot\|_{K(\mathbb{R}^3)}$ *denotes Kato norm*

$$\|V\|_{K(\mathbb{R}^3)} := \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}d^3y, \tag{6.75}$$

then $K_m(V) < \infty$.

Proof. Recall that

$$K_m(V(x)) := \sup_{\eta\in\mathbb{R}^3}|K_1(V(x),\eta)|$$

and

$$K_1(V(x),\eta) := \max_{l=1,2,3,j=0,1,2}\int_{S^2}d\sigma(\xi)\int_{-\infty}^{\infty}dk\sup_{\epsilon\geq 0}|L_{\eta,l,j}(k,\hat{\xi},\epsilon)|$$

where $L_{\eta,l,j}(k,\hat{\xi},\epsilon)$ denotes the Fourier transform of $\chi(|\xi|\geq 0)|\xi|\partial^j_{\xi\, e_l}[\hat{V}(\xi-\eta)]e^{-\frac{\epsilon}{|\xi|}}$ in $|\xi|$ variable for $l=1,2,3,\ j=0,1,2,\ \eta\in\mathbb{R}^3$.

We start with the case when $l=1, j=0, \epsilon=0$,

$$K^{1,0} := \int_{S^2}d\sigma(\xi)\int_{-\infty}^{\infty}dk\left|\int_0^{\infty}(d|\xi|)|\xi|\hat{V}(\xi-\eta)e^{ik|\xi|}\right|.$$

Doing integration by parts in $|\xi|$ variable twice for $|k|>1$ we have:

$$\begin{aligned}|K^{1,0}| \leq\ & \int_{S^2}d\sigma(\xi)\int_{-1}^{1}dk\int_0^{\infty}(d|\xi|)|\xi||\hat{V}(\xi-\eta)|+\\ & \int_{S^2}d\sigma(\xi)\int_{|k|>1}\frac{dk}{k^2}\int_0^{\infty}(d|\xi|)\left|\partial^2_{|\xi|}[|\xi|\hat{V}(\xi-\eta)]\right|+\\ & \int_{S^2}d\sigma(\xi)\int_{|k|>1}\frac{dk}{k^2}|\hat{V}(-\eta)|(\text{Boundary term}).\end{aligned}$$

By Fubini's Theorem and then changing coordinates from the spherical coordinates to the standard Euclidean coordinates, we get

$$\begin{aligned}|K^{1,0}| \leq\ & \int_{-1}^{1}dk\int d^3\xi\frac{|\hat{V}(\xi-\eta)|}{|\xi|}+\int_{|k|>1}\frac{dk}{k^2}\int d^3\xi\frac{\left|\partial^2_{|\xi|}[|\xi|\hat{V}(\xi-\eta)]\right|}{|\xi|^2}+\int_{S^2}d\sigma(\xi)\int_{|k|>1}\frac{dk}{k^2}|\hat{V}(-\eta)|\\ \leq\ & 2\|\langle|P_\xi|\rangle^4[\hat{V}](\xi)\|_{K(\mathbb{R}^3)}+(2\|\langle|P_\xi|\rangle^4[\hat{V}](\xi)\|_{K(\mathbb{R}^3)}+8\pi\|\hat{V}(\xi)\|_{L^1_\xi})+8\pi\|\hat{V}(\xi)\|_{L^1_\xi}.\end{aligned}$$

Similarly, we will get

$$|K_1(V(x),\eta)| \leq 4\|\langle|P_\xi|\rangle^4[\hat{V}](\xi)\|_{K(\mathbb{R}^3)}+16\pi\|\langle|P_\xi|\rangle^2\hat{V}(\xi)\|_{L^1_\xi} \tag{6.76}$$

and then

$$K_m(V) = \sup_{\eta \in \mathbb{R}^3} |K_1(V(x),\eta)| \leq 4\| |P_\xi|^4 [\hat{V}](\xi)\|_{K(\mathbb{R}^3)} + 16\pi \| |P_\xi|^2 \hat{V}(\xi)\|_{L^\infty_\xi}. \quad (6.77)$$

□

We prove Theorem 6.1.1 in section 6.4 and in section 6.5, Theorem 6.1.2. In section 6.5, we also prove the decay estimates directly for more general Mikhlin-type potentials by using the same method. In section 6.6, we show applications to NLS.

Now we outline the steps of the proof. In section 6.2, based on Cook's method, we show $I : L^1 \cap L^2 \to L^2$ exists. Based on the existence of I, we redefine IK in Abelian limit sense, that is,

$$IK = s\text{-}\lim_{\epsilon \downarrow 0} I_\epsilon, \text{ on } L^1 \cap L^2. \quad (6.78)$$

Then, based on the definition of IK in terms of Abelian limit, we derive a representation formula for IK.

Later, when we prove the L^p boundedness of the wave operators in section 6.3 and section 6.4, we prove that $IK : L^p \to L^p$ is bounded when $V(x,t)$ satisfies certain regularity assumptions. IK is the first non-trivial term in the expansion of the wave operator.

To be precise, the operator IK acts like the generating operator for the wave operator, via the Duhamel representation of Ω_ϵ. We use Duhamel's principle, by iterating it for infinitely many times in the expression of Ω_ϵ:

$$\Omega_\epsilon = \sum_{j=0}^{\infty} i^j I_\epsilon^{(j)}, \quad (6.79)$$

where

$$I_\epsilon^{(j)} := \int_0^\infty dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j e^{-\epsilon t_j} K_{t_j} \cdots K_{t_1} \quad (6.80)$$

or equivalently

$$I_\epsilon^{(j)} := \int_0^\infty dt_j e^{-\epsilon t_j} \int_{t_j}^\infty dt_{j-1} \cdots \int_{t_2}^\infty dt_1 K_{t_j} \cdots K_{t_1} \quad (6.81)$$

with $I_\epsilon^{(0)}$, is the identity. For K_t, see (6.87).

The proofs of Theorem 6.1.1 and Theorem 6.1.8 are based on the fact that $I_\epsilon^{(1)}$ is bounded uniformly in $\epsilon \in [0,1]$, and

$$\|I_\epsilon^{(k+1)}\beta(|P|>M)\|_{L_x^p \to L_x^p} \lesssim \frac{C^k}{M^{k-1}}, \text{ for each } p \in [1, \infty]. \tag{6.82}$$

Here $P_j := e_j \cdot P = -i\partial_{x_j}$, $j = 1,2,3(\beta(H_0 > 4M^2) = \beta(|P| > M)\beta(H_0 > 4M^2))$.

If we choose M large enough such that $\sum_{k=2}^{\infty} \frac{C^k}{M^{k-1}}$ converges, then for $\epsilon \in [0,1]$,

$$\|\Omega_\epsilon \beta(|P|>M)\|_{L^p \to L^p} \lesssim 1 + \sum_{k=1}^{\infty} \frac{C^k}{M^{k-1}} < \infty. \tag{6.83}$$

By the same argument, we get that the maximal Ω transform is L^p bounded, which implies the pointwise convergence in L^p.

Based on the uniform boundedness of Ω_ϵ and pointwise convergence, we get Theorem 6.1.1 and Theorem 6.1.8.

In section 6.5, for self-similar potentials, we only use CL:

$$K_t(V(x,t)) : L_x^p \to L_x^p, \text{ is bounded uniformly in } t, \text{ if } \hat{V}(\xi,t) \in L_t^\infty L_\xi^1. \tag{6.84}$$

Since the other factor is already in L_t^1, we get a bound $\frac{C^k}{k!}$ for each $I_0^{(k)}$; absolute convergence of the sum of $I_0^{(k)}$ over k follows, and we get desired result.

6.2 Improved CL

We introduce further notation used throughout this paper first, and then the CL and improved CL.

6.2.1 Notation

We use n to denote the dimension of the ambient physical space, the configuration space. If $x = (x_1, \dots, x_n)$ and $\xi = (\xi_1, \dots, \xi_n)$ lie in $\mathbb{R}^n$, we use $x \cdot \xi$ to denote the dot product $x \cdot \xi := x_1\xi_1 + \dots + x_n\xi_n$, and $|x|$ to denote the magnitude $|x| := (x_1^2 + \dots + x_n^2)^{1/2}$. We also use $\langle x \rangle$ to denote the inhomogeneous magnitude (Japanese x) $\langle x \rangle := (1 + |x|^2)^{1/2}$ of x. The derivatives will either be interpreted in the classical sense or the distributional sense.

If X and Y are two quantities, we use $X \lesssim Y$ to denote the statement that $X \leq CY$ for some absolute constant $C > 0$. More generally, given some parameters $a_1, \ldots, a_k$, we use $X \lesssim_{a_1,\ldots,a_k} Y$ to denote the statement that $X \leq C_{a_1,\ldots,a_k} Y$ for some constant $C_{a_1,\ldots,a_k} > 0$ which can depend on the parameters $a_1, \ldots, a_k$.

Throughout the paper, $P_j := -i\partial_{x_j}$ and Q_j is multiplication by x_j. Sometimes we use x_j denote the operator of multiplication by x_j. The commutator $i[P_j, Q_k] = \delta_{jk}$ and $P^2 = P_j P_j = -\Delta_x$ where δ_{jk} is the Kronecker delta. $\{e_1, \ldots, e_n\}$ denotes a basis in $\mathbb{R}^n$. τ denotes the operator of dilation $(\tau_\delta f)(x) = f(\delta x)$.

We also assume $\beta(t \leq 1) = 1 - \beta(t > 1)$ and

$$\sup_{n=0,1,2,3,4} \|\beta^{(n)}(t)\|_{L^1_t} \leq C_\beta, \tag{6.85}$$

with

$$\beta^{(n)}(t) := \frac{d^n}{dt^n}[\beta(t)]. \tag{6.86}$$

6.2.2 CL and Improved CL

We start with the introduction of the ***time translated (tT) Potential***, the translation being the flow under the free hamiltonian, the Laplacian:

$$\mathcal{K}_t(V(x,s)) := e^{itH_0} V(Q,s) e^{-itH_0}. \tag{6.87}$$

Since

$$d/dt(e^{itf(P)} g(Q) e^{-itf(P)}) = e^{itf(P)} i[f(P), g(Q)] e^{-itf(P)}, \tag{6.88}$$

we have

$$e^{itH_0} Q e^{-itH_0} = e^{itP^2} Q e^{-itP^2} = Q + \int_0^t (e^{itP^2}(2P)e^{-itP^2})dt = Q + 2tP,$$

which implies

$$e^{itH_0} e^{i\xi\cdot Q} e^{-itH_0} = e^{i\xi\cdot(Q+2tP)}, \text{ for } \xi \in \mathbb{R}^n. \tag{6.89}$$

Based on $i[P_j, Q_j] = 1$, we have

$$[i\xi\cdot Q, it\xi\cdot P] = \sum_{l,j} -t\xi_j\xi_l[Q_j, P_l] = \sum_{l,j} -it\xi_j\xi_l\delta_{jl} = -it\xi^2. \tag{6.90}$$

Then since $[i\xi \cdot Q, it\xi \cdot P]$ is a scalar number, according to Baker-Campbell-Hausdorff formula, we have

$$e^{i\xi\cdot(Q+2tP)} = e^{i\xi\cdot Q} \cdot e^{2it\xi\cdot P} \cdot e^{-\frac{1}{2}[i\xi\cdot Q, 2it\xi\cdot P]} = e^{i\xi\cdot Q} \cdot e^{2it\xi\cdot P} \cdot e^{it\xi^2}. \tag{6.91}$$

Based on identity (6.91), the representation of the tT potential operator follows

$$\mathcal{K}_t(V(x,t)) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int d^n\xi \hat{V}(\xi,t) e^{i\xi\cdot Q} e^{2it\xi\cdot P} \cdot e^{it\xi^2}. \tag{6.92}$$

Hence,the tT potential satisfies:

$$\|\mathcal{K}_t(V(x,t))\|_{L^p_x \to L^p_x} \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \|\hat{V}(\xi,t)\|_{L^1_\xi}. \tag{6.93}$$

If $\hat{V}(\xi,t)$ happens to be a finite measure in ξ and its total variation is denoted by $m(t)$, and if $\sup_{t\in\mathbb{R}} m(t) < \infty$, we also have

$$\|\mathcal{K}_t(V(x,t))\|_{L^p_x \to L^p_x} \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \sup_{t\in\mathbb{R}} m(t). \tag{6.94}$$

Then we get (CL) **cancelation Lemma**:

Proposition 6.2.1 (CL)**.** *If $\hat{V}(\xi,t)$ is assumed to be a finite measure whose total variation is denoted by $m(t)$ and if $\sup_{t\in\mathbb{R}} m(t) < \infty$, then*

$$\sup_{s\in\mathbb{R}} \|\mathcal{K}_s(V(x,t))\|_{L^p_x \to L^p_x} \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \sup_{t\in\mathbb{R}} m(t). \tag{6.95}$$

Proposition 6.2.2. *Recall the definition of $\Omega(0,t)$, see (6.36).*

$$\ln\left(\|\Omega(0,t)\|_{L^p_x \to L^p_x}\right) \leq \int_0^{|t|} du \|\mathcal{K}_u(V(x,u))\|_{L^p_x \to L^p_x}. \tag{6.96}$$

Therefore if $\hat{V}(\xi,t)$ is assumed to be a finite measure whose total variation is denoted by $m(t)$ and if

$$c(t) := \int_0^t ds |m(s)| \lesssim_t 1, \tag{6.97}$$

then for $1 \leq p \leq \infty$,

$$\ln\left(\|\Omega(0,t)\|_{L^p_x \to L^p_x}\right) \leq \frac{c(t)}{(2\pi)^{\frac{n}{2}}} \tag{6.98}$$

or

$$\|\Omega(0,t)\|_{L^p_x \to L^p_x} \leq \exp\left(\frac{c(t)}{(2\pi)^{\frac{n}{2}}}\right). \tag{6.99}$$

Similarly, we have

$$\ln\left(\|\Omega(0,t)\|_{L^p_x \to L^p_x}\right) \le \frac{c(t)}{(2\pi)^{\frac{n}{2}}} \tag{6.100}$$

or

$$\|\Omega(0,t)\|_{L^p_x \to L^p_x} \le \exp\left(\frac{c(t)}{(2\pi)^{\frac{n}{2}}}\right). \tag{6.101}$$

Proof. Since in n dimensions,

$$K_t(V(x,t)) = e^{itH_0}V(x,t)e^{-itH_0} = \frac{1}{(2\pi)^{\frac{n}{2}}}\int d^n\xi \hat{V}(\xi,t)e^{itH_0}e^{ix\cdot\xi}e^{-itH_0} \tag{6.102}$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\int d^n\xi \hat{V}(\xi,t)e^{iQ\cdot\xi}e^{-2it\xi\cdot P}e^{it\xi^2} \tag{6.103}$$

where Q denotes the operator of multiplication by x, we obtain

$$\|e^{itH_0}V(x,t)e^{-itH_0}\|_{L^p \to L^p} \le \frac{|m(t)|}{(2\pi)^{\frac{n}{2}}}. \tag{6.104}$$

Now we prove the boundedness of $\Omega(0,t)$. For $\Omega(0,t)$, we use Duhamel's formula and iterate it for infinitely many times

$$\Omega(0,t) = \sum_{k=0}^{\infty} i^k I^{(k)}(t), \tag{6.105}$$

where

$$I^{(k)}(t) := \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t dt_k K_{t_1}(V(x,t_1)) \cdots K_{t_k}(V(x,t_k)), \quad k = 0, 1, \cdots, \tag{6.106}$$

$I^{(0)}(t)$ denotes the identity. Since

$$\|I^{(k)}(t)\|_{L^p \to L^p} \le \int_0^{|t|} dt_1 \int_{t_1}^{|t|} dt_2 \cdots \int_{t_{k-1}}^{|t|} dt_k \|K_{t_1}(V(x,t_1))\|_{L^p_x \to L^p_x} \cdots \|K_{t_k}(V(x,t_k))\|_{L^p_x \to L^p_x}$$

$$= \frac{1}{k!}\left(\int_0^{|t|} ds \|K_s(V(x,s))\|_{L^p_x \to L^p_x}\right)^k,$$

we have

$$\|\Omega(0,t)\|_{L^p_x \to L^p_x} \le \exp\left(\int_0^{|t|} ds \|K_s(V(x,s))\|_{L^p_x \to L^p_x}\right). \tag{6.107}$$

So if $\hat{V}(\xi,t) \in L^\infty_t L^1_\xi$, due to (6.104), we get

$$\ln\left(\|\Omega(0,t)\|_{L^p_x \to L^p_x}\right) \le \frac{c(t)}{(2\pi)^{\frac{n}{2}}}, \tag{6.108}$$

that is,

$$\|\Omega(0,t)\|_{L_x^p \to L_x^p} \le \exp\left(\frac{c(t)}{(2\pi)^{\frac{n}{2}}}\right). \tag{6.109}$$

Similarly, since

$$\Omega(0,t) = \sum_{k=0}^{\infty} i^k \left(I^{(k)}\right)(t), \tag{6.110}$$

where

$$\left(I^{(k)}\right)(t) := \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k K_{t_1}(V(x,t_1)) \cdots K_{t_k}(V(x,t_k)), \quad k = 0, 1, \cdots, \tag{6.111}$$

we have

$$\begin{aligned}\|\left(I^{(k)}\right)(t)\|_{L^p \to L^p} &\le \int_0^{|t|} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k \|K_{t_1}(V(x,t_1))\|_{L_x^p \to L_x^p} \cdots \|K_{t_k}(V(x,t_k))\|_{L_x^p \to L_x^p} \\ &= \frac{1}{k!}\left(\int_0^{|t|} ds \|K_s(V(x,s))\|_{L_x^p \to L_x^p}\right)^k\end{aligned}$$

and therefore

$$\|\Omega(0,t)\|_{L_x^p \to L_x^p} \le \exp\left(\int_0^{|t|} ds \|K_s(V(x,s))\|_{L_x^p \to L_x^p}\right). \tag{6.112}$$

So if $\hat{V}(\xi,t) \in L_t^\infty L_\xi^1$, due to (6.104), we get

$$\ln\left(\|\Omega(0,t)\|_{L_x^p \to L_x^p}\right) \le \frac{c(t)}{(2\pi)^{\frac{n}{2}}}, \tag{6.113}$$

that is,

$$\|\Omega(0,t)\|_{L_x^p \to L_x^p} \le \exp\left(\frac{c(t)}{(2\pi)^{\frac{n}{2}}}\right). \tag{6.114}$$

□

It implies immediately the global boundedness of $\Omega(0,T)$ for Schrödinger equations with general potentials; for example, quasi-periodic in x, on L^∞ space in one dimension:

Corollary 6.2.1. *In one dimension, if $V(x)$ is quasi periodic, (in other word, if $V(x)$ is a finite sum of terms of the form $a\cos(bx)$ or $a\sin(bx)$) and if the initial data is de^{icx} for some $c, d \in \mathbb{R}$, then $\Omega(0,t)\psi(0)$ of*

$$i\partial_t \psi(x,t) = (H_0 + V(x))\psi(x,t) \tag{6.115}$$

exists in L^∞ and is a sum of sine and cosine terms only, and is bounded for all times.

Proof. Assume

$$V(x) = \sum_{k=0}^{N} a_k \cos(b_k x) + c_k \sin(d_k x) \tag{6.116}$$

The boundedness follows from (6.109) with

$$c(t) \quad t \sum_{k=0}^{N} j a_k j + j c_k j. \tag{6.117}$$

The solution is a sum of sine and cosine terms only since

$$K_t(e^{iax})\psi(0) = K_t(e^{iax})(de^{icx}) = de^{ita^2} e^{iax} e^{ic(x+2ta)} = de^{i(ta^2+2tac)} e^{ix(a+c)}. \tag{6.118}$$

□

In particular, if both initial data $\psi(x, 0)$ and the potential $V(x, t)$ are smooth in x, then so is the solution:

Corollary 6.2.2. *If both initial data* $\psi(x, 0)$ *and the potential* $V(x, t)$ *are smooth in* x*, then so is the solution of* (6.115)*.*

Proof. If the initial data $\psi(x, 0)$is smooth in x, then in (6.106), take nth order derivative on both sides and on the right hand side; one can commute through the derivative. It hits the potential term. So if $V(x, t)$ is smooth in x, then so is the solution for all times. □

We would like to introduce the *Integrated tT Potential operator*

$$IK \; := \int_0^{1} dt K_t(V(x, t)) \tag{6.119}$$

which is relevant to the study of L^p boundedness of the wave operator. Based on Cook's method, one can prove the existence of $IK \; : L^1_x \setminus L^2_x \; ! \; L^2_x$ when $V(x, t) \; 2 \; L^1_t L^1_x \setminus L^1_t L^2_x$.

Lemma 6.2.1. *When* $V(x, t) \; 2 \; L^1_t L^1_x \setminus L^1_t L^2_x(\mathrm{R} \quad \mathrm{R}^n)$*,* $n \quad 3$*,* $I : L^1_x \setminus L^2_x \; ! \; L^2_x$ *exists and is bounded.*

Proof. Let $\psi \; 2 \; L^1_x \setminus L^2_x$. Since

$$k e^{itH_0} V(Q, t) e^{\;\, itH_0} \psi k_{L^2_x} \; \cdot_n \; \frac{1}{hti^{n/2}} k V(x, t) k_{L^1_t L^2_x \setminus L^1_t L^1_x} k \psi(x) k_{L^2_x \setminus L^1_x} \tag{6.120}$$

where we use that e^{itH_0} is unitary on L^2 and the L^∞ decay estimates of e^{itH_0} on L^1, we have

$$\|IK\|_{L^2_x\cap L^1_x\to L^2_x} \leq_n \|V(x,t)\|_{L^\infty_t L^2_x\cap L^\infty_t L^1_x}\int_0^\infty \frac{dt}{\langle t\rangle^{n/2}} \leq_n \|V(x,t)\|_{L^\infty_t L^2_x\cap L^\infty_t L^1_x}. \tag{6.121}$$

□

Once we know the existence of IK on $L^1_x\cap L^2_x$, we can redefine IK in Abelian limit sense

$$IK = s\text{-}\lim_{\epsilon\downarrow 0} IK_\epsilon, \text{ on } L^1_x\cap L^2_x \tag{6.122}$$

where

$$IK_\epsilon := \int_0^\infty dt e^{-\epsilon t} K_t(V(x,t)). \tag{6.123}$$

There is no confusion about this limit taken in strong sense. Due to a similar argument in Lemma 6.2.1 we have that $IK_\epsilon : L^1_x\cap L^2_x \to L^2_x$ is uniformly bounded in $\epsilon\in[0,1]$. Based on this definition of IK, when V is time-independent, we get the following representation of IK:

Lemma 6.2.2. *If $\hat{V}(\xi)\in L^1_\xi$, then for $\epsilon>0$,*

$$IK_\epsilon = \frac{1}{(2\pi)^{n/2}}\int d^3\xi \hat{V}(\xi)e^{ix\cdot\xi}\frac{1}{i(\xi^2+2\xi\cdot P)-\epsilon} \tag{6.124}$$

Proof. It suffices to check on a dense set of $L^1_x\cap L^2_x$. Choose $\psi\in L^\infty_x\cap L^1_x$. According to the identity (6.91),

$$IK_\epsilon\psi(x) = \frac{1}{(2\pi)^{n/2}}\int_0^\infty dt\int d^n\xi \hat{V}(\xi)e^{ix\cdot\xi}e^{it\xi^2-\epsilon t}\psi(x+2t\xi). \tag{6.125}$$

That $\psi\in L^\infty_x$, $e^{-\epsilon t}\in L^1_t[0,\infty)$ and $\hat{V}(\xi)\in L^1_\xi$ imply

$$\hat{V}(\xi)e^{ix\cdot\xi}e^{it\xi^2-\epsilon t}\psi(x+2t\xi)\in L^1_t[0,\infty)L^1_\xi. \tag{6.126}$$

Then by Fubini's theorem, we change the order of the integral and then take the integral over t

$$IK_\epsilon\psi = \frac{1}{(2\pi)^{n/2}}\int d^n\xi \hat{V}(\xi)e^{ix\cdot\xi}\frac{1}{i(\xi^2+2\xi\cdot P)-\epsilon}\psi. \tag{6.127}$$

□

IK is regarded as the limit of IK_ϵ as $\epsilon \downarrow 0$ in strong topology. Based on Lemma 6.2.2,

$$IK = \frac{1}{(2\pi)^{n/2}} \int d^n\xi \hat{V}(\xi) e^{ix\cdot\xi} \frac{1}{i(\xi^2 + 2\xi \cdot P) - 0}. \tag{6.128}$$

For the construction of the wave operator, we have to introduce another representation formula for IK_ϵ. Choose $V(x,t) \in S_t S_x$. For $\psi \in L^\infty_x \cap L^p_x$, in identity (6.125), we use Fubini's theorem to integrate over t first, use spherical coordinates of ξ, then change variables from $t \to u = |\xi| t$ and then change the order of the integral over $|\xi|$ and u

$$IK_\epsilon \psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{S^2} d\sigma(\xi) \int_0^\infty du \int_0^\infty d|\xi| |\xi| \hat{V}(\xi, \frac{u}{|\xi|}) e^{-\frac{\epsilon u}{|\xi|} + i(x\cdot\xi + u|\xi|)} \psi(x + 2u\hat{\xi}). \tag{6.129}$$

Then for $\psi \in L^p$ and general $V(x,t)$, we have a representation

$$IK_\epsilon \psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{S^2} d\sigma(\xi) \int_0^\infty du \int_0^\infty d|\xi| |\xi| \hat{V}(\xi, \frac{u}{|\xi|}) e^{-\frac{\epsilon u}{|\xi|} + i(x\cdot\xi + u|\xi|)} \psi(x + 2u\hat{\xi}). \tag{6.130}$$

6.2.3 Improved CL For Time Dependent Potentials

For tT time-dependent Potentials, we cannot prove the improved cancellation lemma (ICL) without regularity assumptions in x. .*To be precise, if we just assume $V(x,t) \in C_t L^1_x$, the improved cancellation lemma fails!*. Here is a proof:

Let $B_{\infty,2}(T)(T > 0)$ denote the space of bounded linear transformation from $C_t([-T,T])L^2_x$ to $L^p_t([-T,T])L^2_x(p > 1)$ and its standard norm is denoted by $\| \cdot \|_{B_{\infty,2}(T)}$. Now we consider the following linear transformation

$$L_T : D_T \to B_{\infty,2}(T), \quad V(x,t) \mapsto K_t(V(x,t)) \tag{6.131}$$

where

$$D_T := \{V(x,t) \in C_t([-T,T])L^1_x : \|L_T(V(x,t))\|_{B_{\infty,2}(T)} < \infty\}. \tag{6.132}$$

The following lemma reveals the unbounded nature of L_T:

Lemma 6.2.3. *For all $T > 0$, L_T defined in* (6.131) *is unbounded.*

Proof. Prove by contradiction. Assume there exists $T_0 > 0$ such that

$$L_{T_0} := \|L_{\frac{T_0}{2}}\|_{D_{T_0/2} \to B_{\infty,2}(T_0/2)} < \infty \tag{6.133}$$

and therefore $D_{T_0} = C_t([-T_0, T_0])L^1_x$. According to the definition of L_T, we have

$$L_{t_1} \le L_{t_2}, \text{ if } 0 < t_1 < t_2, \tag{6.134}$$

which implies $L_T < \infty$ if $T \le T_0$. In the following, we are going to use this to get a contradiction. We consider a NLS system

$$i\partial_t \psi(t) = H_0\psi(t) + |\psi(t)|^{p-1}\psi(t), \text{ with } p = 3, n = 3. \tag{6.135}$$

We will show that if (6.133) holds, it implies the local wellposedness of this NLS in $L^2_x(\mathbb{R}^n)$. This violates the known result that well-posedness in $H^s_x(\mathbb{R}^n)$ holds, if and only if $s \ge \max(s_c, 0)$, where $s_c := \frac{d}{2} - \frac{2}{p-1}$.

For $\psi(0) = \psi_0 \in L^2_x(\mathbb{R}^n)$, let us consider the following iteration

$$\phi_k(t) = e^{-itH_0}\psi_0 + (-i)\int_0^t ds e^{-itH_0} K_s(|\phi_{k-1}(s)|^2)e^{isH_0}\phi_k(s) \tag{6.136}$$

with $\phi_0 = e^{-itH_0}\psi_0$. Since due to the definition of L_T and Hölder's inequality,

$$\left\|\int_0^t ds e^{-itH_0} K_s(|\phi_{k-1}(s)|^2) f(x,s)\right\|_{C_t([-T,T])L^2_x} \le T^{p'} L_T \||\phi_{k-1}(t)|^2\|_{C_t([-T,T])L^1_x} \|f(x,t)\|_{C_t([-T,T])L^2_x}, \tag{6.137}$$

due to Corollary 6.2.2,

$$\|\phi_k(t)\|_{C_t([-T,T])L^2_x} \le \|\phi_0\|_{L^2_x} \exp\left(T^{p'} \||\phi_{k-1}(t)|^2\|_{C_t([-T,T])L^1_x} L_T\right), \tag{6.138}$$

if $\phi_{k-1}(t) \in C_t([-T,T])L^2_x$. Since $\phi_0 = e^{-itH_0}\psi_0 \in C_t([-T,T])L^2_x$, due to conservation law, we have

$$\|\phi_k(t)\|_{L^2_x} = \|\psi_0\|_{L^2_x}, \text{ for all } k = 0, \cdots. \tag{6.139}$$

Since

$$\begin{aligned} &K_t(|\phi_{k-1}|^2)e^{itH_0}\phi_k - K_t(|\phi_k|^2)e^{itH_0}\phi_{k+1} \\ =&K_t((\phi_{k-1} - \phi_k)\phi_{k-1})e^{itH_0}\phi_k + K_t(\phi_k(\phi_{k-1} - \phi_k))e^{itH_0}\phi_k + K_t(|\phi_k|^2)e^{itH_0}(\phi_k - \phi_{k+1}), \end{aligned}$$

applying estimate (6.137), we get

$$\begin{aligned}&\|\phi_k(t)-\phi_{k+1}(t)\|_{C_t([-T,T])L^2_x}\\ \le\;& 2T^{p'}L_T\|\psi_0\|^2_{L^2_x}\|\phi_k(t)-\phi_{k-1}(t)\|_{C_t([-T,T])L^2_x}+T^{p'}L_T\|\psi_0\|^2_{L^2_x}\|\phi_k(t)-\phi_{k+1}(t)\|_{C_t([-T,T])L^2_x},\end{aligned}$$

which implies

$$\|\phi_k(t)-\phi_{k+1}(t)\|_{C_t([-T,T])L^2_x} \tag{6.140}$$

$$\le 2T^{p'}L_T\|\psi_0\|^2_{L^2_x}\|\phi_k(t)-\phi_{k-1}(t)\|_{C_t([-T,T])L^2_x}+T^{p'}L_T\|\psi_0\|^2_{L^2_x}\|\phi_k(t)-\phi_{k+1}(t)\|_{C_t([-T,T])L^2_x}. \tag{6.141}$$

Choose T small enough such that $T^{p'}L_T\|\psi_0\|^2_{L^2_x}\le\frac{1}{8}$. Then

$$\|\phi_k(t)-\phi_{k+1}(t)\|_{C_t([-T,T])L^2_x}\le\frac{1}{2}\|\phi_k(t)-\phi_{k-1}(t)\|_{C_t([-T,T])L^2_x}. \tag{6.142}$$

By the contraction mapping principle, we get local wellposedness in L^2_x. Then based on the same argument, we get global existence of (6.135). Contradiction since in [60], Merle, Raphaël and Szeftel showed there is a solution $u\in C_t([0,T))H^1_x\subset L^2_x$ which blows up in L^2_x at time T. Also, in [11], Christ, Colliander and Tao sketched the proof of the ill-posedness in L^2_x.

□

Remark 35. *Lemma 6.2.3 implies the failure of local smoothing property for some $C_tL^1_x$ localization. In other word, for some $V(x,t)\in C_tL^1_x$, any $A>0$, the map $C: C_t([-A,A])L^2_x\to L^1_t([-A,A])L^2_x$, $f\mapsto V(x,t)e^{-itH_0}f$, is unbounded.*

By applying a similar argument, we get a useful bound for deriving for decay estimates for rough potentials:

Lemma 6.2.4. *If $V(x,t)\in L^\infty_tL^q_x(\mathbb{R}^3)$ for $q\in(\frac{4}{3},2]$, then for $t\in(0,1]$, $s\in[\frac{t}{2},t)$, $K_se^{itH_0}:L^1_x\cap L^2_x(\mathbb{R}^3)\to L^\infty_x$ is bounded with*

$$\|K_se^{itH_0}\|_{L^1_x\cap L^2_x\to L^\infty_x}\lesssim\frac{1}{t^{3/2}}\,\frac{1}{(t-s)^{1-\epsilon}} \tag{6.143}$$

for some $\epsilon=\epsilon(q)\in(0,1]$.

Proof. Let $\psi \in S$ and $\hat{V}(\xi,t) \in L^\infty_t L^1_\xi$. According to the same computation above,

$$K_s e^{itH_0}\psi = \frac{1}{(2\pi)^{3/2}}\int d^3\xi\, \hat{V}(\xi,s) e^{ix\cdot\xi} e^{is|\xi+P|^2} e^{i(t-s)P^2}\psi. \tag{6.144}$$

Let $\psi_{t-s} := e^{i(t-s)P^2}\psi$. Then $\psi_{t-s} \in L^2_x \cap L^\infty_x$ when $s<t$.

$$e^{is|\xi+P|^2} e^{i(t-s)P^2}\psi = \frac{1}{(2\pi i s)^{3/2}}\int d^3k\, e^{-i\frac{k^2}{2s}}\psi_{t-s}(x-k) e^{-ix\cdot\xi} e^{i(x-k)\cdot\xi}. \tag{6.145}$$

Hence,

$$K_s e^{itH_0}\psi = \frac{1}{(2\pi)^{3/2}}\cdot\frac{1}{(2\pi i s)^{3/2}}\int d^3\xi\, d^3k\, \hat{V}(\xi,s) e^{i(x-k)\cdot\xi} e^{-i\frac{k^2}{2s}}\psi_{t-s}(x-k). \tag{6.146}$$

$\psi \in S$ implies $\psi_{t-s}(x) \in L^1_x$. Then $\hat{V}(\xi,s)\psi_{t-s}(x-k) \in L^1_\xi L^1_k$. By Fubini's theorem, we change the order of the integral and integrate over ξ first

$$K_s e^{itH_0}\psi = \frac{1}{(2\pi i s)^{3/2}}\int d^3k\, e^{-i\frac{k^2}{2s}}\psi_{t-s}(x-k)V(x-k,s). \tag{6.147}$$

Then when $V(x,t) \in L^\infty_t L^q_x$ for $q \in (\frac{4}{3},2]$, by Hölder's inequality,

$$\|\psi_{t-s}(x-k)V(x-k,s)\|_{L^1_k} \le \|\psi_{t-s}(x-k)\|_{L^{q'}_k}\|V(x-k,t)\|_{L^\infty_t L^q_k} \lesssim \frac{\|V(x-k,t)\|_{L^\infty_t L^q_k}\|\psi\|_{L^1_x\cap L^2_x}}{(t-s)^{3(2-q')/2}}. \tag{6.148}$$

$q \in (\frac{4}{3},2]$ implies $3(2-q')/2 \in [0,1)$. Then we use the B.L.T. twice and get the same inequality (6.148) for $\psi \in L^1_x \cap L^2_x(\mathbb{R}^3)$, $V \in L^\infty_t L^q_x(\mathbb{R}^3)$. Combining this inequality with (6.146), we complete the proof. □

For the construction of the wave operator, we also need to introduce the following operators

$$I^{(k)}_\epsilon := \int_0^\infty dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{k-1}}dt_k\, e^{-\epsilon t_1}K_{t_k}(V(x,t_k))\cdots K_{t_1}(V(x,t_1)),\ \text{for } k=1,2,\cdots. \tag{6.149}$$

6.3 Time-independent potentials in $\mathbb{R}^3$

In this section, we prove the $L^p(\mathbb{R}^3)$ boundedness of the wave operator Ω for time-independent potentials $V(x)$. We consider only high-frequency part of the solution. We assume

$$\begin{cases} K_m(V(x)) = \sup_{\eta\in\mathbb{R}^3}|K_1(V(x),\eta)| < \infty, \\ \hat{V}_a(\xi) \in L^1_\xi. \end{cases} \tag{6.150}$$

Recall that $L_{\eta,l,j}(k,\hat{\xi},\epsilon)$ denotes the Fourier transform of $|\xi|\partial^j_{\xi\cdot e_l}[\hat{V}(\xi-\eta)]e^{-\frac{\epsilon}{|\xi|}}$ in $|\xi|$ variable for $l=1,2,3,\ j=0,1,2,$

$$K_1(V(x),\eta)=\max_{l=1,2,3,j=0,1,2}\int_{S^2}d\sigma(\xi)\int_{-\infty}^{\infty}dk\sup_{\epsilon\geq 0}|L_{\eta,l,j}(k,\hat{\xi},\epsilon)| \tag{6.151}$$

and

$$\hat{V}_a(\xi)=\sum_{j,l=0}^{2}\sum_{r,m=1}^{3}|\partial^j_{\xi\cdot e_r}\partial^l_{\xi\cdot e_m}\hat{V}(\xi)|,\ \text{with a basis } \{e_1,e_2,e_3\}. \tag{6.152}$$

6.3.1 Some basic lemmas

For the L^p estimates for $I-K$ and wave operator in the following context we need:

Lemma 6.3.1. *Let $f(u)\in L^1_u(\mathbb{R})$. Then the operator $T_{\hat{\xi}}:L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$*

$$T_{\hat{\xi}}(\psi)(x):=\int_0^{\infty}dk f(x\cdot\hat{\xi}+k)\psi(x+2k\hat{\xi}) \tag{6.153}$$

is uniformly bounded in $\hat{\xi}\in S^{n-1}$ for $1\leq p\leq\infty$ with upper bound $\|f(k)\|_{L^1_k}$.

Proof. Write $x:=\sum_{j=1}^{n}x_je_j=(x_1,\cdots,x_n)$ with $e_1:=\hat{\xi}$. We do a change of variables $k\to u=k+x\cdot\hat{\xi}$

$$T_{\hat{\xi}}(\psi)(x)=\int_{x\cdot\hat{\xi}}^{\infty}du f(u)\psi(2u-x_1,x_2,\cdots,x_n). \tag{6.154}$$

Then by Minkowski's integral inequality,

$$\|T_{\hat{\xi}}(\psi)(x)\|_{L^p_x}\leq\int|f(u)|\|\psi(2u-x_1,x_2,\cdots,x_n)\|_{L^p_x}du=\|f(u)\|_{L^1_u}\|\psi(x)\|_{L^p_x}. \tag{6.155}$$

□

Lemma 6.3.2. *For $d\in\{1,2,3,4\}$, $j\in\{0,1,2\}$, $M>1$, $\epsilon\in\mathbb{R}$, $1\leq p\leq\infty$, let*

$$P_{jd}(M,\epsilon):=\frac{\beta^{(j)}(|P|>2M)}{(P+i\epsilon)^d}:L^p(\mathbb{R})\to L^p(\mathbb{R}), \tag{6.156}$$

a Fourier multiplier. Then $\|P_{jd}(M,\epsilon)\|_{L^p\to L^p}\lesssim\frac{1}{M^d}$. In addition, for $\psi\in L^p$,

$$\|\sup_{\epsilon\in[0,1]}|P_{jd}(M,\epsilon)\psi(x)|\|_{L^p_x}\lesssim\frac{1}{M^d}\|\psi(x)\|_{L^p_x}. \tag{6.157}$$

Proof. When $d=1$, it suffices to show that it is the Fourier transform of some finite Borel measure μ_M whose total variation is less than C/M. Let

$$\mu(x) := \mathcal{F}_q^{-1}\Big[\frac{\beta(|q|>2)}{q+i\epsilon/M}\Big](x), \text{ and then } \mu_M(x) = \Big[\mathcal{F}^{-1}\tau_{1/M}\mathcal{F}\Big[\frac{\mu}{M}\Big]\Big](x) = [\tau_M\mu](x) = \mu(Mx) \tag{6.158}$$

since $\mathcal{F}\sigma_\delta = |\delta|^{-1}\tau_{\delta^{-1}}\mathcal{F}$. We will show that $M\int|d\mu_M(x)| \lesssim 1$ for $d=1$, and the other cases will follow in the same way. Since for q large, $\frac{1}{q+i\epsilon/M} \sim \frac{1}{q}$, then for $|x| \le 1$, $|d\mu(x)| \lesssim \ln|x|dx$. For $|x|>1$, since $|d\mu(x)| \lesssim_N \frac{1}{|x|^N}dx$ for any $N \ge 1$, by the use of integration by parts, then $|\mu(x)| \lesssim \frac{1}{x^2}$. Hence,

$$\int|d\mu_M(x)| = \frac{1}{M}\int M|d\mu(Mx)| = \frac{1}{M}\int|d\mu(x)| \lesssim \frac{1}{M}. \tag{6.159}$$

□

In [48], Journé, Soffer and Sogge proved that the high energy cutoff function $\gamma(H/M): L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is bounded for each $M>0$, when $\gamma \in C^\infty(\mathbb{R})$ satisfying $\gamma(\lambda)=1$ for $\lambda \ge 1$, and $\beta(\lambda)=0$ for $-\infty<\lambda<1/2$; $H = H_0+V(x)$ for some nice $V(x)$ including the case when $H=H_0$.

When $H=H_0$, this high energy function $\gamma(H_0>M)$ is Fourier multiplier, and it implies that $\beta(|P|>M)$ is also bounded on L^1 by taking $\gamma(H_0/M^2) = \beta(\sqrt{H_0/M^2})$. By duality, we get the L^p boundedness of $\beta(|P|>M)$ for all $1 \le p \le \infty$. We will use the L^p boundedness of $\beta(|P|>M)$ throughout the following context. Let

$$E_{n,M} := \max\Big(\|\beta(|P|>M)\|_{L^p_x(\mathbb{R}^n)\to L^p_x(\mathbb{R}^n)}, \|\beta(|P|\le M)\|_{L^p_x(\mathbb{R}^n)\to L^p_x(\mathbb{R}^n)}\Big) \tag{6.160}$$

in dimension n.

Lemma 6.3.3. *If $T(\eta): L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, is bounded with*

$$A := \sup_{\eta\in\mathbb{R}^n}\|T(\eta)\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}, \tag{6.161}$$

then for $f(\xi) \in L^1_\xi(\mathbb{R}^n)$, we have

$$\left\|\int d^n\xi_1\cdots d^n\xi_n f(\xi_1)f(\xi_2-\xi_1)\cdots f(\xi_k-\xi_{k-1})T(\xi_k)\right\|_{L^p\to L^p} \le A\|f(\xi)\|^k_{L^1_\xi}. \tag{6.162}$$

Proof. It follows from

$$
\begin{aligned}
&\left\| \int d^n\xi_1 \cdots d^n\xi_n f(\xi_1) f(\xi_2 - \xi_1) \cdots f(\xi_k - \xi_{k-1}) T(\xi_k) \right\|_{L^p \to L^p} \\
&\leq \int d^n\xi_1 \cdots d^n\xi_n |f(\xi_1) f(\xi_2 - \xi_1) \cdots f(\xi_k - \xi_{k-1})| \sup_{\eta \in \mathbb{R}^n} \|T(\eta)\|_{L^p \to L^p} \\
&= A \|f(\xi)\|^k_{L^1_\xi}.
\end{aligned}
$$

□

6.3.2 L^p boundedness for $I^{(\,)}$

Let

$$I^{(\,)}\psi(x) = \sup_{\epsilon \geq 0} |I_\epsilon \psi(x)|, \text{ for } \psi \in L^p. \tag{6.163}$$

Theorem 6.3.1. *If $K_1(V(x), 0) < \infty$, then for $1 \leq p \leq \infty$, $\psi \in L^p$,*

$$\|I^{(\,)}\psi(x)\|_{L^p_x} \lesssim K_1(V(x), 0) \|\psi(x)\|_{L^p_x}. \tag{6.164}$$

Proof. According to equation (6.130),

$$I_\epsilon \psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{S^2} d\sigma(\xi) \int_0^\infty du \int_0^\infty d|\xi| |\xi| \hat{V}(\xi) e^{-\frac{\epsilon u}{|\xi|} + i(x \cdot \xi + u|\xi|)} \psi(x + 2u\hat{\xi}). \tag{6.165}$$

Then

$$I^{(\,)}\psi(x) \leq \int_{S^2} d\sigma(\xi) \int_{-\infty}^{\infty} du \left(\sup_{\epsilon \geq 0} |L_{0,1,0}(x \cdot \hat{\xi} + u, \hat{\xi}, \epsilon)| \right) |\psi(x + 2u\hat{\xi})| \tag{6.166}$$

where we use

$$L_{0,1,0}(k, \hat{\xi}, u\epsilon) = \mathcal{F}_{|\xi|}(\chi(|\xi| \geq 0)|\xi| \hat{V}(\xi) e^{-\frac{u\epsilon}{|\xi|}}) \tag{6.167}$$

and

$$\sup_{\epsilon \geq 0} |L_{0,1,0}(k, \hat{\xi}, u\epsilon)| = \sup_{\epsilon \geq 0} |L_{0,1,0}(k, \hat{\xi}, \epsilon)|, \text{ for } u > 0. \tag{6.168}$$

Due to Lemma 6.3.1, we have

$$\|I^{(\,)}\psi(x)\|_{L^p_x} \lesssim K_1(V(x), 0) \|\psi(x)\|_{L^p_x}. \tag{6.169}$$

□

Recall that

$$\mathcal{K}_t(V(x,s)) = e^{itH_0}V(Q,s)e^{-itH_0}. \tag{6.170}$$

To Proceed, we need more general operators

$$T_\epsilon(\eta) := \int_0^\infty dt e^{-\epsilon t}\mathcal{K}_t(V(x)e^{i\eta\cdot x}), \tag{6.171}$$

and

$$\partial^l_{\eta\cdot e_j}[T_\epsilon(\eta)] := \int_0^\infty dt e^{-\epsilon t}\mathcal{K}_t((ix\cdot e_j)^l V(x)e^{i\eta\cdot x}), \text{ for } \epsilon \geq 0. \tag{6.172}$$

The corresponding maximal T transform is

$$[T_{j,l}(\eta)]^{(*)}\psi(x) = \sup_{\epsilon\geq 0}|\partial^l_{\eta\cdot e_j}[T_\epsilon(\eta)]\psi(x)|. \tag{6.173}$$

Corollary 6.3.1. *If $V(x)$ satisfies condition (6.72), then*

$$\|[T_{j,l}(\eta)]^{(*)}\psi(x)\|_{L^p_x} \lesssim K_m\|\psi(x)\|_{L^p_x}, \quad j=1,2,3,\ l=0,1,2. \tag{6.174}$$

Proof. Repeating the proof of Theorem 6.3.1 by replacing $\hat{V}(\xi)$ with $\partial^l_{\eta\cdot e_j}[\hat{V}(\xi-\eta)]$, and taking the supremum over $\eta \in \mathbb{R}^3$, we will get the same an upper bound, with K_m instead of K_1. □

6.3.3 L^p boundedness of $I_M^{(*,k)}$

Let

$$I_M^{(*,k)}\psi(x) := \sup_{\epsilon\geq 0}|I_\epsilon^{(k)}\beta(|P|>M)\psi(x)|, \text{ for } \psi \in L^p_x. \tag{6.175}$$

Before controlling the L^p_x norm of $I_M^{(*,k)}\psi(x)$, we introduce the following expression:

Lemma 6.3.4 (Representation formula 1)**.** *For $\xi_i \in \mathbb{R}^n$, $i=1,\cdots,k$ ($k\in\mathbb{N}^+$), $\psi(x)\in L^p_x(\mathbb{R}^n)$, we have*

$$\mathcal{K}_{t_k}(e^{i(\xi_k-\xi_{k-1})\cdot x})\cdots\mathcal{K}_{t_1}(e^{i(\xi_1-\xi_0)\cdot x})\psi = \left[e^{i(Q\cdot\xi_k)}e^{it_k(\xi_k^2+2\xi_k\cdot P)}\Pi_{j=1}^{k-1}e^{i(t_j-t_{j+1})(\xi_j^2+2\xi_j\cdot P)}\right]\psi \tag{6.176}$$

with $\xi_0 = 0 \in \mathbb{R}^n$.

Proof. We prove by induction. When $k = 1$,it follows from equations (6.89) and (6.91). Assume that when $k = m$, the representation formula holds. When $k = m + 1$,

$$
\begin{aligned}
&K_{t_{m+1}}(e^{i(\xi_{m+1}-\xi_m)\cdot x}) \cdots K_{t_1}(e^{i(\xi_1-\xi_0)\cdot x})\psi \\
=&K_{t_{m+1}}(e^{i(\xi_{m+1}-\xi_m)\cdot x}) \left[e^{i(Q\cdot\xi_m)}e^{it_m(\xi_m^2+2\xi_m\cdot P)}\Pi_{j=1}^{m-1}e^{i(t_j-t_{j+1})(\xi_j^2+2\xi_j\cdot P)}\right]\psi \\
=&e^{iQ\cdot(\xi_{m+1}-\xi_m)}e^{it_{m+1}[(\xi_{m+1}-\xi_m)^2+2(\xi_{m+1}-\xi_m)\cdot P]}\left[e^{i(Q\cdot\xi_m)}e^{it_m(\xi_m^2+2\xi_m\cdot P)}\Pi_{j=1}^{m-1}e^{i(t_j-t_{j+1})(\xi_j^2+2\xi_j\cdot P)}\right]\psi \\
=&\left[e^{i(Q\cdot\xi_{m+1})}e^{it_{m+1}(\xi_{m+1}^2+2\xi_{m+1}\cdot P)}\Pi_{j=1}^{m}e^{i(t_j-t_{j+1})(\xi_j^2+2\xi_j\cdot P)}\right]\psi.
\end{aligned}
$$

By induction, we finish the proof. □

Choose $V(x) \in S_x$. For $\xi = (\xi_1, \cdots, \xi_k) \in \mathbb{R}^{3k}$, let

$$V(\xi, k) := \frac{1}{(2\pi)^{\frac{3k}{2}}}\hat{V}(\xi_1)\hat{V}(\xi_2-\xi_1)\cdots\hat{V}(\xi_k-\xi_{k-1}). \tag{6.177}$$

Writing $K_{t_j}(V(x))$ as

$$K_{t_j}(V(x)) = \frac{1}{(2\pi)^{3/2}}\int d^3\xi_j\hat{V}(\xi_j-\xi_{j-1})K_{t_j}(e^{ix\cdot(\xi_j-\xi_{j-1})}),\ \text{for } j = 1, \cdots, k,$$

and applying Lemma 6.3.4, we have

$$
\begin{aligned}
I_\epsilon^{(k)}\psi(x) = &\int_0^\infty dt_k\int_{t_k}^\infty dt_{k-1}\cdots\int_{t_2}^\infty e^{-\epsilon t_1}dt_1\int d^3\xi_1\cdots d^3\xi_k V(\xi, k) \\
&\int d^3q e^{i(x\cdot(\xi_k+q)+t_k(\xi_k^2+2q\cdot\xi_k)+(t_{k-1}-t_k)(\xi_{k-1}^2+2q\cdot\xi_{k-1})+\cdots+(t_1-t_2)(\xi_1^2+2\xi_1\cdot q))}\frac{\hat{\psi}(q)}{(2\pi)^{\frac{3}{2}}}
\end{aligned}
$$

where $\frac{1}{(2\pi)^{\frac{3}{2}}}$ comes from the inverse Fourier transform in q. It is sufficient to work with $\psi \in \beta(|P| > 32M)S_x, V(x,t) \in S_tS_x$ to get a concise representation of $I_\epsilon^{(k)}\psi(x)$. This can then be extended to all of L_x^p and general V. For any $\epsilon > 0$,

$$\int_0^\infty dt_k\cdots\int_{t_2}^\infty dt_1\int d^3\xi_1\cdots d^3\xi_k d^3q e^{-\epsilon t_1}|V(\xi,k)||\hat{\psi}(q)| \lesssim \frac{1}{(2\pi)^{3k/2}\epsilon^k}\|\hat{V}(\xi)\|_{L_\xi^1}^k\|\hat{\psi}(q)\|_{L_q^1} < \infty. \tag{6.178}$$

Due to Fubini's theorem, we can change the order of the integral over ξ_j, t_j and q when needed. We change variables from t_k, to $t_k = s_k$, t_j, to $t_j = s_k + \cdots + s_j$, $j = 1, \cdots, k-1$ with Jacobian 1,

$$
\begin{aligned}
I_\epsilon^{(k)}\psi(x) = &\int_0^\infty e^{-\epsilon s_k}ds_k\int_0^\infty e^{-\epsilon s_{k-1}}ds_{k-1}\cdots\int_0^\infty e^{-\epsilon s_1}ds_1\int d^3\xi_1\cdots d^3\xi_k d^3q V(\xi, k) \\
&e^{i(x\cdot(\xi_k+q)+(s_k\xi_k^2+\cdots+s_1\xi_1^2)+2(s_k\xi_k+\cdots+s_1\xi_1)\cdot q)}\frac{\hat{\psi}(q)}{(2\pi)^{\frac{3}{2}}}.
\end{aligned}
$$

The L^p_x estimates of $I^{(k)}_\epsilon$ are based on bounding following operators

$$J_\epsilon(\xi) := \int_0^\infty ds e^{-\epsilon s + i(s|\xi|^2 + 2s\xi\cdot P)}, \quad J^{(k)}_\epsilon(\xi_1, \cdots, \xi_k) := \Pi^k_{j=1} J_\epsilon(\xi_j), \text{ for } \xi_j \in \mathbb{R}^3. \tag{6.179}$$

Recall the definition of the operator $T_\epsilon(\eta)$(see equation (6.171))and then rewrite $I^{(k+1)}_\epsilon$ as

$$I^{(k+1)}_\epsilon = \int d^3\xi_1 \cdots d^3\xi_k V(\xi, k) T_\epsilon(\xi_k) J^{(k)}_\epsilon(\xi), \text{ for } \xi = (\xi_1, \cdots, \xi_k). \tag{6.180}$$

We have the following representation and estimates for $\int d^3\xi f(\xi) J_\epsilon(\xi)$.

Lemma 6.3.5. *Assume $f(\xi) \in C^2_\xi(\mathbb{R}^3)$. For $1 \le p < \infty$, $\int d^3\xi f(\xi) J_\epsilon(\xi) : \beta(|P| > 32M) L^p_x \to L^p_x$ and $\int d^3\xi f(\xi) J_\epsilon(\xi) : \beta(|P| > 32M) C_0 \to C_0$; preserves the frequency and for ψ in the given space,*

$$\int d^3\xi f(\xi) J_\epsilon(\xi)\psi(x) = \int d^3\xi f(\xi) Q_0 \psi(x) + \sum_{j=1}^3 \sum_{l=0}^2 \int d^3\xi \partial^l_{\xi\cdot e_j}[f(\xi)] Q_{3(j-1)+l+1}\psi(x) \tag{6.181}$$

for some operator $Q_j = Q_j(\xi, \epsilon) : L^q_x \to L^q_x$, $1 \le q \le \infty$, with $\|Q_j(\xi,\epsilon)\beta(|P| > 32M)\|_{L^q_x \to L^q_x} \lesssim \frac{1}{M}$. Moreover, for $\psi \in L^p_x$,

$$\|Q^{(\cdot)}_l(\xi)\psi(x)\|_{L^p_x} := \|\sup_{\epsilon \ge 0} |Q_l(\xi,\epsilon)\psi(x)|\|_{L^p_x} \lesssim \frac{1}{M}\|\psi(x)\|_{L^p_x}. \tag{6.182}$$

Remark 36. *Here we regard $f(\xi)$ as a multiplier.*

Proof. Since J_ϵ is a composition of translation operators, $\int d^3\xi f(\xi) J_\epsilon(\xi)$ preserves the support of the frequency. Now we would like to get a detailed formula. We choose $\psi \in \beta(|P| > 32M) S_x$. According to the similar transformation used for $I^{(k)}_\epsilon\psi$, we have

$$\int d^3\xi f(\xi) J_\epsilon(\xi)\psi = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\xi d^3q \int_0^\infty ds f(\xi) e^{-\epsilon s + i(x\cdot q + s|\xi|^2 + 2sq\cdot\xi)} \hat{\psi}(q). \tag{6.183}$$

Recall that $\{e_1, e_2, e_3\}$ is a basis in $\mathbb{R}^3$. Let $\xi_j = \xi \cdot e_j$. We claim that for all $\xi \neq 0$,

$$\beta(|P| > 32M) = \left[\sum_{j=1}^3 \beta(|\xi_j + P_j| > 2M)\beta_j(\xi, P, 2M) + \beta(||\xi| + 2P\cdot\hat{\xi}| > 2M)\gamma(\xi, P, 2M)\right]$$

$$\beta(|P| > 32M) =: \beta_{1,1} + \beta_{1,2} + \beta_{1,3} + \beta_{1,4}$$

where

$$\beta_j(\xi, P, 2M) := \Pi^{j-1}_{l=1}\beta(|\xi_l + P_l| \le 2M), \text{ for } j = 1, 2, 3, \text{ with } \Pi^0_{l=1} = 1 \tag{6.184}$$

$$\gamma(\xi, P, 2M) := \Pi_{j=1}^{3} \beta(|\xi_j + P_j| \leq 2M). \tag{6.185}$$

We prove the claim first.

Proof of the claim

Proof. We have

$$1 = \sum_{j=1}^{3} \beta(|\xi_j + P_j| > 2M)\beta_j(\xi, P, 2M)$$
$$+ \beta(||\xi| + 2P \cdot \hat{\xi}| > 2M)\gamma(\xi, P, 2M) + \beta(||\xi| + 2P \cdot \hat{\xi}| \leq 2M)\gamma(\xi, P, 2M).$$

Then, in order to prove the claim(since for $q \in \mathbb{R}^3$, $\beta(|q| > 32M)$ implies $|q| > 16M$), it suffices to prove that

$$\{q : |q| > 16M, |q_j + \xi_j| \leq 2M, j = 1, \cdots, 3\} \bigcap \{||\xi| + 2q \cdot \hat{\xi}| \leq 2M\} = \emptyset. \tag{6.186}$$

Assume that $|\xi_j + q_j| \leq 2M$, $|q| > 16M$. Then

$$|\xi + q| \leq \sqrt{\sum_{j=1}^{3} (\xi_j + q_j)^2} \leq 2\sqrt{3}M < 4M < \frac{|q|}{4}, \tag{6.187}$$

which implies

$$|\xi| \geq |q| - |\xi + q| > \frac{3|q|}{4}, \text{ and } |\xi| \leq |q| + |\xi + q| < \frac{7|q|}{4}. \tag{6.188}$$

Then according to equation (6.187), (6.188),

$$|\xi^2 + 2\xi \cdot q| = |(\xi + q)^2 - q^2| \geq \frac{15q^2}{16} > \frac{15|\xi||q|}{28} > \frac{60|\xi|M}{7} > 2|\xi|M. \tag{6.189}$$

Hence,

$$||\xi| + 2q \cdot \hat{\xi}| > 2M \tag{6.190}$$

which proves identity (6.186). Then when multiplied by $\beta(|q| > 32M)$, the factor $\beta(||\xi| + 2q \cdot \hat{\xi}| \leq 2M)\gamma(\xi, q, 2M)$ drops and thus the claim follows. □

ψ can then be written as a sum of 4 parts:

$$\psi = \beta_{1,1}\psi + \beta_{1,2}\psi + \beta_{1,3}\psi + \beta_{1,4}\psi =: \psi_1 + \psi_2 + \psi_3 + \psi_4. \tag{6.191}$$

For ψ_j, $j = 1, 2, 3$,

$$\psi_j(x) = \beta(|\xi_j + P_j| > 2M)\beta_j(\xi, P, 2M)\psi(x) =: \beta(|\xi_j + P_j| > 2M)\psi_{j,1}(x). \tag{6.192}$$

Recalling the definition of $E_{n,M}$ in equation (6.160), we have

$$\|\psi_{j,1}(x)\|_{L^p_x} \lesssim E^{j-1}_{3,2M}\|\psi(x)\|_{L^p_x}, \text{ and } \|\psi_j(x)\|_{L^p_x} \lesssim E^{j}_{3,2M}\|\psi(x)\|_{L^p_x}. \tag{6.193}$$

Since $\beta(|\xi_j + P_j| > 2M)$ implies $|\xi_j + q_j| > M$(q denotes the argument of $\hat{\psi}$), for $s \geq \frac{1}{M}$ we do integration by parts in ξ_j twice, by setting

$$e^{is(\xi_j^2+2\xi_j q_j)} = \frac{1}{2is(\xi_j + q_j)}\partial_{\xi_j}[e^{is(\xi_j^2+2\xi_j q_j)}] \tag{6.194}$$

and have

$$\begin{aligned}&\int d^3\xi f(\xi)\beta_j(|\xi_j + q_j| > 2M)e^{is(\xi_j^2+2\xi_j q_j)}\\ =&\frac{1}{(2is)^2}\int d^3\xi \partial_{\xi_j}[\frac{1}{(\xi_j+q_j)} \partial_{\xi_j}[\frac{f(\xi)\beta(|\xi_j+q_j| > 2M)}{\xi_j+q_j}]]e^{is(\xi_j^2+2\xi_j q_j)}\end{aligned}$$

with

$$\partial_{\xi_j}[\frac{1}{(\xi_j+q_j)} \partial_{\xi_j}[\frac{f(\xi)\beta(|\xi_j+q_j| > 2M)}{\xi_j+q_j}]] \tag{6.195}$$

$$=\partial^2_{\xi_j}[f(\xi)]\frac{\beta(|\xi_j+q_j| > 2M)}{(\xi_j+q_j)^2} + f(\xi)\partial_{\xi_j}[\frac{1}{(\xi_j+q_j)} \partial_{\xi_j}[\frac{\beta(|\xi_j+q_j| > 2M)}{\xi_j+q_j}]]+ \tag{6.196}$$

$$\partial_{\xi_j}[f(\xi)]\left[\partial_{\xi_j}[\frac{\beta(|\xi_j+q_j| > 2M)}{(\xi_j+q_j)^2}] + \frac{1}{(\xi_j+q_j)}\partial_{\xi_j}[\frac{\beta(|\xi_j+q_j| > 2M)}{\xi_j+q_j}]\right] \tag{6.197}$$

$$=:\partial^2_{\xi_j}[f(\xi)]\mathcal{F}[J_2](\xi_j+q_j) + f(\xi)\mathcal{F}[J_0](\xi_j+q_j) + \partial_{\xi_j}[f(\xi_j)]\mathcal{F}[J_1](\xi_j+q_j). \tag{6.198}$$

Then take the integral over q and we have

$$\begin{aligned}\int d^3\xi f(\xi)\mathcal{J}_\epsilon(\xi)\psi_j(x) =& \sum_{l=0}^{2}\int d^3\xi\int_{\sqrt{\frac{1}{M}}}^{\infty}\frac{ds}{(2is)^2}\partial^l_{\xi_j}[f(\xi)]e^{-\epsilon s+is\xi^2}\int dk J_l(k)e^{-i\xi_j k}\psi_{j,1}(x+2s\xi - ke_j)\\ &+\int d^3\xi\int_0^{\sqrt{\frac{1}{M}}} ds f(\xi)e^{-\epsilon s+is\xi^2}\psi_j(x+2s\xi) =: \sum_{l=0}^{2}\int d^3\xi\partial^l_{\xi_j}[f(\xi)]Q_{3(j-1)+l+1}(\xi,\epsilon)\psi(x)\end{aligned}$$

where for $\psi \in L^q$, $1 \leq q \leq \infty$, $j = 1, 2, 3$, $l = 1, 2$,

$$Q_{3(j-1)+0+1}(\xi,\epsilon)\psi(x) := \int_0^{\frac{1}{M}} ds e^{-\epsilon s+is\xi^2}\psi_j(x+2s\xi)+ \tag{6.199}$$

$$\int_{\frac{1}{M}}^{\infty}\frac{ds}{(2is)^2}e^{-\epsilon s+is\xi^2}\int dk J_0(k)e^{-i\xi_j k}\psi_{j,1}(x+2s\xi - ke_j), \tag{6.200}$$

$$Q_{3(j-1)+l+1}(\xi,\epsilon)\psi(x) := \int_{\frac{1}{M}}^{\infty} \frac{ds}{(2is)^2} e^{-\epsilon s+is\xi^2} \int dk J_l(k) e^{-i\xi_j k} \psi_{j,1}(x+2s\xi-ke_j) \tag{6.201}$$

and for the definition of $\psi_j, \psi_{j,1}$, see equation (6.192).

For ψ_4,

$$\psi_4 = \beta(||\xi|+2q\cdot\hat{\xi}| > 2M)\gamma(\xi,q,2M)\psi =: \beta(||\xi|+2\hat{\xi}\cdot P| > 2M)\psi_{4,1}, \tag{6.202}$$

with $\|\psi_{4,1}(x)\|_{L^p_x} \leq E^3_{3,2M}\|\psi(x)\|_{L^p_x}$. For $\int d^3\xi f(\xi) J_\epsilon(\xi)\psi_4$, we take the integral over s:

$$\int_0^{\infty} dse^{-\epsilon s+is(\xi^2+2\xi\cdot q)}\beta(||\xi|+2\hat{\xi}\cdot q| > 2M) = \frac{\beta(||\xi|+2\hat{\xi}\cdot q| > 2M)}{|\xi|(-\frac{\epsilon}{|\xi|}+i(|\xi|+2\hat{\xi}\cdot q))} =: \frac{1}{|\xi|}\mathcal{F}[J_{4,\epsilon/|\xi|}](|\xi|+2q\cdot\hat{\xi}|). \tag{6.203}$$

Let

$$J_{4,\epsilon}(\lambda) := \mathcal{F}_k^{-1}[\frac{\beta(|k| > 2M)}{\epsilon+ik}](\lambda). \tag{6.204}$$

Then

$$\int_0^{\infty} dse^{-\epsilon s+is(\xi^2+2\xi\cdot q)}\beta(||\xi|+2\hat{\xi}\cdot q| > 2M) = \frac{\beta(||\xi|+2\hat{\xi}\cdot q| > 2M)}{|\xi|(-\frac{\epsilon}{|\xi|}+i(|\xi|+2\hat{\xi}\cdot q))} = \frac{1}{|\xi|}\mathcal{F}[J_{4,\epsilon/|\xi|}](|\xi|+2q\cdot\hat{\xi}|). \tag{6.205}$$

In this case, since $|\xi+q| \geq 2\sqrt{3}M$, $|\xi| \geq |q| - 2\sqrt{3}M > M > 1$. Then

$$\int d^3\xi f(\xi) J_\epsilon(\xi)\psi_4 = \int d^3\xi f(\xi) Q_0(\xi,\epsilon)\psi(x), \tag{6.206}$$

where

$$Q_0(\xi,\epsilon)\psi(x) := \frac{\beta(|\xi| > 1)}{2|\xi|}\int dk J_{4,\epsilon/|\xi|}(k/2)e^{-i|\xi|k/2}\psi_{4,1}(x-k\hat{\xi}). \tag{6.207}$$

Due to Lemma 6.3.2, we have

$$\|J_j(k)\|_{L^1_k(\mathbb{R})} \lesssim \frac{1}{M^2}, \text{ and } \|J_{4,\epsilon}(k)\|_{L^1_k(\mathbb{R})} \lesssim \frac{1}{M}, j = 0,1,2. \tag{6.208}$$

Hence, combining with 6.160 and equation (6.208), for $1 \leq q \leq \infty$,

$$\|Q_l(\xi,\epsilon)\|_{L^q_x \to L^q_x} \lesssim \frac{1}{M}, \text{ for } l = 0,1,2,\cdots,9. \tag{6.209}$$

According to the expression of $Q_l(\xi,\epsilon), l = 1,\cdots,9$ and Lemma 6.3.2,

$$\|Q_l^{(*)}(\xi)\psi(x)\|_{L^p_x} := \|\sup_{\epsilon\geq 0}|Q_l(\xi,\epsilon)\psi(x)|\|_{L^p_x} \lesssim \frac{1}{M}\|\psi(x)\|_{L^p_x} \tag{6.210}$$

and finish the proof. □

Next, we prove the L^p_x estimates for $I^{(k+1)}_\epsilon \beta(|P| > M)$. We will show that for some sufficiently large $M > 0$, $\|I^{(k+1)}_\epsilon \beta(|P| > M)\|_{L^p_x \to L^p_x} \lesssim \frac{C^k}{M^k}$ uniformly in $\epsilon \in [0, 1]$. Then according to the same process, L^p_x boundedness of $I^{(\,,k+1)}\beta(|P| > M)$ follows as a corollary. We consider s_l, ξ_l, with $l = 1, \cdots, k+1$. When $l = 1, \cdots, k$ and when we look at ξ_l, s_l, we have to deal with

$$\int d^3\xi_l \hat{V}(\xi_{l+1} - \xi_l)\hat{V}(\xi_l - \xi_{l-1}) J_\epsilon(\xi_l)\psi(x). \tag{6.211}$$

Applying Lemma 6.3.5 to (6.211), we obtain that the expression (6.211) is equal to

$$\sum_{j_l=0}^{9} \int d^3\xi_l Q_{j_l,1}(\xi_l)[\hat{V}(\xi_{l+1} - \xi_l)\hat{V}(\xi_l - \xi_{l-1})]Q_{j_l}(\xi_l, \epsilon)\psi(x) \tag{6.212}$$

where $Q_{0,1} := identity$ and for $j_l = 1, \cdots, 9$,

$$Q_{j_l,1}(\xi_l) := \partial^{m_l}_{\xi_l \cdot e_{r_l}}, \text{ with } m_l := [j_l - 1]_3, \quad r_l := \frac{j_l - 1 - m_l}{3} + 1. \tag{6.213}$$

Now we introduce further notation. For $j = (j_1, \cdots, j_k) \in \{0, \cdots, 9\}^k := \alpha^k, \xi = (\xi_1, \cdots, \xi_k) \in \mathbb{R}^{3k}, \epsilon > 0, k \in \mathbb{N}^+$, define

$$Q_j(\xi, \epsilon, k) := \Pi^k_{l=1} Q_{j_l}(\xi_l, \epsilon), \; Q_{j,1}(\xi, k) := \Pi^k_{l=1} Q_{j_l,1}(\xi_l). \tag{6.214}$$

Remark 37. *Here, since $Q_{j_l}(\xi_l, \epsilon)$ commutes with $Q_{j_{l'}}(\xi_{l'}, \epsilon)$ and $Q_{j_l,1}(\xi_l)$ commutes with $Q_{j_{l'},1}(\xi_{l'})$ for $l \neq l'$, there is no confusion about $\Pi^k_{l=1} Q_{j_l}(\xi_l, \epsilon)$ and $\Pi^k_{l=1} Q_{j_l,1}(\xi_l)$.*

Then for $\psi \in \beta(|P| > 32M)S_x$,

$$I^{(k+1)}_\epsilon \psi(x) = \sum_{j \in \alpha^k} \int d^3\xi_1 \cdots d^3\xi_k Q_{j,1}(\xi, k)[V_k(\xi) T_\epsilon(\xi_k)] Q_j(\xi, \epsilon, k)\psi(x) \tag{6.215}$$

where, recall that

$$\partial^l_{\xi_j}[T_\epsilon(\xi)] = \int_0^\infty dt e^{itH_0} V(x) \partial^l_{\xi_j}[e^{i\xi_j \cdot x}] e^{-itH_0}, \tag{6.216}$$

which is equivalent to the potential $(ix \cdot e_m)^l V(x) e^{i\xi_j \cdot x}$. Now let us look at the L^p_x estimates of $I^{(k)}_\epsilon$ on $\beta(|P| > 32)S_x$.

Lemma 6.3.6. *If $V(x)$ satisfies the assumptions of Theorem 6.1.8 and if*

$$|||V(x)|||_{in} := \max(\|\hat{V}_a(\xi)\|_{L^1_\xi}, K_m), \tag{6.217}$$

then for $\psi(x) \in \beta(|P| > 32M)L_x^p$, $k \geq 1$, $M > 1$, *there exists some constant* $C > 0$ *such that*

$$\|I_\epsilon^{(k+1)}\psi(x)\|_{L_x^p} \lesssim \frac{C^k |||V(x)|||_{in}^{k+1}}{M^k}\|\psi(x)\|_{L_x^p} \tag{6.218}$$

for $1 \leq p \leq \infty$, $\epsilon \in [0,1]$.

Proof. For $p \neq \infty$, choose $\psi \in \beta(|P| > 32M)S_x$. For $l = 0,1,2, j = 1,2,3$, due to Corollary 6.3.1 and Lemma 6.3.5,

$$\|\partial^l_{\xi_k \cdot e_j}[T_\epsilon(\xi_k)]Q_j(\xi,\epsilon,k)\psi(x)\|_{L_x^p} \lesssim K_m\|Q_j(\xi,\epsilon,k)\psi(x)\|_{L_x^p} \leq \frac{C^k K_m}{M^k}\|\psi(x)\|_{L_x^p}. \tag{6.219}$$

The expression

$$\int d^3\xi_1 \cdots d^3\xi_k Q_{j,1}(\xi)[V(\xi,k)T_\epsilon(\xi_k)] \tag{6.220}$$

is the sum of L many terms ($L \leq 4^k$) with each term having the form:

$$\frac{1}{(2\pi)^{\frac{3k}{2}}}\int d^3\xi_1 \cdots d^3\xi_k P^{l_1}_{\xi_1 \cdot e_{j_1}}[\hat{V}(\xi_1)] \cdots P^{l_k}_{\xi_k \cdot e_{j_k}}[\hat{V}(\xi_k - \xi_{k-1})]\partial^{l_{k+1}}_{\xi_k \cdot e_{j_{k+1}}}[T_\epsilon(\xi_k)], \tag{6.221}$$

for $j_m \in \{1,2,3\}$, $l_m \in \{0,1,2,3,4\}$, $m = 1,\cdots,k$, $l_{k+1} \in \{0,1,2\}$, $j_{k+1} \in \{1,2,3\}$. According to equation (6.219) and Lemma 6.3.3,

$$\left\|\int d^3\xi_1 \cdots d^3\xi_k Q_{j,1}(\xi,k)[V_k(\xi)T_\epsilon(\xi_k)]Q_j(\xi,\epsilon,k)\psi(x)\right\|_{L_x^p} \lesssim \frac{C_2^k |||V(x)|||_{in}^k K_m}{M^k}\|\psi(x)\|_{L_x^p}. \tag{6.222}$$

Then according to equation (6.222) and (6.215),

$$\|I_\epsilon^{(k+1)}\psi(x)\|_{L^p} \lesssim \sum_{j \in \alpha^k}\frac{C_2^k |||V(x)|||_{in}^k K_m}{M^k}\|\psi(x)\|_{L_x^p} \lesssim \frac{C^k |||V(x)|||_{in}^{k+1}}{M^k}\|\psi(x)\|_{L_x^p} \tag{6.223}$$

for some constant $C > 0$. Then by B.L.T. theorem [66], the conclusion follows for $1 \leq p < \infty$. For $p = \infty$, we work on $\beta(|P| > 32M)C_0$ first. Then by using duality twice, we get the estimates for $p = \infty$. □

Corollary 6.3.2. *If* $V(x)$ *satisfies the assumptions in Theorem 6.1.8 and*

$$|||V(x)|||_{in} := \max(\|\hat{V}_a(\xi)\|_{L_\xi^1}, K_m), \tag{6.224}$$

then for $\psi(x) \in \beta(|P| > 32M)L^p_x$, $k \geq 1$, $M > 1$, *there exists some constant* $C > 0$ *such that*

$$\|I^{(\epsilon,k+1)}\psi(x)\|_{L^p_x} \lesssim \frac{C^k |||V(x)|||_{in}^{k+1}}{M^k}\|\psi(x)\|_{L^p_x} \tag{6.225}$$

for $1 \leq p \leq \infty$.

Proof. It follows from the same proof of Lemma 6.3.6 by replacing $I^{(k+1)}_\epsilon\psi(x)$ with $I^{(\epsilon,k+1)}\psi(x)$. □

Now we prove Theorem 6.1.8.

Proof. According to Lemma 6.3.6, we have that for $M > C|||V(x)|||_{in}$ and for $\psi \in S$, $1 \leq p \leq \infty$, any $\epsilon \in [0,1]$,

$$\|\Omega_\epsilon\beta(|P| > 32M)\psi(x)\|_{L^p_x} \lesssim \left(1 + \frac{|||V(x)|||_{in}}{1 - \frac{C|||V(x)|||_{in}}{\sqrt{M}}}\right) E_3\|\psi(x)\|_{L^p_x} \tag{6.226}$$

and

$$\|\Omega^{(\epsilon)}\beta(|P| > 32M)\psi(x)\|_{L^p_x} \lesssim \left(1 + \frac{|||V(x)|||_{in}}{1 - \frac{C|||V(x)|||_{in}}{\sqrt{M}}}\right) E_3\|\psi(x)\|_{L^p_x} \tag{6.227}$$

which completes the proof of $\Omega_\epsilon\beta(|P| > 32M) \to \Omega_0\beta(|P| > 32M) = \Omega\beta(|P| > 32M)$ in strong L^p-sense, provided that the almost everywhere convergence of $\Omega_\epsilon\beta(|H_0| > M)$ to $\Omega\beta(|H_0| > M)$. It is a consequence of the L^p boundedness of $\Omega^{(\epsilon)}\beta(|H_0| > M)$ and of Theorem 2.1.14 in [34]. By duality, we get the same result for $\beta(|P| > 32M)\Omega^*$ and we finish the proof. □

Remark 38. *From the proof, based on the definition of* Ω_ϵ, *we see that the result follows from the fact that* $\Omega_\epsilon : L^p \to L^p$, *is bounded uniformly in* $\epsilon \in [0,1]$.

Corollary 6.3.3. *If* $V(x)$ *satisfies the condition in Theorem 6.1.8, asymptotic completeness holds for the Schrödinger equation on high frequency subspace.*

Corollary 6.3.4. *If* $V(x)$ *satisfies the assumptions in Theorem 6.1.8 and*

$$|||V(x)|||_{in} := \max(\|\hat{V}_a(\xi)\|_{L^1_\xi}, K_m), \tag{6.228}$$

then for $\psi(x) \in L^p_x$, $k \geq 1$, $M > 1$, *there exists some constant* $C > 0$ *such that*

$$\|\beta(|P| > 32M)\left(I^{(\cdot,k+1)}\right)^* \psi(x)\|_{L^p_x} \lesssim \frac{C^k \||V(x)|\|^{k+1}_{in}}{M^k} \|\psi(x)\|_{L^p_x} \tag{6.229}$$

for $1 \leq p \leq \infty$, *where*

$$\left(I^{(\cdot,k+1)}\right)^* := \max_{\epsilon>0} \left(I^{(k+1)}_\epsilon\right)^* . \tag{6.230}$$

Proof. According to Lemma 6.3.6, for $1 \leq p < \infty$, by duality, the conclusion follows. When $p = \infty$, choose $\phi \in L^1_x, \psi \in L^\infty_x$

$$\left|\langle\phi(x), \beta(|P| > 32M)\left(I^{(\cdot,k+1)}\right)^* \psi(x)\rangle_{L^2_x}\right| \tag{6.231}$$

$$= \left|\langle I^{(\cdot,k+1)}\beta(|P| > 32M)\phi(x), \psi(x)\rangle_{L^2_x}\right| \tag{6.232}$$

$$\leq \|I^{(\cdot,k+1)}\beta(|P| > 32M)\|_{L^1_x \to L^1_x} \|\phi(x)\|_{L^1_x} \|\psi(x)\|_{L^\infty_x} . \tag{6.233}$$

So we conclude that for $\psi \in L^\infty_x$,

$$\|\beta(|P| > 32M)\left(I^{(\cdot,k+1)}\right)^* \psi(x)\|_{L^\infty_x} \lesssim \frac{C^k \||V(x)|\|^{k+1}_{in}}{M^k} \|\psi(x)\|_{L^\infty_x} . \tag{6.234}$$

□

Corollary 6.3.5. *If* $V(x)$ *satisfies the assumptions of Theorem 6.1.8, there exists* $M = M(V(x)) > 0$*, such that*

$$\sup_{T \in \mathbb{R}} \|U(0,T)e^{-iTH_0}\beta(|P| > M)\|_{L^p_x \to L^p_x} < C. \tag{6.235}$$

Proof. Due to Theorem 6.1.1, there exists $M > 0$ such that

$$\Omega\beta(|P| > M) = s\text{-}\lim_{\epsilon \downarrow 0} \Omega_\epsilon \beta(|P| > M). \tag{6.236}$$

Then

$$\lim_{\epsilon \downarrow 0}(f, \int_0^\infty dt e^{-\epsilon t}\Omega'(t)\beta(|P| > M)g)_{L^2_x} = (f, \lim_{\epsilon \downarrow 0} \int_0^\infty dt e^{-\epsilon t}\Omega'(t)\beta(|P| > M)g)_{L^2_x}. \tag{6.237}$$

Let

$$a(T,f,g) := (f, U(0,T)e^{-iTH_0}\beta(|P| > M)g)_{L^2_x}. \tag{6.238}$$

Then for each $f \in L^p$, $g \in L^q$, $a(T,f,g)$ is continuous in T since for $t_1, t_2 \in \mathbb{R}$,

$$\|\int_{t_1}^{t_2} dt\Omega'(t)\beta(|P| > M)\|_{L^p_x \to L^p_x} < \infty \tag{6.239}$$

and goes to 0 as $t_1 \to t_2$. Due to Theorem 6.1.1, we have $\lim_{T\to\infty} a(T,f,g)$ exists for each pair f, g. Combining with the continuity, for each $g \in L^q$,

$$\sup_{T\in\mathbb{R}^+} |a(T,f,g)| < C(f,g). \tag{6.240}$$

By Principle of uniform boundedness,

$$\sup_{\|g\|_{L^q}\le 1} \sup_{T\in\mathbb{R}^+} |a(T,f,g)| < C(f), \tag{6.241}$$

that is,

$$\sup_{T\in\mathbb{R}^+} \|U(0,T)e^{-iTH_0}\beta(|P|>M)f\|_{L^p_x} < C(f). \tag{6.242}$$

Then by Principle of uniform boundedness again and duality,

$$\sup_{T\in\mathbb{R}^+} \|U(0,T)e^{-iTH_0}\beta(|P|>M)\|_{L^p_x\to L^p_x} < C. \tag{6.243}$$

Similarly, we have

$$\sup_{T\in\mathbb{R}^-} \|U(0,T)e^{-iTH_0}\beta(|P|>M)\|_{L^p_x\to L^p_x} < C. \tag{6.244}$$

Thus,

$$\sup_{T\in\mathbb{R}} \|U(0,T)e^{-iTH_0}\beta(|P|>M)\|_{L^p_x\to L^p_x} < C. \tag{6.245}$$

□

Corollary 6.3.6. *If $V(x)$ satisfies the assumptions of Theorem 6.1.8 and $V(x)$ is sufficiently small, then*

$$\Omega = s\text{-}\lim_{\epsilon\downarrow 0}\Omega_\epsilon, \text{ in } L^p_x, 1\le p\le\infty \tag{6.246}$$

and $\Omega_\epsilon, \Omega : L^p_x \to L^p_x$ are bounded.

Proof. In $I_\epsilon^{(k)}$, for s_j, ξ_j, we have to deal with

$$\int d^3\xi_j \int_0^\infty ds_j \hat{V}(\xi_{j+1}-\xi_j)\hat{V}(\xi_j-\xi_{j-1})e^{-s_j\epsilon+is_j(\xi_j^2+2\xi_j\cdot P)}. \tag{6.247}$$

We do change of variables $s_j \to u_j = s_j|\xi_j|$, $j = 1, \cdots, k-1$. For $u_j \le 1$, we leave as is. For $u_j > 1$, we do integration by parts in $|\xi_j|$ twice by setting

$$e^{iu_j|\xi_j|} = \frac{1}{iu_j}\partial_{|\xi_j|}[e^{iu_j|\xi_j|}].$$

For $j = k$, we apply Corollary 6.3.1 and for $I_\epsilon^{(k)}$,

$$\|I_\epsilon^{(k)}\|_{L^p_x \to L^p_x} \le C^k |||V(x)|||_{in}^k, \text{ for some } C, \text{ independent on } V(x). \tag{6.248}$$

and

$$\|I^{(\ ,k)}\|_{L^p_x \to L^p_x} \le C^k |||V(x)|||_{in}^k. \tag{6.249}$$

Then if $|||V(x)|||_{in}$ is sufficiently small, the conclusion follows. □

6.4 L^p boundedness of wave operator for some time-dependent potentials

In this section, we begin the analysis of time-dependent potentials. We will show the L^p boundedness of the wave operator on the high frequency subspace for a class of Mikhlin-type potentials $V(x,t)$ satisfying

$$\sup_{t\in\mathbb{R}} \frac{(1+|t|)^a}{a!} \sum_{l,j=0}^{2} \sum_{m,r=1}^{3} \left|\frac{\partial^a}{\partial t^a}\left[\partial^l_{\xi\cdot e_r}\partial^j_{\xi\cdot e_m}\hat{V}(\xi,t)\right]\right| \le c^a \hat{V}_0(\xi), \text{ for all } a \in \mathbb{N}, \text{ some } c \ge 1 \tag{6.250}$$

with $\hat{V}_0(\xi) \in L^1_\xi(\mathbb{R}^3) \cap L^\infty_\xi(\mathbb{R}^3)$.

6.4.1 L^p boundedness for $I - K$

We show the L^p boundedness of $I - K$ when $V(x,t)$ satisfies

$$|||V(x,t)|||_{W1} := \|\sup_{t\in\mathbb{R}} 4\pi \sum_{l,j=0}^{2}\sum_{m=1}^{3} (|t|+1)^l |\partial^j_{\xi\cdot e_m}\partial^l_t \hat{V}(\xi,t)|\|_{L^1_\xi \cap L^\infty_\xi} < \infty. \tag{6.251}$$

Lemma 6.4.1. *If $V(x,t)$ satisfies assumption (6.250), then*

$$|||(x\cdot e_j)^l V(x,t) e^{i\eta\cdot x}|||_{W1} < 4\pi(1+c+2c^2)\|\hat{V}_0(\xi)\|_{L^1_\xi \cap L^\infty_\xi} \tag{6.252}$$

for any $\eta \in \mathbb{R}^3$, $j = 1,2,3$, $l = 0,1,2$.

Proof. Due to assumption (6.250) and the definition of $|||\cdot|||_{W1}$,

$$|||(x\cdot e_j)^l V(x,t)e^{i\eta\cdot x}|||_{W1} \le 4\pi\|(0!c^0+1!c+2!c^2)\hat{V}_0(\xi-\eta)\|_{L^1_\xi\cap L^\infty_\xi} = 4\pi(1+c+2c^2)\|\hat{V}_0(\xi)\|_{L^1_\xi\cap L^\infty_\xi}. \tag{6.253}$$

□

Theorem 6.4.1. *If $V(x,t)$ satisfies assumption (6.251), then $I_\epsilon : L^p_x \to L^p_x$ is uniformly bounded in $\epsilon \in [0,1]$ for $1 \le p \le \infty$.*

Proof. By the same transformation as in equation (6.130), we get

$$I_\epsilon\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{S^2} d\sigma(\xi)\int_0^\infty du \int_0^\infty d|\xi||\xi|\hat{V}(\xi,\frac{u}{|\xi|})e^{-\epsilon\frac{u}{|\xi|}+i(x\cdot\xi+u|\xi|)}\psi(x+2u\hat{\xi}). \tag{6.254}$$

Rewrite $I_\epsilon\psi(x)$ as

$$\begin{aligned} I_\epsilon\psi(x) =& \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\xi\int_0^\infty \chi(|x\cdot\hat{\xi}+u|\le 1)du\frac{\hat{V}(\xi,\frac{u}{|\xi|})}{|\xi|}e^{-\epsilon\frac{u}{|\xi|}+i(x\cdot\xi+t\xi^2)}\psi(x+2u\hat{\xi})+\\ &\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{S^2} d\sigma(\xi)\int_0^\infty du\int_0^\infty d|\xi|\chi(|x\cdot\hat{\xi}+u|>1)|\xi|\hat{V}(\xi,\frac{u}{|\xi|})e^{-\epsilon\frac{u}{|\xi|}+i(x\cdot\xi+u|\xi|)}\psi(x+2u\hat{\xi})\\ :=& I_{1\epsilon}\psi(x)+I_{2\epsilon}\psi(x). \end{aligned}$$

For $I_{1\epsilon}\psi(x)$, due to Lemma 6.3.1 and for any $\hat{\xi}$ direction($\chi(|x\cdot\hat{\xi}+u|\le 1)f(\frac{u}{|\xi|})\in L^1_u$),

$$\|I_{1\epsilon}\psi(x)\|_{L^p_x} \lesssim \|\sup_{u\in\mathbb{R}}\frac{|\hat{V}(\xi,\frac{u}{|\xi|})|}{|\xi|}\|_{L^1_\xi}\|\psi(x)\|_{L^p_x} \lesssim |||V(x,t)|||_{W1}\|\psi(x)\|_{L^p_x} \tag{6.255}$$

where we use the inequality

$$\begin{aligned} &\|\frac{\sup_{u\in\mathbb{R}}|\hat{V}(\xi,\frac{u}{|\xi|})|}{|\xi|}\|_{L^1_\xi} = \|\frac{\chi(|\xi|\ge 1)\sup_{u\in\mathbb{R}}|\hat{V}(\xi,\frac{u}{|\xi|})|}{|\xi|}\|_{L^1_\xi}+\|\frac{\chi(|\xi|<1)\sup_{u\in\mathbb{R}}|\hat{V}(\xi,\frac{u}{|\xi|})|}{|\xi|}\|_{L^1_\xi}\\ \le&\|\sup_{u\in\mathbb{R}}|\hat{V}(\xi,\frac{u}{|\xi|})|\|_{L^1_\xi}+\int_{S^2}d\sigma(\xi)\int_0^1(d|\xi|)|\xi|\|\sup_{u\in\mathbb{R}}|\hat{V}(\xi,\frac{u}{|\xi|})|\|_{L^\infty_\xi} \lesssim |||V(x,t)|||_{W1}. \end{aligned}$$

For $I_{2\epsilon}\psi(x)$, we do integration by parts in $|\xi|$ in the same way as the time-independent case and we have,

$$\begin{aligned} I_{2\epsilon}\psi(x) =& \frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^\infty du\int_{S^2}d\sigma(\xi)\frac{\chi(|x\cdot\hat{\xi}+u|>1)}{i(x\cdot\hat{\xi}+u)}\psi(x+2u\hat{\xi})\\ &\left[\int_0^{\frac{1}{\sqrt{|x\cdot\hat{\xi}+u|}}}d|\xi|\partial_{|\xi|}[|\xi|\hat{V}(\xi,\frac{u}{|\xi|})e^{-\epsilon\frac{u}{|\xi|}}]e^{i(x\cdot\xi+u|\xi|)}+\int_{\frac{1}{\sqrt{|x\cdot\hat{\xi}+u|}}}^\infty d|\xi|\partial_{|\xi|}[|\xi|\hat{V}(\xi,\frac{u}{|\xi|})e^{-\epsilon\frac{u}{|\xi|}}]e^{i(x\cdot\xi+u|\xi|)}\right]\\ :=& I_{21\epsilon}\psi(x)+I_{22\epsilon}\psi(x) \end{aligned}$$

where we drop the boundary terms, near infinity and near 0 due to our assumptions:

$$\|\sup_{t\in\mathbb{R}}|\hat{V}(\xi,t)|\|_{L^\infty_\xi} \Longrightarrow |\xi|\hat{V}(\xi,\frac{u}{|\xi|})|_{|\xi|=0}=0 \tag{6.256}$$

and using the definition of $|||\cdot|||_{W^1}$,

$$\hat{V}_0(\xi) := \|\sup_{t\in\mathbb{R}}|\hat{V}(\xi,t)|\|_{L^1_\xi} \Longrightarrow \exists r_n (r_n \to \infty \text{ as } n \to \infty) s.t.$$

$$r_n|\hat{V}(r_n\hat{\xi},\frac{u}{r_n})| \leq r_n\hat{V}_0(r_n\hat{\xi}) \to 0, \text{ as } n \to \infty.$$

For $I_{21\epsilon}\psi(x)$, we have

$$\|I_{21\epsilon}\psi(x)\|_{L^p_x} \leq \left\|\int_0^\infty \frac{du}{(2\pi)^{3/2}}\int_{S^2} d\sigma(\xi)\frac{\chi(|x\cdot\hat{\xi}+u|>1)}{|x\cdot\hat{\xi}+u|^{\frac{3}{2}}}\|\partial_{|\xi|}[|\xi|\hat{V}(\xi,\frac{u}{|\xi|})e^{-\epsilon\frac{u}{|\xi|}}]\|_{L^1_{|\xi|}[0,1]}|\psi(x+2u\hat{\xi})|\right\|_{L^p_x}$$

$$\text{(Lemma 6.3.1)} \lesssim \int_{S^2} d\sigma(\xi)\|\sup_{u\in\mathbb{R}^+}|\partial_{|\xi|}[|\xi|\hat{V}(\xi,\frac{u}{|\xi|})e^{-\epsilon\frac{u}{|\xi|}}]|\|_{L^1_{|\xi|}[0,1]}\|\psi(x)\|_{L^p_x}$$

$$\lesssim |||V(x,t)|||_{W^1}\|\psi(x)\|_{L^p_x}$$

where from the second line to the third line, we used $(\partial_1[\hat{V}(\xi,t)] := \partial_{|\xi|}[\hat{V}(\xi,t)]$, $\partial_2[\hat{V}(\xi,t)] := \partial_t[\hat{V}(\xi,t)]$.)

$$\|\sup_{u\in\mathbb{R}^+}|\partial_{|\xi|}[|\xi|\hat{V}(\xi,\frac{u}{|\xi|})e^{-\epsilon\frac{u}{|\xi|}}]|\|_{L^1_{|\xi|}[0,1]}$$

$$\leq \|\sup_{u\in\mathbb{R}^+}\left[|(1+\frac{\epsilon u}{|\xi|})|\hat{V}(\xi,\frac{u}{|\xi|})|e^{-\epsilon\frac{u}{|\xi|}} + |\xi||\partial_1[\hat{V}(\xi,\frac{u}{|\xi|})]| + \frac{|u|}{|\xi|}|\partial_2[\hat{V}(\xi,\frac{u}{|\xi|})]|\right]\|_{L^1_{|\xi|}[0,1]}$$

$$\lesssim |||V(x,t)|||_{W^1}.$$

For $I_{22\epsilon}\psi(x)$, we do integration by parts in $|\xi|$ in the same way again, and have

$$I_{22\epsilon}\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^\infty du\int_{S^2} d\sigma(\xi)\frac{\chi(|x\cdot\hat{\xi}+u|>1)}{(i(x\cdot\hat{\xi}+u))^2}\psi(x+2u\hat{\xi})$$

$$\left[\partial_{|\xi|}[|\xi|\hat{V}(\xi,\frac{u}{|\xi|})e^{-\epsilon\frac{u}{|\xi|}}]e^{i(x\cdot\xi+u|\xi|)}\Big|_{|\xi|=\frac{1}{\sqrt{|x\cdot\hat{\xi}+u|}}}^{|\xi|=\infty} - \int_{\frac{1}{\sqrt{|x\cdot\hat{\xi}+u|}}}^\infty d|\xi|\partial^2_{|\xi|}[|\xi|\hat{V}(\xi,\frac{u}{|\xi|})e^{-\epsilon\frac{u}{|\xi|}}]e^{i(x\cdot\xi+u|\xi|)}\right].$$

Then similarly, take absolute value of the integrand, use Lemma 6.3.1 and estimate the L^p_x norm of $I_{22\epsilon}\psi(x)$

$$\|I_{22\epsilon}\psi(x)\|_{L^p_x} \lesssim |||V(x,t)|||_{W^1}\|\psi(x)\|_{L^p_x}. \tag{6.257}$$

Then we have

$$\|I_{2\epsilon}\psi(x)\|_{L^p_x} \lesssim |||V(x,t)|||_{W^1}\|\psi(x)\|_{L^p_x}. \tag{6.258}$$

Hence, according to equation (6.255) and equation (6.258),

$$\|I_\epsilon\psi(x)\|_{L^p_x} \lesssim |||V(x,t)|||_{W^1}\|\psi(x)\|_{L^p_x}. \tag{6.259}$$

□

Corollary 6.4.1. *Let*

$$T_\epsilon^{[k]}(\eta)\psi(x) = \int_0^\infty dt e^{iH_0 t} f^{[k]}(t) e^{-\epsilon t}(x\cdot e_m)^l V(x,t) e^{ix\cdot\eta} e^{-iH_0 t}\psi(x), \tag{6.260}$$

for $\psi \in L^p, a_j \geq 0, \eta \in \mathbb{R}^3,\ k \in \mathbb{N}^+, l = 0,1,2, e_m \in S^2$, *with*

$$f^{[k]}(t) = \Pi_{j=1}^k f_j(a_j+t),\ a_j \geq 0, \quad \sup_{t\in\mathbb{R}^+} |t|^a |f_j^{(a)}(t)| \leq C_j,\ \text{for } a = 0,1,2, \text{ and for some } C_j > 1. \tag{6.261}$$

If $V(x,t)$ *satisfies condition* (6.250), *then* $T_\epsilon^{[k]} : L_x^p \to L_x^p$ *is uniformly bounded in* $\epsilon \in [0,1]$, *for* $1 \leq p \leq \infty$ *and*

$$\|T_\epsilon^{[k]}(\eta)\|_{L_x^p \to L_x^p} \lesssim c^2 k^2 (\Pi_{j=1}^k C_j)\|\hat{V}_0(\xi)\|_{L_\xi^1 \cap L_\xi^\infty}. \tag{6.262}$$

Proof. Replace $V(x,t)$ with $V(x,t)e^{i\eta\cdot x} f^{[k]}(t)$ in the proof of Theorem 6.4.1. Since for $t \geq 0$,

$$\left|t^j \frac{d^j[f_l(t+a_j)]}{dt}\right| \leq |(t+a_j)^j \frac{d^j[f_l(t+a_j)]}{dt^j}| \leq C_l, \text{ for } j = 0,1,2, \quad l = 1,\cdots,k, \tag{6.263}$$

based on Leibniz formula,

$$\left|t^j \frac{d^j[f^{[k]}(t)]}{dt}\right| \leq k^j \Pi_{l=1}^k C_l, \text{ for } j = 0,1,2, a \geq 0. \tag{6.264}$$

Then for $l = 0,1,2$,

$$4\pi \sum_{u=0}^{2}\sum_{r=1}^{3} (|t|+1)^u |\partial_{\xi\cdot e_r}^j \partial_t^u [f^{[k]}(t)\partial_{\xi\cdot e_m}^l[\hat{V}(\xi-\eta,t)]]| \tag{6.265}$$

$$\leq \sum_{u=0}^{2}\sum_{r=1}^{3}\sum_{l_1=0}^{u}\begin{pmatrix} u \\ l_1 \end{pmatrix} 4\pi(|t|+1)^{l_1}|\partial_{\xi\cdot e_r}^j \partial_t^{l_1}\partial_{\xi\cdot e_m}^l[\hat{V}(\xi-\eta,t)]| \leq k^2 \Pi_{l=1}^k C_l. \tag{6.266}$$

Hence, due to Lemma 6.4.1 and equation (6.266),

$$|||f^{[k]}(t)(x\cdot e_m)^l V(x,t)e^{ix\cdot\eta}|||_{W_1} \lesssim c^2 k^2 (\Pi_{l=1}^k C_l)\|\hat{V}_0(\xi)\|_{L_\xi^1 \cap L_\xi^\infty} < \infty. \tag{6.267}$$

Apply Theorem 6.4.1 and we finish the proof. □

6.4.2 L^p boundedness for $I_\epsilon^{(k)}$ on high frequency space

In this section, we use the following notation. Let

$$V(\xi,s,k) := \frac{1}{(2\pi)^{\frac{3k}{2}}}\Pi_{j=1}^k \hat{V}(\xi_j - \xi_{j-1}, \sum_{l=j}^k s_l) \tag{6.268}$$

for $\xi = (\xi_1, \cdots, \xi_k) \in \mathbb{R}^{3k}, \xi_0 = 0, s = (s_1, \cdots, s_k) \in \mathbb{R}^k$. For $\psi \in L^q_x$, $1 \leq q \leq \infty$, $j = 1,2,3$, $l = 1,2$, let

$$Q^1_{3(j-1)+l+1}(\xi,\epsilon,s)\psi(x) := \frac{\chi(s>\frac{1}{M})}{(2is)^2} e^{-\epsilon s + is\xi^2} \int dk J_l(k) e^{-i\xi_j k} \psi_{j,1}(x+2s\xi-ke_j), \tag{6.269}$$

$$Q^1_{3(j-1)+0+1}(\xi,\epsilon,s)\psi(x) := \chi(s \leq \frac{1}{M}) e^{-\epsilon s + is\xi^2} \psi_j(x+2s\xi) + \tag{6.270}$$

$$\frac{\chi(s>\frac{1}{M})}{(2is)^2} e^{-\epsilon s + is\xi^2} \int dk J_0(k) e^{-i\xi_j k} \psi_{j,1}(x+2s\xi-ke_j). \tag{6.271}$$

Here we recall the definition of $J_l, \psi_j, \psi_{j,1}$, see (6.198), (6.192), (6.191). Then

$$\int_0^\infty ds Q^1_{3(j-1)+0+1}(\xi,\epsilon,s)\psi(x) = Q_{3(j-1)+0+1}(\xi,\epsilon)\psi(x), \tag{6.272}$$

$$\int_0^\infty ds Q^1_{3(j-1)+l+1}(\xi,\epsilon,s)\psi(x) = Q_{3(j-1)+l+1}(\xi,\epsilon)\psi(x). \tag{6.273}$$

We immediately have the following lemma:

Lemma 6.4.2. *For* $j =, 1,2,3$, $l = 1,2$, $1 \leq p \leq \infty$,

$$\int_0^\infty ds \left\| Q^1_{3(j-1)+0+1}(\xi,\epsilon,s) \right\|_{L^p_x \to L^p_x} \lesssim \frac{1}{M}, \tag{6.274}$$

$$\int_0^\infty ds \left\| Q^1_{3(j-1)+l+1}(\xi,\epsilon,s) \right\|_{L^p_x \to L^p_x} \lesssim \frac{1}{M}. \tag{6.275}$$

Here for C_J*, see Lemma 6.3.5.*

Proof. This follows directly from the proof of Lemma 6.3.5. □

Now we can get the L^p_x estimates of $I^{(k)}_\epsilon$:

Lemma 6.4.3. *If* $V(x,t)$ *satisfies condition* (6.250)*, then for* $M \geq 1$*, when* $\psi \in \beta(|P| > 32M)S_x$*,* $\epsilon \geq 0$*,*

$$\|I^{(k)}_\epsilon \psi(x)\|_{L^p_x} \lesssim \frac{C^k c^{3k+2} k^3 \|\hat{V}_0(\xi)\|^k_{L^1_\xi \cap L^\infty_\xi}}{M^{k-1}} \|\psi(x)\|_{L^p_x}, \tag{6.276}$$

and

$$\|\beta(|P| > 32M)\left(I^{(k)}_\epsilon\right)^*\|_{L^p_x \to L^p_x} \lesssim \frac{C^k c^{3k+2} k^3 \|\hat{V}_0(\xi)\|^k_{L^1_\xi \cap L^\infty_\xi}}{M^{k-1}}, \tag{6.277}$$

for $1 \leq p \leq \infty$, $k \geq 2$.

Proof. According to the same transformation in t_j in section 2, we can rewrite $I_\epsilon^{(k)}\psi(x)$ as

$$I_\epsilon^{(k)}\psi(x) = \sum_{\gamma\in\{0,1\}^{k-1}} \int_0^{\infty} ds_k \cdots \int_0^{\infty} ds_1 \int d^3\xi_1 \cdots d^3\xi_k d^3q \beta^\gamma(\xi,q,k)V(\xi,s,k)$$

$$e^{-s_k\epsilon - \cdots - s_1\epsilon + i(x\cdot(\xi_k+q)+s_k(\xi_k^2+2q\cdot\xi_k)+\cdots+s_1(\xi_1^2+2\xi_1\cdot q))}\frac{\hat{\psi}(q)}{(2\pi)^{\frac{3}{2}}} =: \sum_{\gamma\in\{0,1\}^{k-1}} I_{\gamma\epsilon}^{(k)}\psi(x),$$

where

$$\beta^{\gamma_j}(|\xi_j+q|>2M) = \begin{cases} \beta(|\xi_j+q|>2M) & \text{if } \gamma_j = 0 \\ \beta(|\xi_j+q|\le 2M) & \text{if } \gamma_j = 1 \end{cases}, \quad \beta^\gamma(\xi,q,k) = \Pi_{j=1}^{k-1}\beta^{\gamma_j}(|\xi_j+q|>2M). \tag{6.278}$$

For $I_{\gamma\epsilon}^{(k)}\psi(x)$, if $\gamma_j = 0$ for all $j = 1,\cdots,k-1$, the transformation we take is the same as that in time-independent case. After such a transformation, we use Corollary 6.4.1 instead of Corollary 6.3.1 and get that in this case,

$$\|I_{\gamma\epsilon}^{(k)}\psi(x)\|_{L_x^p} \le \frac{c^2k^2C^k\|\hat{V}_0(\xi)\|^k_{L_\xi^1\cap L_\xi^\infty}}{M^{k-1}}\|\psi(x)\|_{L_x^p} \tag{6.279}$$

for some constant $C>0$. The rest of the task is to deal with $I_{\gamma\epsilon}^{(k)}\psi(x)$ when there exists some j such that $\gamma_j = 1$. In this case, let

$$\{j_1,\cdots,j_r\} := \{j : |\xi_j+q|\le 2M \text{ and } j\in\{1,\cdots,k-1\}\}, \text{ with } j_1<\cdots<j_r, \tag{6.280}$$

where r denotes the number of such j with $|\xi_j+s_j|\le 2M$, $j\le k-1$.

In the following, we will use some transformation to get a desired upper bound for such $I_{\gamma\epsilon}^{(k)}\psi(x)$. This transformation is slightly different from that in time-independent case.

Transformation :

We do the transformation for ξ_l, s_l, with $l\in\{j_1,\cdots,j_r\}$ first. Recall that when $|\xi_l+q|\le 2M$, $||\xi_l|+2q\cdot\hat{\xi}_l|>2M$. We begin with j_1. Look at the integral over s_{j_1}

$$\int_0^{\infty} ds_{j_1} e^{-\epsilon s_{j_1}+is_{j_1}(\xi_{j_1}^2+2\xi_{j_1}\cdot q)}V(\xi,s,k). \tag{6.281}$$

We do integration by parts in s_{j_1} variable by setting

$$e^{-\epsilon s_{j_1}+is_{j_1}(\xi_{j_1}^2+2\xi_{j_1}\cdot q)} = \frac{1}{-\epsilon+i(\xi_{j_1}^2+2\xi_{j_1}\cdot q)}\partial_{s_{j_1}}[e^{-\epsilon s_{j_1}+is_{j_1}(\xi_{j_1}^2+2\xi_{j_1}\cdot q)}] \tag{6.282}$$

and get two terms: boundary term

$$\frac{1}{\epsilon + i(\xi_{j_1}^2 + 2\xi_{j_1} \cdot q)} = \frac{1}{\epsilon + i(\xi_{j_1}^2 + 2\xi_{j_1} \cdot q)} \int_0^1 ds_{j_1} \delta(s_{j_1}) e^{-\epsilon s_{j_1} + i s_{j_1}(\xi_{j_1}^2 + 2\xi_{j_1} \cdot q)} V(\xi, s, k) \tag{6.283}$$

and integral term

$$\frac{1}{\epsilon + i(\xi_{j_1}^2 + 2\xi_{j_1} \cdot q)} \int_0^1 ds_{j_1} e^{-\epsilon s_{j_1} + i s_{j_1}(\xi_{j_1}^2 + 2\xi_{j_1} \cdot q)} \partial_{s_{j_1}} [V(\xi, s, k)]. \tag{6.284}$$

For the boundary term, if $r = 1$, we are done. Otherwise, we move to j_2 and do the same transformation in s_{j_2}. For the integral term, we keep taking integration by parts in s_{j_1} in the same way. We keep doing such transformation for the boundary terms and integration terms for $r + 2$ times, and the terms with $\delta(s_{j_1}) \cdots \delta(s_{j_r})$ are left out. For the rest $j \in \{1, \dots, k-1\}$, the transformation is the same as that in time-independent case. To be precise, here are the full set of steps:

1. **Transformation for $\{j_1, \dots, j_r\}$:**

   Step one: set $l = 1$, $m = 0$ and

   $$F = \beta^{\gamma}(\xi, q, k) V(\xi, s, k) e^{-s_k \epsilon - \cdots - s_1 \epsilon + i(x \cdot (\xi_k + q) + s_k(\xi_k^2 + 2q \cdot \xi_k) + \cdots + s_1(\xi_1^2 + 2\xi_1 \cdot q))}. \tag{6.285}$$

   Step two: set $m = m + 1$ and in $\int_0^1 ds_{j_l} F$, do integration by parts in s_{j_l} variable by setting

   $$e^{-\epsilon s_{j_l} + i s_{j_l}(\xi_{j_l}^2 + 2\xi_{j_l} \cdot q)} = \frac{1}{\epsilon + i(\xi_{j_l}^2 + 2\xi_{j_l} \cdot q)} \partial_{s_{j_l}} [e^{-\epsilon s_{j_l} + i s_{j_l}(\xi_{j_l}^2 + 2\xi_{j_l} \cdot q)}]. \tag{6.286}$$

   We get two terms: boundary term $\int_0^1 ds_{j_l} \delta(s_{j_l}) F_1$ and integral term $\int_0^1 ds_{j_l} F_2$. For example, when $l = 1$, see (6.283) and (6.284). For the boundary term, we go to **Step three** and go to **Step four** for the integral term.

   Step three: for boundary term $\int_0^1 ds_{j_l} \delta(s_{j_l}) F_1$, if $l < r$ and $m < r + 2$, set $F = F_1$, $l = l + 1$ and move back to **Step two**. Otherwise, (($l < r$ and $m = r + 2$) or ($l = r$)) we stop making transformations on the boundary term.

   Step four: for the integral term, if $m < r + 2$, set $F = F_2$ and move back to **Step two**. Otherwise, $m = r + 2$ and we stop taking transformations on the integral term.

After these transformation, we get no more than 2^{r+2} many sub-terms.

Each term has the form of (we call the case when $m = r+2$, type 1)

$$\begin{aligned}(-1)^{r+2}\int_0^1 ds_1 \cdots \int_0^1 ds_k \int d^3q d^3\xi_1 \cdots d^3\xi_k \delta(s_{j_1}) \cdots \delta(s_{j_{m-1}})\partial_{s_{j_1}}^{l_1} \cdots \partial_{s_{j_m}}^{l_m}[V](\xi,s,k)\\ 1/\left[(i(\xi_{j_1}^2+2\xi_{j_1}\cdot q))^{l_1+1}\cdots(i(\xi_{j_{m-1}}^2+2\xi_{j_{m-1}}\cdot q))^{l_{m-1}+1}(i(\xi_{j_m}^2+2\xi_{j_m}\cdot q))^{l_m}\right]\\ \beta^\gamma(\xi,q,k)e^{-s_k\epsilon-\cdots-s_1\epsilon+i(x\cdot(\xi_k+q)+s_k(\xi_k^2+2q\cdot\xi_k)+\cdots+s_1(\xi_1^2+2\xi_1\cdot q))}\frac{\hat{\psi}(q)}{(2\pi)^{\frac{3}{2}}}\end{aligned}$$

with $m-1+\sum_{u=1}^{m} l_u = r+2, 1 \leq m \leq k-1, l_u \geq 0$, or of(we call the case when $m = r+1$, type 2)

$$\begin{aligned}(-1)^{r+1}\int_0^1 ds_1 \cdots \int_0^1 ds_k \int d^3q d^3\xi_1 \cdots d^3\xi_k \delta(s_{j_1}) \cdots \delta(s_{j_r})\partial_{s_{j_1}}^{l_1} \cdots \partial_{s_{j_r}}^{l_r}[V](\xi,s,k)\\ 1/\left[(i(\xi_{j_1}^2+2\xi_{j_1}\cdot q))^{l_1+1}\cdots(i(\xi_{j_r}^2+2\xi_{j_r}\cdot q))^{l_r+1}\right]\\ \beta^\gamma(\xi,q,k)e^{-s_k\epsilon-\cdots-s_1\epsilon+i(x\cdot(\xi_k+q)+s_k(\xi_k^2+2q\cdot\xi_k)+\cdots+s_1(\xi_1^2+2\xi_1\cdot q))}\frac{\hat{\psi}(q)}{(2\pi)^{\frac{3}{2}}}\end{aligned}$$

with $\sum_{u=1}^{r} l_u = 1, l_u \geq 0$, or of(we call the case when $m = r$, type 3)

$$\begin{aligned}(-1)^{r}\int_0^1 ds_1 \cdots \int_0^1 ds_k \int d^3q d^3\xi_1 \cdots d^3\xi_k \delta(s_{j_1}) \cdots \delta(s_{j_r})V(\xi,s,k)\\ 1/\left[(i(\xi_{j_1}^2+2\xi_{j_1}\cdot q))\cdots(i(\xi_{j_r}^2+2\xi_{j_r}\cdot q))\right]\\ \beta^\gamma(\xi,q,k)e^{-s_k\epsilon-\cdots-s_1\epsilon+i(x\cdot(\xi_k+q)+s_k(\xi_k^2+2q\cdot\xi_k)+\cdots+s_1(\xi_1^2+2\xi_1\cdot q))}\frac{\hat{\psi}(q)}{(2\pi)^{\frac{3}{2}}}.\end{aligned}$$

Here each $1/(\xi_{j_u}^2 + 2\xi_{j_u}\cdot q)$ will give us a factor C_1/M for some fixed constant $C_1 > 0$.

2. **Transformation for the rest $j \in \{1, \cdots, k-1\} \setminus \{j_1, \cdots, j_r\}$:**

   When it comes to these j, for each term, we do the same transformation as before and will gain at least $\frac{C_2}{M}$(C_2 is some fixed constant), for each j with this property. And according to the definition of r, we have $k-1-r$ such j and will gain $\frac{C_2^{k-1-r}}{M^{k-1-r}}$ from the transformation here.

Estimates for all three types: the estimates are based on how we deal with $j = k$. For type 1, we do nothing for ξ_k, s_k and defer its L_x^p estimates to the end.

Estimates for type 2: for type 2, after the transformation to case when $|\xi_j+q|>2M$, it becomes the sum of no more than 81^k many terms since for

$$\partial^l_{\xi_j\cdot e_m}[\hat V(\xi_j-\xi_{j-1},\sum_{a=j}^k s_a)]Q_r,\ m\in\{1,2,3\},\ j\in\{1,\cdots,k\},\ l\in\{0,1,2\},\ r\in\{1,\cdots,9\},\tag{6.287}$$

there are 81^k many cases. Here for Q_r, see Lemma 6.3.5. For each term, when it comes to ξ_k, s_k, we have to face

$$\int_0^\infty ds_k\int d^3\xi_k\partial^{l_1}_{s_{j_1}}\cdots\partial^{l_r}_{s_{j_r}}[f^{[k-1]}(\xi,s)\partial^w_{\xi_{k-1}\cdot e_v}[\hat V(\xi_k-\xi_{k-1},s_k)]]e^{iH_0s_k}e^{i\xi_k\cdot Q}e^{-iH_0s_k}\tag{6.288}$$

for some direction e_v, some $w\in\{0,1,2\}$, with

$$f^{[k-1]}(\xi,s)=\partial^{w_{k-1}}_{\xi_{k-1}\cdot e_{m,k-1}}[\hat V(\xi_{k-1}-\xi_{k-2},\sum_{a=k-1}^k s_a)]\cdots\partial^{w_1}_{\xi_1\cdot e_{m,1}}[\hat V(\xi_1-\xi_0,\sum_{a=1}^k s_a)]\tag{6.289}$$

for some $w_j\in\{0,1,2\}, e_{m,j}\in\{1,2,3\}$. Since for type 2, $\sum_{u=1}^r l_u=1$, we have

$$\begin{aligned}\partial^{l_1}_{s_{j_1}}\cdots\partial^{l_r}_{s_{j_r}}[f^{[k-1]}(\xi,s)\partial^w_{\xi_{k-1}\cdot e_v}[\hat V(\xi_k-\xi_{k-1},s_k)]]&=\partial_{s_{j_u}}[f^{[k-1]}(\xi,s)\partial^w_{\xi_{k-1}\cdot e_v}[\hat V(\xi_k-\xi_{k-1},s_k)]]\\&=\sum_{a=1}^{j_u}f_a^{[k-1]}(\xi,s)\partial^w_{\xi_{k-1}\cdot e_v}[\hat V(\xi_k-\xi_{k-1},s_k)],\ \text{for some } u\in\{1,\cdots,r\},\end{aligned}$$

where the difference between $f_a^{[k-1]}$ and $f^{[k-1]}$ is that they have a different ath factor, that is, in $f_a^{[k-1]}$, for the ath factor, it has

$$\partial_{s_{j_u}}\partial^{w_a}_{\xi_a\cdot e_{m,a}}[\hat V(\xi_a-\xi_{a-1},\sum_{b=a}^k s_b)]\tag{6.290}$$

instead of

$$\partial^{w_a}_{\xi_a\cdot e_{m,a}}[\hat V(\xi_a-\xi_{a-1},\sum_{b=a}^k s_b)].\tag{6.291}$$

Since for $b=0,1$, $j=0,1,2$, $a=1,\cdots,k$,

$$\sup_{s_k\in\mathbb{R}^+}|s_k|^j|\partial^j_{s_k}\partial^b_{s_{j_u}}\partial^{w_j}_{\xi_j\cdot e_{m,j}}[\hat V(\xi_j-\xi_{j-1},\sum_{b=j}^k s_b)]|\leq c^3\hat V_0(\xi_j-\xi_{j-1}),\tag{6.292}$$

we can apply Corollary 6.4.1, Lemma 6.3.3 and have

$$\|\text{type } 2\|_{L^p_x}\lesssim\frac{C_3^k c^{2k-k^2}81^k(c^3\|\hat V_0(\xi)\|_{L^1_\xi\cap L^\infty_\xi})^k}{M^{r+1+(k-r-1)}}\|\psi(x)\|_{L^p_x}$$

where we have another k since $j_u \leq k - 1 < k$. Therefore

$$\|\text{type } 2\|_{L^p_x} \lesssim \frac{C_4^k c^{3k+2} k^3 \|\hat{V}_0(\xi)\|^k_{L^1_\xi \cap L^\infty_\xi}}{M^k} \|\psi(x)\|_{L^p_x}. \tag{6.293}$$

Estimates for type 3: for type 3, similarly, after the transformation to case when $|\xi_j + q| > 2M$, it becomes the sum of no more than 9^k many terms. For each term, when it comes to ξ_k, s_k, we have to face the operator

$$\int_0^\infty ds_k \int d^3\xi_k f^{[k-1]}(\xi, s) \partial^w_{\xi_{k-1} e_v} [\hat{V}_{\alpha_k}(\xi_k - \xi_{k-1}, s_k)] e^{iH_0 s_k} e^{i\xi_k \cdot Q} e^{-iH_0 s_k} \tag{6.294}$$

with $f^{[k-1]}$ satisfying equation (6.289). Due to inequality (6.292), Lemma 6.3.3 again, we have

$$\|\text{type } 3\|_{L^p_x} \lesssim \frac{C_5^k c^2 81^k k^2 (c^3 \|\hat{V}_0(\xi)\|_{L^1_\xi \cap L^\infty_\xi})^k}{M^r M^{k-1-r}} \|\psi(x)\|_{L^p_x} \tag{6.295}$$

and therefore

$$\|\text{type } 3\|_{L^p_x} \lesssim \frac{C_6^k c^{3k+2} k^2 \|\hat{V}_0(\xi)\|^k_{L^1_\xi \cap L^\infty_\xi}}{M^{k-1}} \|\psi(x)\|_{L^p_x}. \tag{6.296}$$

Estimates for type 1: it requires the following lemma:

Lemma 6.4.4. *For* $1 \leq j_1 < \cdots < j_m < k$, $\mathbb{N} = \{0, 1, \cdots\}$, *let*

$$L_m := \Pi_{l=1}^m f(s_{j_l} + s_{j_l+1} + \cdots + s_k) \tag{6.297}$$

and for $\gamma \in \mathbb{N}^m$,

$$L_m^\gamma := \Pi_{l=1}^m \frac{1}{\gamma_l!} f^{(\gamma_l)}(s_{j_l} + s_{j_l+1} + \cdots + s_k). \tag{6.298}$$

If $l_1 + \cdots + l_m \leq k + 1$, *then*

$$\partial^{l_1}_{s_{j_1}} \cdots \partial^{l_m}_{s_{j_m}} [L_m] = \sum_{\gamma \in \mathbb{N}^m, |\gamma| = l_1 + \cdots + l_m} c_\gamma L_m^\gamma \tag{6.299}$$

with

$$\sum_{\gamma \in \mathbb{N}^m, |\gamma| = l_1 + \cdots + l_m} |c_\gamma| \leq (2k)^{l_1 + \cdots + l_m}. \tag{6.300}$$

Proof. Let

$$M := \Pi_{l=1}^m f(s + a_l), \text{ for } a_l > 0 \tag{6.301}$$

and for $\gamma \in \mathbb{N}^m$,

$$M^\gamma := \Pi_{l=1}^m \frac{1}{\gamma_l!} f^{(\gamma_l)}(s + a_l). \tag{6.302}$$

Since

$$\partial_s[\mathsf{M}^{\gamma}] = \sum_{l=1}^{m}(\gamma_l+1)\,\mathsf{M}^{\eta(l)} \tag{6.303}$$

for $\eta(l) \in \mathbb{N}^m$, with

$$\gamma_j = \eta(l)_j, \quad j \in \{1, \cdots, l-1, l+1, \cdots, m\}, \quad \gamma_l + 1 = \eta(l)_l, \tag{6.304}$$

then $\partial_s[\mathsf{M}^{\gamma}]$ can be regarded as the sum of

$$\sum_{l=1}^{m}(\gamma_l+1) = m + \sum_{l=1}^{m}\gamma_l \tag{6.305}$$

many terms, with each term having the form of M^{η} with

$$\gamma_{j_0} + 1 = \eta_{j_0}, \quad \gamma_j = \eta_j, \quad j \in \{1, \cdots, m\} \setminus \{j_0\}, \text{ for some } j_0 \in \{1, \cdots, m\}. \tag{6.306}$$

Then $\partial^{l_1}_{s_{j_1}} \cdots \partial^{l_m}_{s_{j_m}}[\mathsf{L}_m]$ can be regarded as the sum of no more than

$$\Pi_{j=0}^{l_1+\cdots+l_m-1}(m+j) \tag{6.307}$$

many terms, with each term having the form of M^{η} with $|\eta| = l_1 + \cdots + l_m$. Since $m \le k-1$, therefore

$$\Pi_{j=0}^{l_1+\cdots+l_m-1}(m+j) \le (2k)^{l_1+\cdots+l_m}, \tag{6.308}$$

we have

$$\sum_{\gamma\in\mathbb{N}^m, |\gamma|=l_1+\cdots+l_m} |c_\gamma| \le (2k)^{l_1+\cdots+l_m} \tag{6.309}$$

and finish the proof. □

Then for type 1, we do transformation in the following order: take the integral over s_{j_l} for $l \le m-1$, use Lemma 6.4.4 and condition (6.250), use

$$\sup_{t\in\mathbb{R}} \frac{1}{a!}\sum_{l,j=0}^{2}\sum_{m,r=1}^{3}\left|\frac{\partial^a}{\partial t^a}\left[\partial^l_{\xi\cdot e_r}\partial^j_{\xi\cdot e_m}\hat{V}(\xi,t)\right]\right| \le \frac{c^a\hat{V}_0(\xi)}{(1+|t|)^a} \text{ and } \frac{1}{(1+s+a)^j} \le \frac{1}{(1+s)^j}, \text{ for } s,a,j>0, \tag{6.310}$$

take the integral over $\xi_1, \cdots, \xi_k, s_j$(such s_j with $|\xi_j + q| > 2M$) and we have

$$\begin{aligned}\|\text{type } 1\|_{L^p_x} \le & \int_0^\infty ds_{j_m} \cdots \int_0^\infty ds_{j_r} \int_0^\infty ds_k (2k)^{r+3-m} \\ & \frac{c^{r+3-m}\|\hat{V}_0(\xi)\|^k_{L^1_\xi}}{(1+s_{j_m}+\cdots+s_{j_r}+s_k)^{r+3-m}}\frac{C_1^{r+2}}{(2\pi)^{3k/2}M^{r+2}} \le \frac{81^k C_2^{k-1-r}}{M^{k-1-r}}\|\psi(x)\|_{L^p_x}.\end{aligned}$$

Since

$$\int_0^\infty ds_{j_m} \cdots \int_0^\infty ds_{j_r} \int_0^\infty ds_k (2k)^{r+3-m} \frac{1}{(1+s_{j_m}+\cdots+s_{j_r}+s_k)^{r+3-m}}$$
$$= \frac{(2k)^{r+3-m}}{(r+2-m)!} \leq 2ke^{2k},$$

we have

$$\|\text{type } 1\|_{L^p_x} \leq \frac{2c^{k+1}kC_3^{k+1}\|\hat{V}_0(\xi)\|^k_{L^1_\xi \cap L^\infty_\xi}}{M^{k+1}}\|\psi(x)\|_{L^p_x}. \tag{6.311}$$

Estimates for $I^{(k)}_\epsilon\psi(x)$: combining the estimates for type 1, type 2 and type 3, we have

$$\|I^{(k)}_{\gamma\epsilon}\psi(x)\|_{L^p_x} \lesssim \frac{c^{3k+2}k^3C_4^k\|\hat{V}_0(\xi)\|^k_{L^1_\xi \cap L^\infty_\xi}}{M^{k-1}}\|\psi(x)\|_{L^p_x}. \tag{6.312}$$

Hence,

$$\|I^{(k)}_{\epsilon}\psi(x)\|_{L^p_x} \lesssim \frac{c^{3k+2}k^3C^k\|\hat{V}_0(\xi)\|^k_{L^1_\xi \cap L^\infty_\xi}}{M^{k-1}}\|\psi(x)\|_{L^p_x}. \tag{6.313}$$

Similarly,

$$\left\|\beta(|P| > 32M)\left(I^{(k)}_\epsilon\right)\right\|_{L^p_x \to L^p_x} \lesssim \frac{c^{3k+2}k^3C^k\|\hat{V}_0(\xi)\|^k_{L^1_\xi \cap L^\infty_\xi}}{M^{k-1}}. \tag{6.314}$$

□

Now we can prove Theorem 6.1.1:

Proof. The proof is the same as Theorem 6.1.8 by applying Lemma 6.4.3, Theorem 6.4.1 instead. □

Similarly, we get asymptotic completeness on high frequency subspace.

Corollary 6.4.2. *If $V(x,t)$ satisfies the condition in Theorem 6.1.1, the Schrödinger equation has asymptotic completeness on high frequency subspace.*

Now let us think about

$$\Omega_T := s\text{-}\lim_{t\to\infty} U(T+t,T)e^{-itH_0}, \text{ on } L^2, \text{ for } T \geq 0. \tag{6.315}$$

Assume

$$\Omega_T(t) = U(T+t,T)e^{-itH_0}. \tag{6.316}$$

$$\Omega_{T,\epsilon} = I + (-i)\int_0^\infty dt\, e^{-\epsilon t}\Omega_T(t)e^{itH_0}V(x,t+T)e^{-itH_0}. \tag{6.317}$$

By the same argument, we also have its L^p boundedness on high-frequency subspace:

Corollary 6.4.3. *If $V(x,t)$ satisfies condition (6.250), there exists $M = M(V(x,t)) > 0$ such that for all $1 \leq p \leq \infty$,*

$$\Omega_T\beta(|H_0| > M^2) = s\text{-}\lim_{\epsilon\downarrow 0}\Omega_{T,\epsilon}\beta(|H_0| > M^2), \text{ on } L^p, \tag{6.318}$$

and $\beta(|H_0| > M^2)\Omega_T, \Omega_T\beta(|H_0| > M^2)$ are bounded on L^p.

Proof. Since Ω_T is obtained by replacing $V(x,t)$ with $V(x,T+t)$ in Ω and since

$$\frac{(1+t)^a}{a!} \leq \frac{(1+t+T)^a}{a!}, \text{ for } t, T \geq 0, \tag{6.319}$$

then following the same argument in Theorem 6.1.1, the conclusion follows. □

Similarly, we have the following corollary:

Corollary 6.4.4. *If $V(x,t)$ satisfies the assumptions in Theorem 6.1.1, there exists $M = M(V(x,t)) > 0$, such that*

$$\sup_{T\in\mathbb{R}}\|U(T,0)e^{-iTH_0}\beta(|P| > M)\|_{L^p_x\to L^p_x} < C. \tag{6.320}$$

This can be extended to the case when

$$V(x,t) = \chi(|t| < T_0)B(x,t) + \chi(|t| \geq T_0)V_1(x,t), \tag{6.321}$$

with $V_1(x,t)$ satisfying the assumption in Theorem 6.4.1, $\hat{B}(\xi,t) \in L^\infty_t L^1_\xi$. This application is based on the following operators

$$I^{(k)}_\epsilon(T_0) := \int_{T_0}^\infty dt_k\int_{t_k}^\infty dt_{k-1}\cdots\int_{t_2}^\infty e^{-\epsilon t_1}dt_1 K_{t_k}(V(x,t_k))\cdots K_{t_1}(V(x,t_1)) \tag{6.322}$$

and

$$J^{(k)}_\epsilon(T_0) := \int_0^{T_0} dt_k\int_{t_k}^{T_0} dt_{k-1}\cdots\int_{t_2}^{T_0} e^{-\epsilon t_1}dt_1 K_{t_k}(V(x,t_k))\cdots K_{t_1}(V(x,t_1)). \tag{6.323}$$

Then

$$I^{(k)}_\epsilon = \sum_{j=0}^{k} J^{(j)}_\epsilon(T_0)I^{(k-j)}_\epsilon(T_0). \tag{6.324}$$

Corollary 6.4.5. *If $V_1(x,t)$ satisfies the assumptions in Theorem 6.1.1, $\hat{B}(\xi,t)\in L_t^\infty L_\xi^1$, then there exists some large M such that for all $1\le p\le\infty$, $\Omega\beta(|P|>32M): L_x^p\to L_x^p$ is bounded.*

Proof. Similarly, we have that for $\psi\in\beta(|P|>32M)S_x$,

$$I_\epsilon^{(k)}(T_0)\psi(x)=\int_0^\infty e^{-\epsilon s_k}ds_k\cdots\int_0^\infty e^{-\epsilon s_1}ds_1\int d^3\xi_1\cdots d^3\xi_k d^3q e^{i(x\cdot(\xi_k+q)+2(s_k\xi_k+\cdots+s_1\xi_1)\cdot q)}$$
$$V(\xi,k)e^{i(s_k\xi_k^2+\cdots+s_1\xi_1^2)}\frac{\hat{\psi}(q)}{(2\pi)^{\frac{3}{2}}}-\int_0^{T_0}e^{-\epsilon s_k}ds_k\cdots\int_0^\infty e^{-\epsilon s_1}ds_1\int d^3\xi_1\cdots d^3\xi_k d^3q V(\xi,k)\frac{\hat{\psi}(q)}{(2\pi)^{\frac{3}{2}}}$$
$$e^{i(x\cdot(\xi_k+q)+(s_k\xi_k^2+\cdots+s_1\xi_1^2)+2(s_k\xi_k+\cdots+s_1\xi_1)\cdot q)}:=L_1^{(k)}\psi(x)+L_2^{(k)}\psi(x).$$

We apply Lemma 6.4.3 to $L_1^{(k)}\psi(x)$ and have

$$\|L_1^{(k)}\psi(x)\|_{L_x^p}\le\frac{(2k)^3C_{V_1}^k}{\sqrt{M}^{k-1}}\|\psi(x)\|_{L_x^p},\ \text{for some } C_{V_1}>0. \tag{6.325}$$

For $L_2^{(k)}\psi(x)$, according to the proof of Lemma 6.4.3, we do the same transformation for ξ_j, s_j, $j=1,\cdots,k-1$ while we do nothing for s_k,ξ_k. Similarly, in the end, we will get

$$\|L_2^{(k)}\psi(x)\|_{L_x^p}\le\frac{T_0(2k)^3D_{V_1}^k}{\sqrt{M}^{k-1}}\|\psi(x)\|_{L_x^p},\ \text{for some } D_{V_1}>0. \tag{6.326}$$

Hence,

$$\|I_\epsilon^{(k)}(T_0)\psi(x)\|_{L_x^p}\le\frac{(2k)^3(1+T_0)(D_{V_1}+C_{V_1})^k}{\sqrt{M}^{k-1}}\|\psi(x)\|_{L_x^p}. \tag{6.327}$$

According to the same proof of Corollary 6.2.2, we have that for $\psi\in L^q$,

$$\|J_\epsilon^{(k)}(T_0)\psi(x)\|_{L_x^q}\le\frac{T_0^k\|\hat{V}(\xi,t)\|^k_{L_t^\infty L_\xi^1}}{k!}+\frac{T_0^k\|\hat{B}(\xi,t)\|^k_{L_t^\infty L_\xi^1}}{k!}\|\psi(x)\|_{L_x^p}. \tag{6.328}$$

Then for $\psi\in\beta(|P|>32M)S_x$,

$$\|I_\epsilon^{(k)}\psi(x)\|_{L_x^p}\le\sum_{j=0}^k\frac{M^j}{j!}\frac{(1+T_0)(2k-2j)^3M^{k-j}}{\sqrt{M}^{k-j-1}}\|\psi(x)\|_{L_x^p}\le\frac{(1+T_0)(2k)^3M^k}{\sqrt{M}^{k-1}}\Big(\sum_{j=0}^\infty\frac{\sqrt{M}^j}{j!}\Big)\|\psi(x)\|_{L_x^p}$$
$$\le\frac{(1+T_0)(2k)^3M^k}{\sqrt{M}^{k-1}}\exp(\sqrt{M})\|\psi(x)\|_{L_x^p},$$

where

$$M:=\max\left(T_0\|\hat{B}(\xi,t)\|_{L_t^\infty L_\xi^1},D_{V_1}+C_{V_1}\right). \tag{6.329}$$

Then choose M large enough to make

$$\sum_{k=1}^{\infty} \frac{k^3 M^k}{\sqrt{M}^{k-1}} < \infty \tag{6.330}$$

and then we get the conclusion. □

Corollary 6.4.6. *If $V(x,t)$ satisfies the assumption in Theorem 6.1.1, then when $M > 0$ is sufficiently large,*

$$\sup_{T \in \mathbb{R}} \|U(T,0)e^{-iTH_0}\beta(|P| > M)\|_{L^p_x \to L^p_x} < \infty, \text{ for } 1 \leq p \leq \infty. \tag{6.331}$$

Therefore,

$$\sup_{T \in \mathbb{R}} |T|^{3/2}\|U(T,0)\beta(|P| > M)\|_{L^p_x \to L^{p'}_x} < \infty, \text{ for } 1 \leq p \leq 2. \tag{6.332}$$

Proof. The proof is the same as that of Corollary 6.3.5. □

6.4.3 Examples

In this subsection, we are considering the potential $V(x,t)$ satisfying

$$V(x,t) = \sum_{j=0}^{\infty} V_j(x)\frac{1}{(1+t)^j}, \text{ for } t > \frac{T_0}{2}, \text{ for some } T_0 > 0. \tag{6.333}$$

If

$$\sum_{b=0}^{\infty} \frac{2^b}{(1+T_0)^b} \sum_{l,j=0}^{2} \sum_{m,r=1}^{3} |\partial^l_{\xi_{e_r}}\partial^j_{\xi_{e_m}}\hat{V}_a(\xi)| \in L^1_\xi \cap L^\infty_\xi, \tag{6.334}$$

and $\hat{V}(\xi,t) \in L^\infty_t(0,T_0)L^1_\xi$, then we choose $B(x,t) = \chi(t < T_0)V(x,t)$ and $V_1(x,t) = \chi(t \geq T_0)V(x,t)$ with

$$\frac{(1+t)^a}{a!}\left|\sum_{l,j=0}^{2}\sum_{m,r=1}^{3}\left|\frac{\partial^a}{\partial t^a}\left[\partial^l_{\xi_{e_r}}\partial^j_{\xi_{e_m}}\hat{V}(\xi,t)\right]\right|\right| \leq \sum_{b=0}^{\infty}\frac{\begin{pmatrix} b+a-1 \\ a \end{pmatrix}}{(1+t)^b}\sum_{l,j=0}^{2}\sum_{m,r=1}^{3}|\partial^l_{\xi_{e_r}}\partial^j_{\xi_{e_m}}\hat{V}_a(\xi)|$$
$$\leq 2^a\sum_{b=0}^{\infty}\frac{2^b}{(1+T_0)^b}\sum_{l,j=0}^{2}\sum_{m,r=1}^{3}|\partial^l_{\xi_{e_r}}\partial^j_{\xi_{e_m}}\hat{V}_a(\xi)|.$$

Then we can choose $c = 2$ and

$$\hat{V}_0(\xi) = \sum_{b=0}^{\infty}\frac{2^b}{(1+T_0)^b}\sum_{l,j=0}^{2}\sum_{m,r=1}^{3}|\partial^l_{\xi_{e_r}}\partial^j_{\xi_{e_m}}\hat{V}_a(\xi)|. \tag{6.335}$$

Apply Corollary 6.4.5 and we have the following corollary:

Corollary 6.4.7. *Assume* $V(x,t)$ *has the form of* (6.333) *and satisfies condition* (6.334)*, then* $\Omega\beta(|P|>M): L^p_x \to L^p_x$ *is bounded for some sufficiently large* M.

Next, we consider the potential $V(x,t)$ satisfying

$$V(x,t)=\sum_{j=0}^{\infty} V_j(x)f_j(t), \tag{6.336}$$

when $t>\frac{T_0}{2}$ for some $T_0>0$. If $\hat{V}(\xi,t)\in L^{\infty}_t(0,T_0/2)L^1_\xi$ and if for $b\in\mathbb{N}$,

$$\sup_{t\in[T_0/2,\infty)}\frac{(t+1)^b}{b!}|f_j^{(b)}(t)|\le c_j^b,\quad \sum_{a=0}^{\infty}c_a^b\sum_{l,j=0}^{2}\sum_{m,r=1}^{3}|\partial^l_{\xi\cdot e_r}\partial^j_{\xi\cdot e_m}\hat{V}_a(\xi)|<\infty, \tag{6.337}$$

we will get a similar result:

Corollary 6.4.8. *Assume* $V(x,t)$ *has the form of* (6.336) *and satisfies condition* (6.337)*, then* $\Omega\beta(|P|>M): L^p_x \to L^p_x$ *is bounded for some sufficiently large* M.

Here are some other examples.

Example 6.4.2 (quench potentials)**.** *A quench potential has the form of* $V(x,t)=\chi(t-d)V_1(x)$ *or* $V(x,t)=\beta(t>2d)V_1(x)$ *for some* $d>0$. *If* $\sum_{l,j=0}^{2}\sum_{m,r=1}^{3}|\partial^l_{\xi\cdot e_r}\partial^j_{\xi\cdot e_m}\hat{V}_1(\xi)|\in L^1_\xi\cap L^\infty_\xi$*, then* $\Omega\beta(|P|>M)$ *is bounded on* L^p_x *for some sufficiently large* M.

Proof. Choose $B(x,t)=V(x,t)$, $T_0=d, c=1$, $V_1(x,t)=V_1(x)$. When we take the derivative with respect to t, it is 0 and of course satisfies the condition (6.250). □

Example 6.4.3 (Hyperbolic potentials)**.** *A hyperbolic potential has the form of* $V(x,t)=\tanh(t)V_1(x)+V_0(x)$. *If* $\sum_{l,j=0}^{2}\sum_{m,r=1}^{3}|\partial^l_{\xi\cdot e_r}\partial^j_{\xi\cdot e_m}\hat{V}_a(\xi)|\in L^1_\xi\cap L^\infty_\xi$, $a=0,1$*, then* $\Omega\beta(|P|>M)$ *is bounded on* L^p_x *for some sufficiently large* M.

Proof. Since for $a\in\mathbb{N}^+$, $t\ge 1$,

$$\frac{(1+t)^j}{j!}\frac{d^j}{dt^j}[\tanh t]=\frac{(1+t)^j}{j!}\frac{d^j}{dt^j}[1-2e^{-2t}\sum_{l=0}^{\infty}(-1)^l e^{-2lt}]=\sum_{l=0}^{\infty}(-1)^l\frac{[-2(l+1)(1+t)]^j}{j!}e^{-2(l+1)t}, \tag{6.338}$$

we can choose $c=4$ and

$$\sup_{t\in[1,\infty)}\frac{(1+t)^j}{j!}\left|\frac{d^j}{dt^j}[\tanh t]\right|\le 4^j\sum_{l=0}^{\infty}e^{-(l+1)t}=\frac{4^j e^{-t}}{1-e^{-t}}<4^j. \tag{6.339}$$

For $t\in[0,1)$, it satisfies the condition for some time. By Corollary 6.4.8, we get the result. □

6.5 Moving and self-similar potentials

A fundamental class of time dependent potentials is called moving potentials, of the form $\sum_i V_i(x - c_i(t))$. They appear naturally in charge transfer models, soliton dynamics, models of Atom+Radiation and more. The mathematical analysis of such potentials has been carried out for certain classes, mostly when

$$c_i(t) = ct + f(t) \tag{6.340}$$

with $f(t)$ decaying fast, see e.g. [104], [33], [72] and [63]. More general movement was considered in [5],[6] and [7], but it was limited to ONE potential term. Moreover it was assumed that the velocity goes to zero, or random in other cases. The more difficult cases when the movement is not linear is treated in this section. But the case $c(t) = t$ does not satisfy our condition, if there is another potential added. For more information about charge transfer models, see [9], [8] and [12].

We prove Theorem 6.1.2(the self-similar example) first.

Theorem 6.5.1. *If $V(x,t)$ is defined in equation (6.31) and satisfies condition (6.32), then*

$$\lim_{T\to\infty} \|U(0,T)e^{-iTH_0} - \Omega\|_{L^p\to L^p} = 0, \quad \|\Omega\|_{L^p\to L^p} \lesssim \exp\left(\frac{\|h(t)\|_{L^1_t(0,\infty)}}{(2\pi)^{\frac{n}{2}}}\right). \tag{6.341}$$

Proof. In this case,

$$K_t(V(x,t)) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int d^n\xi \hat{V}_1(\xi,t)e^{iH_0t}e^{i\xi\cdot g(t)x}e^{-iH_0t} + \sum_{j=1}^{\infty} f_j(t)e^{iH_0t}e^{ia_j\cdot g_j(t)x}e^{-iH_0t}. \tag{6.342}$$

According to the same computation in section 1 and the proof of Corollary 6.2.2, we have that for $T_0 \in [0,\infty]$,

$$\|\sum_{k=0}^{\infty} i^k I(T_0)^{(k)}\|_{L^p_x\to L^p_x} \lesssim \exp\left(\frac{\|h(t)\|_{L^1_t[0,\infty)}}{(2\pi)^{\frac{n}{2}}}\right) \tag{6.343}$$

where

$$I(T_0)^{(j)} := \int_0^{T_0} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j K_{t_j}(V(x,t_j)) \cdots K_{t_1}(V(x,t_1)), \tag{6.344}$$

and as $T_0 \to \infty$,

$$\Big\| \sum_{k=0}^{\infty} I(\infty)^{(k)} - \sum_{k=0}^{\infty} I(T_0)^{(k)} \Big\|_{L^p_x \to L^p_x} \leq \frac{\|h(t)\|_{L^1_t[T_0,\infty)}}{(2\pi)^{\frac{n}{2}}} \exp\left(\frac{\|h(t)\|_{L^1_t[0,\infty)}}{(2\pi)^{\frac{n}{2}}} \right) \to 0. \tag{6.345}$$

Then $\sum\limits_{k=0}^{\infty} I(T_0)^{(k)} \to \sum\limits_{k=0}^{\infty} I(\infty)^{(k)}$ in norm. Hence

$$\Omega = \sum_{k=0}^{\infty} I(\infty)^{(k)}, \quad \|\Omega\|_{L^p_x \to L^p_x} \leq \exp\left(\frac{\|h(t)\|_{L^1[0,\infty)}}{(2\pi)^{\frac{n}{2}}} \right). \tag{6.346}$$

□

Corollary 6.5.1. *If $V(x,t)$ satisfies the assumption in Theorem 6.1.2, then*

$$\sup_{T \in \mathbb{R}} \|U(0,T)e^{-iTH_0}\|_{L^p_x \to L^p_x} < \infty, \text{ for } 1 \leq p \leq \infty. \tag{6.347}$$

Therefore,

$$\sup_{T \in \mathbb{R}} |T|^{3/2} \|U(T,0)\|_{L^p_x \to L^{p'}_x} < \infty, \text{ for } 1 \leq p \leq 2. \tag{6.348}$$

Proof. The proof is the same as that of Theorem 6.1.2. □

Here is an example where $f(t)$ does not even have a limit in $\mathbb{R}^3$ as $t \to \infty$ and it is not just limited to one potential:

Example 6.5.2. *Assume a potential has the form of $V(x,t) = V_1(x - \sin(\ln(1+|t|))v) + V_0(x)$ for some $v \in \mathbb{R}^3$. Then, if $\sum\limits_{l,j=0}^{2} \sum\limits_{m,r=1}^{3} |\partial^l_{\xi_{e_r}} \partial^j_{\xi_{e_m}} \hat{V}_a(\xi)| \in L^1_\xi \cap L^\infty_\xi$, $a = 0, 1$, and the support of $\hat{V}_1$ is contained in a ball B_R centered at the origin with a radius R, then $\Omega\beta(|P| > M)$ is bounded on L^p_x for some sufficiently large M.*

Proof. In this case,

$$\hat{V}(\xi,t) = \hat{V}_0(\xi) + \hat{V}_1(\xi) e^{-\sin(\ln(1+|t|)) i\xi \cdot v}. \tag{6.349}$$

For $t \geq 0$, $a \in \mathbb{N}^+$,

$$\left| \partial^a_t \partial^l_{\xi_{e_r}} \partial^j_{\xi_{e_m}} [\hat{V}(\xi,t)] \right| \leq \sum_{b=0}^{4} (R|v|)^b \left| \partial^a_t [\sin(\ln(1+t))^b e^{-\sin(\ln(1+t)) i\xi \cdot v}] \right| \sum_{l,j=0}^{2} \sum_{m,r=1}^{3} |\partial^l_{\xi_{e_r}} \partial^j_{\xi_{e_m}} \hat{V}_1(\xi)|. \tag{6.350}$$

Since for $a_1, a_2, a_3 \in \mathbb{R}$,

$$\frac{d}{dt}[e^{(a_1 i - a_2)\ln(1+t) - i\sin(\ln(1+t))a_3}] = (a_1 i - a_2)e^{(a_1 i - a_2 - 1)\ln(1+t) - i\sin(\ln(1+t))a_3}$$
$$-\frac{i}{2}e^{((a_1+1)i - a_2 - 1)\ln(1+t) - i\sin(\ln(1+t))a_3} - \frac{i}{2}e^{((a_1-1)i - a_2 - 1)\ln(1+t) - i\sin(\ln(1+t))a_3},$$

we can regard it as the sum of $|a_1| + |a_2| + 1$ terms, with each term having the form

$$e^{(b_1 i - a_2 - 1)\ln(1+t) - i\sin(\ln(1+t))a_3}, \quad ie^{(b_1 i - a_2 - 1)\ln(1+t) - i\sin(\ln(1+t))a_3}$$

with $|b_1 - a_1| = 0$ or $|b_1 - a_1| = 1$. Hence, for $b \in \{-4, -3, \cdots, 3, 4\}$,

$$\frac{(1+t)^a}{a!}\left|\frac{d^a}{dt^a}[e^{bi\ln(1+t) - \sin(\ln(1+t))iv\cdot\xi}]\right| \le \frac{1}{a!}\Pi_{j=0}^{a-1}(|b| + 1 + 2j) \le 2^{2a+3}. \tag{6.351}$$

Then, there exists a constant C independent of a such that

$$\left|\partial_t^a \partial_{\xi\cdot e_r}^l \partial_{\xi\cdot e_m}^j [\hat{V}(\xi, t)]\right| \le \sum_{b=0}^{4}(R|v|)^b \left|\partial_t^a[\sin(\ln(1+t))^b e^{-\sin(\ln(1+t))i\xi\cdot v}]\right| \le C\frac{(4|v|R)^a}{(1+t)^a} \tag{6.352}$$

which implies $V(x,t)$ satisfies condition (6.250) and finish the proof.

□

In the following, we apply the same argument as in previous sections, to prove decay estimates for potentials $V(x - \sqrt{1+|t|}v)$ on high frequency subspace for $v \in \mathbb{R}^3$, which satisfies assumption 6.38.

Remark 39. *This case is a new class, since $\sqrt{1+|t|}$ is not Mikhlin-type anymore, and the derivative of $\sqrt{1+t}(t > 0)$ is not in $L_t^2(0, \infty)$.*

We stick to $t > 0$. Let

$$G_{\le 2M}(\eta, t) := \beta(|P| \le 2M)e^{itH_0}e^{i\eta\cdot x}e^{-itH_0}, \text{ for } \eta \in \mathbb{R}^3, \tag{6.353}$$

$$G_{>2M}(\eta, t) := \beta(|P| > 2M)e^{itH_0}e^{i\eta\cdot x}e^{-itH_0}, \tag{6.354}$$

$$G_M(\xi^j, t_{k+j+1}, s^j, k) = G_{\le 2M}(\xi_{k+j} - \xi_{k+j-1}, t_{k+j+1} + s_{k+j}) \tag{6.355}$$

$$G_{>2M}(\xi_{k+j-1} - \xi_{k+j-2}, t_{k+j+1} + \sum_{l=k-1}^{k} s_{l+j}) \cdots G_{>2M}(\xi_{1+j} - \xi_j, t_{k+j+1} + \sum_{l=1}^{k} s_{l+j}), \tag{6.356}$$

for $\xi \in \mathbb{R}^{3(k+j)}, s \in \mathbb{R}^{k+j}, t_{k+j+1} \in \mathbb{R}$, $j \in \mathbb{N}$, with $\xi_0 = 0$,

$$V(\xi, t_{k+1}, s, k) := \Pi_{j=1}^{k} \mathcal{F}\left[V\left(x - \left(\sqrt{1+t_{k+1}+\sum_{l=j}^{k} s_l}\right) v\right)\right](\xi_j - \xi_{j-1}) \tag{6.357}$$

and let

$$J_{M,\epsilon}^{(k+1)} := \frac{1}{(2\pi)^{3k/2}} \int d^3\xi_1 \cdots d^3\xi_k \int_0^\infty dt_{k+1} e^{-\epsilon t_{k+1}} U(0, t_{k+1}) e^{i\xi_k \cdot x} V(x - \sqrt{1+t_{k+1}} v) \tag{6.358}$$

$$e^{-it_{k+1}H_0} \int_0^\infty e^{-\epsilon s_k} ds_k \int_0^\infty e^{-\epsilon s_{k-1}} ds_{k-1} \cdots \int_0^\infty e^{-\epsilon s_1} ds_1 V(\xi, t_{k+1}, s, k) G_M(\xi', t_{k+1}, s', k), \tag{6.359}$$

$$K^{(k)}(T) := \int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k e^{it_k H_0} V(x - \sqrt{1+|t_k|} v) e^{-it_k H_0} \beta(|P| > 2M) \cdots \tag{6.360}$$

$$e^{it_1 H_0} V(x - \sqrt{1+|t_1|} v) e^{-it_1 H_0} \beta(|P| > 2M). \tag{6.361}$$

The proof is based on following lemma and the estimates for $J_{M,\epsilon}^{(k+1)}, K^{(k)}(T)$:

Lemma 6.5.1 (Representation formula 2)**.** *For $\xi_i \in \mathbb{R}^n$, $i = 1, \cdots, k$ $(k \in \mathbb{N}^+)$, $\psi(x) \in L_x^p(\mathbb{R}^n)$, we have*

$$G_{\le 2M}(\xi_k - \xi_{k-1}, t_k) G_{>2M}(\xi_{k-1} - \xi_{k-2}, t_{k-1}) \cdots G_{>2M}(\xi_{k-1} - \xi_{k-2}, t_{k-1}) \psi(x)$$
$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int d^n q e^{i(x \cdot (\xi_k+q) + t_k(\xi_k^2 + 2q\cdot\xi_k) + (t_{k-1} - t_k)(\xi_{k-1}^2 + 2q\cdot\xi_{k-1}) + \cdots + (t_1 - t_2)(\xi_1^2 + 2\xi_1 \cdot q))}$$
$$\beta(|\xi_k + q| \le 2M) \Pi_{j=1}^{k-1} \beta(|\xi_j + q| > 2M) \hat{\psi}(q).$$

Proof. It follows directly from

$$f(|P|) e^{ix\cdot\xi} = e^{ix\cdot\xi} f(|P+\xi|) \tag{6.362}$$

and Lemma 6.3.4. □

Lemma 6.5.2. *If $V(x - \sqrt{1+t}v)$ satisfies assumption (6.38), then when M is large enough,*

$$\sup_{T\in\mathbb{R}} |T|^{3/2} \| J_{M,\epsilon}^{(k+1)} e^{iTH_0} \|_{L_x^1 \to L_x^\infty} \lesssim \frac{k^5 (C|||V(x)|||_p)^k}{\sqrt{M}^k} \tag{6.363}$$

for some constant C.

Proof. Due to Lemma 6.5.1, for s_k, ξ_k, we have a factor $\beta(j\xi_k + Pj \quad 2M)$. We deal with them first. **Step one:** in this case, we have to face

$$\int_0^1 ds_k e^{is_k(\xi_k^2+2\xi_k \ q) \quad \epsilon s_k} \left(\Pi_{l=1}^k e^{\ i\sqrt{1+\sum_{l=j}^{k+1} s_l v \ (\xi_j \quad \xi_j \quad 1)}} \right). \tag{6.364}$$

We do the same transformation as before, that is,

$$e^{is_k(\xi_k^2+2\xi_k \ q) \quad \epsilon s_k} = \frac{1}{i(\xi_k^2 + 2\xi_k \quad q) \quad \epsilon} \partial_{s_k}[e^{is_k(\xi_k^2+2\xi_k \ q) \quad \epsilon s_k}].$$

Then we will get two terms: boundary term

$$\frac{1}{i(\xi_k^2 + 2\xi_k \quad q) \quad \epsilon} e^{is_k(\xi_k^2+2\xi_k \ q) \quad \epsilon s_k} \Pi_{l=1}^k e^{\ i\sqrt{1+\sum_{l=j}^{k+1} s_l v \ (\xi_j \quad \xi_j \quad 1)}} j_{s=0}$$

and the integral term

$$\frac{1}{i(\xi_k^2 + 2\xi_k \quad q)} \int_0^1 ds_k e^{is_k(\xi_k^2+2\xi_k \ q) \quad \epsilon s_k} \partial_{s_k} \left(\Pi_{l=1}^k e^{\ i\sqrt{1+\sum_{l=j}^{k+1} s_l v \ (\xi_j \quad \xi_j \quad 1)}} \right).$$

For the integral term, we keep doing this transformation until we reach $\partial_{s_k}^5$ ($\partial_{s_k}^5$ will bring no more than $(2k)^5$ many terms with each term controlled by $1/(1 + s_k + t_{k+1})^5$).

Step two: we keep doing transformation for the boundary terms. For each boundary term, we break it into two terms ($G \ {}_{2M}(\xi_{k+1} \quad \xi_k, t_{k+1})$ and $G_{>2M}(\xi_{k+1} \quad \xi_k, t_{k+1})$).

Step three: for the term with $G_{>2M}(\xi_{k+1} \quad \xi_k, t_{k+1})$, we keep using Duhamel's formula

$$\mathbb{1} \qquad + i \int_0^1 dt_{k+2} U(t_{k+2}, 0) \qquad . \tag{6.365}$$

For the $\mathbb{1}$ term, it has the same form as $I_\epsilon^{(k+1)} e^{iTH_0}$. For the integral term, we break it into two terms ($G \ {}_{2M}(\xi_{k+2} \quad \xi_{k+1}, t_{k+2})$ and $G_{>2M}(\xi_{k+2} \quad \xi_{k+1}, t_{k+2})$). We keep doing this until we gain $G \ {}_{2M}(\xi_{k+j} \quad \xi_{k+j} \ {}_1, t_{k+j})$ for some j 2 N^+(**type one**) or there is no $U(0, t_{k+j})$(**type two**) in it.

Step four: for the term with $G \ {}_{2M}(\xi_{k+j} \quad \xi_{k+j} \ {}_1, t_{k+j})$, we use Duhamel's formula one more time. Then for the integral term, after change of variables $t_{k+l} = t_{k+j+1} +$

$\sum_{m=k+l}^{k+j} s_m$, $l = 1, \quad , j$, we get

$$\int d^3\xi_1 \quad d^3\xi_{k+j} \int_0^1 dt_{k+j+1} e^{\quad \epsilon t_{k+j+1}} U(t_{k+j+1}, 0) e^{i\xi_{k+j+1}\, x} V(x \quad \sqrt{1+t_{k+j+1}} v) e^{\quad it_{k+j+1}H_0}$$

$$\int_0^1 e^{\quad \epsilon s_{k+j}} ds_{k+j} \quad \int_0^1 e^{\quad \epsilon s_1} ds_1 \delta(s_k) \partial_{s_k}^{b_k} [V(\xi, t_{k+j+1}, s, k+j)] G_M(\xi, t_{k+j+1}, s, k, j) \frac{1}{(2\pi)^{3(k+j)/2}}$$

$$(\quad 1)^{b_k+1} / (i(\xi_k^2 + 2\xi_k \quad P)^{b_k+1}) \beta(jPj > 32M),$$

for some b_k 2 $f0, 1, 2, 3, 4g$, where

$$G_M(\xi, t_{k+j+1}, s, k, j) := G_M(\xi^k, t_{k+j+1}, s^k, j) G_M(\xi^0, t_{k+j+1} + \sum_{l=k+1}^{k+j} s_l, s^0, k). \quad (6.366)$$

Then for ξ_{k+j}, s_{k+j}, we do the same transformation as ξ_k, s_k except that for ξ_{k+j}, s_{k+j}, we stop integration by parts after we gain $\partial_{s_{k+j}}^{b_{k+j}}$ with $b_{k+j} = 5 \quad b_k$. For the boundary terms, we do the same transformation as step two to step four except that we stop until we gain $\partial_{s_{k+j_1}}^{b_{k+j_1}} \quad \partial_{s_{k+j_l}}^{b_{k+j_l}}$ with $b_{k+j_1} + \quad + b_{k+j_l} = 5$. After these transformations, we will get many terms having the following form:

case one:

$$\int d^3\xi_1 \quad d^3\xi_{k+jjj} \int_0^1 dt_{k+jjj+1} e^{\quad \epsilon t_{k+jjj+1}} U(t_{k+jjj+1}, 0) e^{i\xi_{k+jjj+1}\, x} V(x \quad \sqrt{1+t_{k+jjj+1}} v) e^{\quad it_{k+jjj+1}H_0}$$

$$\int_0^1 e^{\quad \epsilon s_{k+jjj}} ds_{k+jjj} \quad \int_0^1 e^{\quad \epsilon s_1} ds_1 \delta(s_k) \delta(s_k + j_1) \quad \delta(s_k + j_1 + \quad + j_{l \quad 1})$$

$$\partial_{s_k}^{b_k} \quad \partial_{s_k+j_1+ \quad +j_l}^{b_{k+l}} [V(\xi, t_{k+jjj+1}, s, k+jjj)] G_M(\xi, t_{k+jjj+1}, s, k, j, l) \frac{1}{(2\pi)^{3(k+jjj)/2}} \beta(jPj > 32M)$$

$$(\quad 1)^{l+b_k+ \quad +b_l} \quad 1/(i(\xi_{k+jjj}^2 + 2\xi_{k+jjj} \quad P))^{b_{k+l}} \quad \Pi_{m=0}^{l \quad 1} 1/(i(\xi_{k+j_1+ \quad j_m}^2 + 2\xi_{k+j_1+ \quad j_m} \quad P))^{b_{k+m}+1}$$

for $b_k + \quad + b_{k+l} = 5$, b_{k+m} 2 N$, m = 0, \quad , m$, where $j = (j_1, \quad , j_l)$ 2 N^l,

$$G_M(\xi, t_{k+jjj+1}, s, k, j, l) := G_M(\xi^{k+j_1+ \quad +j_l \quad 1}, t_{k+jjj+1}, s^{k+j_1+ \quad +j_l \quad 1}, j_l) \quad (6.367)$$

$$G_M(\xi^k, t_{k+jjj+1} + \sum_{m=k+j_1+1}^{k+jjj} s_m, s^k, j_1) G_M(\xi^0, t_{k+jjj+1} + \sum_{l=k+1}^{k+jjj} s_l, s^0, k); \quad (6.368)$$

case two:

$$\int d^3\xi_1 \quad d^3\xi_{k+jjj} \int_0^1 dt_{k+jjj+1} e^{\quad \epsilon t_{k+jjj+1}} e^{i(k+jjj+1)H_0} e^{i\xi_{k+jjj+1}\, x} V(x \quad \sqrt{1+t_{k+jjj+1}} v) e^{\quad it_{k+jjj+1}H_0}$$

$$\int_0^1 e^{\quad \epsilon s_{k+jjj}} ds_{k+jjj} \quad \int_0^1 e^{\quad \epsilon s_1} ds_1 \delta(s_k) \delta(s_k + j_1) \quad \delta(s_k + j_1 + \quad + j_l)$$

$$\partial_{s_k}^{b_k} \quad \partial_{s_k+j_1+ \quad +j_l}^{b_{k+l}} [V(\xi, t_{k+jjj+1}, s, k+jjj)] G_M(\xi, t_{k+jjj+1}, s, k, j, l) \frac{1}{(2\pi)^{3(k+jjj)/2}}$$

$$\Pi_{m=0}^{l} 1/(i(\xi_{k+j_1+ \quad j_m}^2 + 2\xi_{k+j_1+ \quad j_m} \quad P))^{b_{k+m}+1} \beta(jPj > 32M)$$

for $b_k + \cdots + b_{k+l} \le 4$.

Now we deal with ξ_j, s_j with $\beta(|\xi_j + q| > 2M)$. In this case, we do the same transformation as before except that for $s_j \ge 1/\sqrt{M}$, after taking integration by parts in $\xi_{j,l} := \xi_j \cdot e_l$ for some direction e_l, we gain

$$\frac{iv_l\xi_{j,l}(\sqrt{1+t_{k+1}+s_k+\cdots+s_{j+1}} - \sqrt{1+t_{k+1}+s_k+\cdots+s_j})}{s_j} e^{-i\sqrt{1+\sum_{l=j}^{k+1} s_l}\, v\cdot(\xi_j - \xi_{j-1})} \tag{6.369}$$

which means that for some terms, we can only gain

$$\frac{1}{\sqrt{1+t_{k+1}+s_k+\cdots+s_{j+1}} + \sqrt{1+t_{k+1}+s_k+\cdots+s_j}}$$

since we have

$$\mathcal{F}[V(x-\sqrt{1+s}v)](\xi) = \hat{V}(\xi)e^{-i\sqrt{1+s}v\cdot\xi}. \tag{6.370}$$

For these terms, we keep doing the same transformation until we gain

$$\frac{1}{(\sqrt{1+t_{k+1}+s_k+\cdots+s_{j+1}} + \sqrt{1+t_{k+1}+s_k+\cdots+s_j})^a}\cdot\frac{1}{s_j^b} \text{ for } a/2+b>1, \text{ for some } a,b\in\mathbb{N}$$

which means we do this transformation for no more than 3 times. In the end, we deal with $t_{k+|j|+1}$. For case two, we have to face

$$D(T) := \int_0^\infty dt_{k+|j|+1}e^{it_{k+|j|+1}H_0}e^{ix\cdot\xi_{k+|j|}}V(x-\sqrt{1+t_{k+|j|+1}}v)e^{i(T-t_{k+|j|+1})H_0} \tag{6.371}$$

since due to Lemma 6.5.1, other parts are reduced to be translation. We need following lemma:

Lemma 6.5.3. *If $\hat{V}(\xi)\in L^1_\xi$ and $V(x)\in L^1_x$, then*

$$\sup_{T\in\mathbb{R}}|T|^{3/2}\|D(T)\|_{L^1_x\to L^\infty_x} \le C(\|\hat{V}(\xi)\|_{L^1_\xi}+\|V(x)\|_{L^1_x}), \text{ for some } C>0. \tag{6.372}$$

Proof. For $t_{k+|j|+1}\in(1,T-1)\cup(T+1,\infty)$, we use

$$\|e^{it_{k+|j|+1}H_0}e^{ix\cdot\xi_{k+|j|}}V(x-\sqrt{1+t_{k+|j|+1}}v)e^{i(T-t_{k+|j|+1})H_0}\|_{L^1_x\to L^\infty_x} \lesssim \frac{\|V(x)\|_{L^1_x}}{|t_{k+|j|+1}|^{3/2}|T-t_{k+|j|+1}|^{3/2}} \tag{6.373}$$

while for $t_{k+|j|+1}\in(0,1]\cup[T-1,T+1]$, we use cancellation lemma 6.2.1. Then the result follows. □

After all these transformations, based on Lemma 6.5.3, we will obtain no more than $C_1^{k+|j|}$ many terms for some C_1. Then for each term, we have a bound $C_1^{k+|j|}|||V(x)|||_p^{k+|j|+1}/\sqrt{M}^{k+|j|}$. Hence, we have

$$\sup_{T\in\mathbb{R}}|T|^{3/2}\|\text{case two}^{(k+|j|+1)}(T)\|_{L^1_x\to L^\infty_x}\le\frac{(k+|j|)^4C^{k+|j|+1}|||V(x)|||_p^{k+|j|+1}}{\sqrt{M}^{k+|j|}}\tag{6.374}$$

where $(k+|j|)^4$ comes from that for $a:=b_k+\cdots+b_{k+l}\le 4$,

$$\begin{aligned}&|\partial_{s_k}^{b_k}\cdots\partial_{s_k+j_1+\cdots+j_l}^{b_{k+l}}[\Pi_{m=1}^{k+|j|}e^{-i(t_{k+|j|+1}+s_{k+|j|}+\cdots+s_m)(\xi_m-\xi_{m-1})\cdot v}]|\\ \le&\frac{C(k+|j|)^4\max_{j=1,\cdots,k+|j|}(|\xi_j-\xi_{j-1}|+1)^4}{(1+t_{k+|j|+1})^a},\ \text{for some } C>0.\end{aligned}$$

For case one, we need following lemma:

Lemma 6.5.4. *If $V\in L^\infty_tL^1_x\cap L^2_x$ and $\hat V(\xi,t)\in L^\infty_tL^1_\xi$, then*

$$B:=\sup_{|s-t|\le 1}\|U(s,t)\|_{L^1_x\to L^\infty_x}<\infty.\tag{6.375}$$

Proof. By using Duhamel's formula twice,

$$\begin{aligned}U(s,t)=&e^{-i(t-s)H_0}+(-i)\int_0^{t-s}due^{-i[(t-s)-u]H_0}V(x,s+u)e^{-iuH_0}-\int_0^{t-s}du\int_0^u dwe^{-i[(t-s)-u]H_0}\\ &V(x,s+u)U(s+w,s+u)V(x,s+w)e^{-iwH_0}=:A_1+A_2+\int_0^{t-s}du\int_0^u dwA_3(u,w,s,t).\end{aligned}$$

For the first two terms, it is clear when $V(x,t)\in L^\infty_tL^1_x$ and $\hat V(\xi,t)\in L^\infty_tL^1_\xi$. For the last term, when $u\le 1$, we use

$$\sup_{|a|\le 1}\|U(s+w,a+s+w)e^{iaH_0}\|_{L^1_x\to L^\infty_x}<C,\ \text{for some constant } C.\tag{6.376}$$

So in the following, we stick to $u\ge 1$. When there is no singularity, since $U(s,t)$ is unitary on L^2_x, we have

$$\|A_3(u,w,s,t)\|_{L^1_x\to L^\infty_x}\lesssim\frac{\|V(x,t)\|_{L^2_xL^\infty_t}}{|w|^{3/2}|t-s-u|^{3/2}}\tag{6.377}$$

and then it is integrable over $\int_0^{t-s}du\int_0^u dw$ when there is no singularity. When there is a singularity $1/w$, we use

$$U(s+w,s+u)V(x,s+w)e^{-iwH_0}=U(s+w,s+u+w)[U(s+u+w,s+u)e^{-iwH_0}][e^{iwH_0}V(x,s+w)e^{-iwH_0}].\tag{6.378}$$

Since Corollary 6.2.2 tells us $U(s+u+w,s+u)e^{-iwH_0} : L^p_x \to L^p_x$, is bounded by e if w is small enough, we have

$$\|A_3(u,w,s,t)\|_{L^1_x \to L^1_x} \le \frac{C_1(B+1)\|\hat{V}(\xi,t)\|_{L^1_t L^1_\xi}\|V(x,t)\|_{L^1_t L^1_x}}{|t-s-u|^{3/2}} \tag{6.379}$$

for some constant C, where we use

$$\|U(s+w,s+u+w)\|_{L^1_x \to L^1_x} \le B+\frac{1}{u^{3/2}}\sup_{|a|\le 1}\|U(s+w,a+s+w)e^{iaH_0}\|_{L^1_x \to L^1_x} \tag{6.380}$$

Then this part can be controlled by $\int_0^{c_1} dwC_2B$. We choose c_1 small enough such that $C_2c_1 < 1/4$. Similarly, when there is a singularity for $1/(t-s-u)$, we use

$$e^{-i[(t-s)-u]H_0}V(x,s+u)U(s+w,s+u) = [e^{-i[(t-s)-u]H_0}V(x,s+u)e^{i[(t-s)-u]H_0}]$$
$$[e^{-i[(t-s)-u]H_0}U(s+w,s+w+u-(t-s))]U(s+w+u-(t-s),s+u)$$

and then

$$\int_{t-s-c_2}^{t-s} du\int_0^u dw\|A_3(u,w,s,t)\|_{L^1_x \to L^1_x} \le \int_{t-s-c_2}^{t-s} du\int_{c_1}^u dw\frac{C_3B}{|w|^{3/2}}+$$
$$\int_{t-s-c_2}^{t-s} du\int_{u-1)}^u dw\frac{C_3}{(|t-s-w|^{3/2})|w|^{3/2}} \le C_4(B+1)(c_2+c_2^{1/2}).$$

We can choose c_2 small enough such that $C_4(c_2+c_2^{1/2}) < 1/4$. If we have a singularity both for $1/w$ and $1/(t-s-u)$, then we use

$$e^{-i[(t-s)-u]H_0}V(x,s+u)U(s+w,s+u)V(x,s+w)e^{-iwH_0} = [e^{-i[(t-s)-u]H_0}V(x,s+u)e^{i[(t-s)-u]H_0}]$$
$$[e^{-i[(t-s)-u]H_0}U(s+w,s+w+u-(t-s))]U(s+w+u-(t-s),s+u+w)$$
$$[U(s+u+w,s+u)e^{-iwH_0}][e^{iwH_0}V(x,s+w)e^{-iwH_0}].$$

Then we get

$$\|A_3(u,w,s,t)\|_{L^1_x \to L^1_x} \le \frac{C_5B}{|t-s|^{3/2}} \le C_5B. \tag{6.381}$$

Then we choose c_3 small enough in $\int_{t-s-c_3}^{t-s} du\int_0^{c_1} dw$ such that $c_3c_1C_5 < 1/4$. So we have that for each pair s,t with $|s-t| \le 1$,

$$\|U(s,t)\|_{L^1_x \to L^1_x} \le 3/4B+C. \tag{6.382}$$

Take the supremum over $\{(s,t) : |s-t| \le 1\}$ on the left in equation (6.382) and we have

$$B \le 4C. \tag{6.383}$$

Then the conclusion follows. $\square$

Due to Lemma 6.5.4, we have

$$\sup_{T\in\mathbb{R}}|T|^{3/2}\int_0^{\infty}\frac{dt_{k+|j|+1}}{(1+t_{k+|j|+1})^{3/2}}$$
$$\|U(t_{k+|j|+1},0)e^{i\xi_{k+|j|}\cdot x}V(x-\sqrt{1+t_{k+|j|+1}}v)e^{i(T-t_{k+|j|+1})H_0}\|_{L^1\to L^\infty}<\infty$$

where we have $1/(1+t_{k+|j|+1})^{3/2}$ since from $b_k+\cdots+b_l=5$, we gain $1/(1+t_{k+|j|+1}+s_{k+|j|})^{5/2}$. After taking the integral over $s_{t+|k|}$, we have $1/(1+t_{k+|j|+1})^{3/2}$. Hence,

$$\sup_{T\in\mathbb{R}}|T|^{3/2}\|\text{case one}^{(k+|j|+1)}(T)\|_{L^1_x\to L^\infty_x}\lesssim\frac{(k+|j|)^5C^{k+|j|+1}\|\langle x\rangle V(x)\|_p^{k+|j|+1}}{\sqrt{M}^{k+|j|}}. \quad (6.384)$$

Fix $|j|$. For case one, $l\in\{0,1,\cdots,|j|\}$ and for each l and there are $\binom{5+l}{l}\leq 2^{5+|j|}$ many solutions of $(b_k,b_{k+1},\cdots,b_{k+l})\in\mathbb{N}^{l+1}$ satisfying

$$b_k+b_{k+1}+\cdots+b_{k+1}=5.$$

So for $k+|j|$, there are no more than $|j|\cdot 2^{5+|j|}$ many case one terms. For case two, $l\in\{0,1,\cdots,|j|\}$ and for each l and there are $\binom{b+l}{l}\leq 2^{4+|j|}$ many solutions of $(b_k,b_{k+1},\cdots,b_{k+l})\in\mathbb{N}^{l+1}$ satisfying

$$b_k+b_{k+1}+\cdots+b_{k+1}=b,\ \text{for } b=0,1,2,3,4.$$

So there are no more than $5|j|\cdot 2^{4+|j|}$ many case one terms. Thus,

$$\sup_{T\in\mathbb{R}}|T|^{3/2}\|J_{M,T}^{(k+1)}\beta(|P|>32M)\|_{L^1_x\to L^\infty_x}\lesssim\sum_{|j|=1}^{\infty}|j|\cdot 2^{5+|j|}\frac{(k+|j|)^5C^{k+|j|+1}\|\langle x\rangle V(x)\|_p^{k+|j|+1}}{\sqrt{M}^{k+|j|}}$$
$$+5|j|\cdot 2^{4+|j|}\frac{(k+|j|)^4C^{k+|j|+1}\|\langle x\rangle V(x)\|_p^{k+|j|+1}}{\sqrt{M}^{k+|j|}}\lesssim\frac{k^5(C\|\langle x\rangle V(x)\|_p)^k}{\sqrt{M}^k}$$

if M is large enough. $\square$

Lemma 6.5.5. *If $V(x-\sqrt{1+|t|}v)$ satisfies assumption 6.38, then*

$$\sup_{T\in\mathbb{R}}|T|^{3/2}\|K^{(k)}(T)e^{iTH_0}\|_{L^1_x\to L^\infty_x}\lesssim\frac{(C\|\langle x\rangle V(x)\|_p)^k}{\sqrt{M}^{k-1}},\ \text{for } k\in\mathbb{N}^+. \quad (6.385)$$

Proof. Apply Lemma 6.5.1 and change of variables from $t_j \to t_j = s_j + \cdots + s_k$. For ξ_j, s_j, $j = 1, \dots, k-1$, it is the case when $\beta(|\xi_j + P| > 2M)$. We do the same transformation as what we do in the proof of Lemma 6.5.2. Then for each j, we will gain $C|||V(x)|||_p/\sqrt{M}$. For s_k, we apply Lemma 6.5.3 and then get the estimate (6.385). □

Now we can prove its decay estimate. According to the definition of $J^{(k+1)}_{M,\epsilon}$, $K^{(k)}(T)$, we have

$$s\text{-}\lim_{T\to\infty} D(T) := s\text{-}\lim_{T\to\infty} U(T,0)e^{-iTH_0} - \sum_{k=1}^{\infty} i^{k+1} J^{(k+1)}_{M,\epsilon} - \sum_{k=1}^{\infty} i^k K^{(k)}(T) = \mathbb{1}. \quad (6.386)$$

Then we have the following result.

Lemma 6.5.6. *If $V(x - \sqrt{1+|t|v})$ satisfies assumption 6.38, we have*

$$\sup_{T\in\mathbb{R}^+} \|D(T)\|_{L^p_x\to L^p_x} < \infty, \text{ for } 1 \le p \le \infty. \quad (6.387)$$

Proof. The proof is the same as that of Corollary 6.3.5. □

Then the decay estimate follows.

Proof. For $T \ge 0$, it follows from

$$U(0,T) = D(T) + \sum_{k=1}^{\infty} i^{k+1} J^{(k+1)}_{M,\epsilon} + \sum_{k=1}^{\infty} i^k K^{(k)}(T) \quad (6.388)$$

and Lemma 6.5.5, Lemma 6.5.2. For $T < 0$, it follows in the same way. □

6.6 NLS equations

6.6.1 L^∞ boundedness for Hartree-type NLS

L^∞ boundedness for some specific Hartree NLS and the proof of Theorem 6.1.4

We start with an example. Consider Hartree NLS equations of the form:

$$i\partial_t\psi(t) = H_0\psi(t) - \lambda[f * |\psi(t)|^2](x)\psi(t), \quad \psi(0) = \psi_0 \text{ for } f(x,t) \in C_t L^2_x. \quad (6.389)$$

We prove Theorem 6.1.5. In other words, we show that $\psi(t)$ is bounded in L^∞_x uniformly in $t \in (-\infty, -c] \cup [c, \infty)$ for any $c > 0$ if $\psi_0 \in L^1_x \cap L^2_x$. We obtain this result by establishing its advanced CL:

Lemma 6.6.1 (Advanced CL). *If $\psi(t) \in C_t([-T,T])L^2_x \cap L^{8/3}_t([-T,T])L^4_x$, then*

$$\int_{-T}^{T} dt\|K_t(f-|\psi(t)|^2)\|_{L^p_x\to L^p_x} \lesssim T^{1/4}\|f(x)\|_{L^2_x}\|\psi(t)\|^2_{L^{8/3}_t([-T,T])L^4_x}. \tag{6.390}$$

In addition,

$$\int_{-T}^{T} dt\|K_t(f-|\psi(t)|^2)\|_{L^4_tL^p_x\to L^p_x} \lesssim 1. \tag{6.391}$$

We defer the proof of Lemma 6.6.1 to the end of the section. We also have to show that the solution $\psi(t)$ to (6.389) satisfies the assumption of Lemma 6.6.1:

Lemma 6.6.2. *If $\psi_0 \in L^2_x$, then for any $T>0, a\in \mathbb{R}$,*

$$\|\psi(t)\|_{L^{8/3}_t([-T+a,T+a])L^4_x} \lesssim_{T,\|\psi_0\|_{L^2_x}} 1. \tag{6.392}$$

The proof of Lemma 6.6.2 is based on the construction of solution to (6.389) by using CL and iteration scheme and we defer the proof to the end of this section.

All the results can be extended to the perturbed NLS.

We are back to proving Theorem 6.1.5. We stick to $t \geq 0$, $f(x,t)=f(x)$; for $t<0$, the results follow from time reversal symmetry. The case for time-dependent f will follow in the same way.

Proof of Theorem 6.1.5. We stick to $t \geq 1$ and the case for $t \geq c>0$ will follow by the same argument. By using Duhamel's formula, rewrite $\psi(t)$ as

$$\begin{aligned}\psi(t) = e^{-itH_0}\psi_0(x) + (-i)\int_0^{t-1/10} ds_1 e^{-i(t-s_1)H_0}[f-|\psi(s_1)|^2](x)\psi(s_1)+\\ (-i)\int_{t-1/10}^{t} ds_1 e^{-i(t-s_1)H_0}[f-|\psi(s_1)|^2](x)\psi(s_1)\\ =: \psi_1(t)+\psi_2(t)+\psi_3(t).\end{aligned} \tag{6.393}$$

For $\psi_1(t)$, its L^∞_x boundedness follows from the decay estimates of e^{-itH_0}. For $\psi_2(t)$, we have

$$
\begin{aligned}
\|\psi_2(t)\|_{L^\infty_x} &\lesssim \int_0^{t-1/10} ds_1 \frac{1}{|t-s_1|^{3/2}} \|[f * |\psi(s_1)|^2](x)\psi(s_1)\|_{L^1_x} \\
\text{(Hölder's inequality)} &\lesssim \int_0^{t-1/10} ds_1 \frac{1}{|t-s_1|^{3/2}} \|[f * |\psi(s_1)|^2](x)\|_{L^2_x} \|\psi(s_1)\|_{L^2_x} \\
&\lesssim \int_0^{t-1/10} ds_1 \frac{1}{|t-s_1|^{3/2}} \|f(x)\|_{L^2_x} \|\psi(s_1)^2\|_{L^1_x} \|\psi(s_1)\|_{L^2_x} \\
\text{(Hölder's inequality)} &\lesssim \int_0^{t-1/10} ds_1 \frac{1}{|t-s_1|^{3/2}} \|f(x)\|_{L^2_x} \|\psi(s_1)\|^3_{L^2_x} \\
&\lesssim \int_0^{t-1/10} ds_1 \frac{1}{|t-s_1|^{3/2}} \|\psi_0\|^3_{L^2_x} \\
&\lesssim \|\psi_0\|^3_{L^2_x}. \qquad (6.394)
\end{aligned}
$$

For $\psi_3(t)$, we use Duhamel's formula again

$$
\begin{aligned}
\psi_3(t) = (-i)\int_{t-1/10}^{t} ds_1 e^{-i(t-s_1)H_0}[f * |\psi(s_1)|^2](x) e^{-is_1H_0}\psi_0(x) + \\
(-i)^2 \int_{t-1/10}^{t} ds_1 \int_0^{s_1-1/10} ds_2 e^{-i(t-s_1)H_0}[f * |\psi(s_1)|^2](x) e^{-i(s_1-s_2)H_0}[f * |\psi(s_2)|^2](x)\psi(s_2) + \\
(-i)^2 \int_{t-1/10}^{t} ds_1 \int_{s_1-1/10}^{s_1} ds_2 e^{-i(t-s_1)H_0}[f * |\psi(s_1)|^2](x) e^{-i(s_1-s_2)H_0}[f * |\psi(s_2)|^2](x)\psi(s_2) \\
=: \psi_{31}(t) + \psi_{32}(t) + \psi_{33}(t). \qquad (6.395)
\end{aligned}
$$

For $\psi_{31}(t)$, using Lemma 6.6.1, Lemma 6.6.2 and the fact that $e^{-itH_0}\psi_0(x) \in L^\infty_x$ for $t \ \ a \ \ \frac{1}{2}$, we have

$$
\|\psi_{31}(t)\|_{L^\infty_x} \lesssim_{\|\psi_0\|_{L^2_x}} \|\psi_0\|_{L^1_x}. \qquad (6.396)
$$

For $\psi_{32}(t)$, using Lemma 6.6.1(regard $t-s_1$ variable as the time variable), Lemma 6.6.2 and applying the same estimate for $\psi_2(t)$ to

$$
\int_0^{s_1-1/10} ds_2 e^{-i(t-s_2)H_0}[f * |\psi(s_2)|^2](x)\psi(s_2), \qquad (6.397)
$$

we have

$$
\|\psi_{32}(t)\|_{L^\infty_x} \lesssim_{\|\psi_0\|_{L^2_x}} 1. \qquad (6.398)
$$

For $\psi_{33}(t)$, we keep using Duhamel's formula in the same way twice. In the end, it is sufficient to deal with

$$\psi_4(t) := \int_{t-1/10}^{t} ds_1 \int_{s_1-1/10}^{s_1} ds_2 \int_{s_2-1/10}^{s_2} ds_3 \int_{s_3-1/10}^{s_3} ds_4 e^{-i(t-s_1)H_0}[f\ast|\psi(s_1)|^2](x) e^{-i(s_1-s_2)H_0} \cdots e^{-i(s_3-s_4)H_0}[f\ast|\psi(s_4)|^2](x)\psi(s_4). \quad (6.399)$$

We use (6.391) in Lemma 6.6.1

$$\begin{aligned}
\|\psi_4(t)\|_{L_x^1} \lesssim & \int_{t-1/10}^{t} ds_1 \int_{s_1-1/10}^{s_1} ds_2 \int_{s_2-1/10}^{s_2} ds_3 \int_{s_3-1/10}^{s_3} ds_4 \|K_{s_1-t}([f\ast|\psi(s_1)|^2](x))\|_{L_x^1\to L_x^1} \\
& \|K_{s_2-t}([f\ast|\psi(s_2)|^2](x))\|_{L_x^1\to L_x^1}\|K_{s_3-t}([f\ast|\psi(s_3)|^2](x))\|_{L_x^1\to L_x^1} \\
& \|e^{i(s_4-t)H_0}[f\ast|\psi(s_4)|^2](x)\psi(s_4)\|_{L_x^1} \\
\lesssim_{\|\psi_0\|_{L_x^2}} & \int_{t-1/10}^{t} ds_1 \int_{s_1-1/10}^{s_1} ds_2 \int_{s_2-1/10}^{s_2} ds_3 \int_{s_3-1/10}^{s_3} ds_4 \|K_{s_1-t}([f\ast|\psi(s_1)|^2](x))\|_{L_x^1\to L_x^1} \\
& \|K_{s_2-t}([f\ast|\psi(s_2)|^2](x))\|_{L_x^1\to L_x^1}\|K_{s_3-t}([f\ast|\psi(s_3)|^2](x))\|_{L_x^1\to L_x^1}\frac{1}{|t-s_4|^{3/2}} \\
\lesssim_{\|\psi_0\|_{L_x^2}} & \int_{t-1/10}^{t} ds_1 \int_{s_1-1/10}^{s_1} ds_2 \int_{s_2-1/10}^{s_2} ds_3 \|K_{s_1-t}([f\ast|\psi(s_1)|^2](x))\|_{L_x^1\to L_x^1} \\
& \|K_{s_2-t}([f\ast|\psi(s_2)|^2](x))\|_{L_x^1\to L_x^1}\|K_{s_3-t}([f\ast|\psi(s_3)|^2](x))\|_{L_x^1\to L_x^1}\frac{1}{|t-s_3|^{1/2}} \\
\lesssim_{\|\psi_0\|_{L_x^2}} & \int_{t-1/10}^{t} ds_1 \int_{s_1-1/10}^{s_1} ds_2 \|K_{s_1-t}([f\ast|\psi(s_1)|^2](x))\|_{L_x^1\to L_x^1} \\
& \|K_{s_2-t}([f\ast|\psi(s_2)|^2](x))\|_{L_x^1\to L_x^1}\left\|\frac{\chi(s_3\in[s_2-1/10,s_2])}{|t-s_3|^{1/2}}\right\|_{L_{s_3}^4} \quad (6.400)
\end{aligned}$$

that is,

$$\begin{aligned}
\|\psi_4(t)\|_{L_x^1} \lesssim_{\|\psi_0\|_{L_x^2}} & \int_{t-1/10}^{t} ds_1 \int_{s_1-1/10}^{s_1} ds_2 \|K_{s_1-t}([f\ast|\psi(s_1)|^2](x))\|_{L_x^1\to L_x^1} \\
& \|K_{s_2-t}([f\ast|\psi(s_2)|^2](x))\|_{L_x^1\to L_x^1}\frac{1}{|t-s_2|^{1/4}} \\
\lesssim_{\|\psi_0\|_{L_x^2}} & \int_{t-1/10}^{t} ds_1 \|K_{s_1-t}([f\ast|\psi(s_1)|^2](x))\|_{L_x^1\to L_x^1}\left\|\frac{\chi(s_3\in[s_1-1/10,s_1])}{|t-s_2|^{1/4}}\right\|_{L_{s_2}^4} \\
\lesssim_{\|\psi_0\|_{L_x^2}} & 1. \quad (6.401)
\end{aligned}$$

We finish the proof.

□

Based on the proof of Theorem 6.1.5, we find that the proof used only that the potential is in L_x^2 and it satisfies advanced CL. Thus, following a similar argument, we can extend the same result to a perturbed one:

Proof of Theorem 6.1.4 part 1. If $\psi(t)$ exists in L^2_x and satisfies local Strichartz estimate, according to 1,2,1-3, we follow a similar argument of Theorem 6.1.5 except that we may have to use Duhamel's formula for $N = [\frac{k_0^0}{2} + 1] + 1$ times, in order to get the L^1_x boundedness result in Theorem 6.1.4. When $N = [\frac{k_0^0}{2} + 1] + 1$

$$\int_{t\ 1}^{t} ds_1 \int_{t\ 1}^{s_1} ds_2 \quad \int_{t\ 1}^{s_N\ 2} ds_N\ {}_1 j \int_{t\ 1}^{s_N\ 1} ds_N \frac{1}{jt \quad s_N j^{3/2}} j^{k_0^0}$$

$$. \int_{t\ 1}^{t} ds_1 \int_{t\ 1}^{s_1} ds_2 \quad \int_{t\ 1}^{s_N\ 2} ds_N\ {}_1 \frac{1}{jt \quad s_N\ {}_1 j^{\frac{k_0^0}{2}}}$$

$$._{k_0} \frac{1}{jt \quad s_1 j^{\frac{k_0^0}{2}\ (N\ 1)}} j_{s_1 = t\ 1}^{s_1 = t} \ ._{k_0} 1 \tag{6.402}$$

where

$$k_0 = \min(k_1, k_2) \quad \text{and} \quad \frac{1}{k_0^0} + \frac{1}{k_0} = 1. \tag{6.403}$$

For k_1, k_2, see 1, 1. So we have to show (6.40) has global wellposedness in L^2_x and local Strichartz estimate. We will show their proof next, see 6.6.1. □

Proof of Lemma 6.6.1. For (6.390), we only have to check if the Fourier transform of the potential is absolutely integrable or not

$$kF[f \quad j\psi(t)j^2](\xi)k_{L^1_\xi} \quad k\hat{f}(\xi)F[j\psi(t)j^2](\xi)k_{L^1_\xi}$$

$$\text{(Hölder's inequality)} \ . \ k\hat{f}(\xi)k_{L^2_\xi} kF[j\psi(t)j^2](\xi)k_{L^2_\xi}$$

$$\text{(Plancherel theorem)} \ . \ kf(x)k_{L^2_x} k\psi(t)k^2_{L^4_x}. \tag{6.404}$$

Thus,

$$\int_T^T dtkK_t(f \quad j\psi(t)j^2)k_{L^p_x\ !\ L^p_x} \ . \int_T^T dtkf(x)k_{L^2_x} k\psi(t)k^2_{L^4_x}$$

$$\text{(Hölder's inequality)} \ . \ T^{1/4} kf(x)k_{L^2_x} k\psi(t)k^2_{L_t^{8/3}([\ T,T])L^4_x}. \tag{6.405}$$

For (6.391), similarly, with $g(x,t)\ 2\ L^4_t L^p_x$,

$$\int_T^T dtkK_t(f \quad j\psi(t)j^2) g(x,t)k_{L^p_x} \ . \int_T^T dtkf(x)k_{L^2_x} k\psi(t)k^2_{L^4_x} kg(x,t)k_{L^p_x}$$

$$\text{(Hölder's inequality)} \ . \ kf(x)k_{L^2_x} k\psi(t)k_{L_t^{8/3}([\ T,T])L^4_x} kg(x,t)k_{L^4_t L^p_x}. \tag{6.406}$$

We finish the proof. □

Proof of Lemma 6.6.2. It is sufficient to check the case when $a = 0$ and $T > 0$ sufficiently small, if the estimate depends only on $\|\psi_0\|_{L^2_x}$. Then we can extend the result all a with fixed T. For anyfinite $T > 0$, we can use

$$\|\psi(t)\|_{L_t^{8/3}([0,T])L_x^4} \le \sum_{j=0}^{N} \|\psi(t)\|_{L_t^{8/3}([T_j,T_{j+1}])L_x^4} \tag{6.407}$$

with $T_0 = 0, T_{N+1} = T$, where N is a sufficiently large number.

Now we go back to prove the case when $a = 0$ and $T > 0$ sufficiently small. It follows from an iteration scheme: set $\psi_1(t) = e^{-itH_0}\psi_0(x)$ and $\psi_{n+1}(t)$ satisfies

$$\begin{cases} i\partial_t \psi_{n+1}(t) = (-\Delta_x + f * |\psi_n(t)|^2)\psi_{n+1}(t) \\ \psi_{n+1}(0) = \psi_0 \end{cases}, \quad t \in [0, T]. \tag{6.408}$$

According to Lemma 6.6.1 and Strichartz estimates for e^{itH_0}, we have

$$\|\psi_{n+1}(t)\|_{L_t^{8/3}L_x^4([0,T]\times\mathbb{R}^3)} \le \sum_{j=0}^{\infty} \left(CT^{1/4}\|f(x)\|_{L^2_x}\|\psi_n(t)\|^2_{L_t^{8/3}([-T,T])L_x^4} \right)^j \tag{6.409}$$

and

$$\|\psi_{n+1}(t)\|_{L^2_x} \le \sum_{j=0}^{\infty} \left(CT^{1/4}\|f(x)\|_{L^2_x}\|\psi_n(t)\|^2_{L_t^{8/3}([-T,T])L_x^4} \right)^j \tag{6.410}$$

for some constant $C > 0$. From (6.409), we see if

$$\|\psi_n(t)\|_{L_t^{8/3}([-T,T])L_x^4} \le 2\|e^{-itH_0}\psi_0\|_{L_t^{8/3}L_x^4} \le 2C_{str}\|\psi_0\|_{L^2_x} \tag{6.411}$$

($C_{str} := \|e^{itH_0}\|_{L^2_x \to L_t^{8/3}L_x^4}$) and if we take $T > 0$ small enough such that

$$4CT^{1/4}\|f(x)\|_{L^2_x}C^2_{str}\|\psi_0\|_{L^2_x} \le \frac{1}{2}, \tag{6.412}$$

then

$$\|\psi_n(t)\|_{L_t^{8/3}L_x^4([0,T]\times\mathbb{R}^3)} \le 2C_{str}\|\psi_0\|_{L^2_x} \tag{6.413}$$

implies

$$\|\psi_{n+1}(t)\|_{L_t^{8/3}L_x^4([0,T]\times\mathbb{R}^3)} \le 2C_{str}\|\psi_0\|_{L^2_x}. \tag{6.414}$$

Since

$$\|\psi_1(t)\|_{L_t^{8/3}L_x^4([0,T]\times\mathbb{R}^3)} \le C_{str}\|\psi_0\|_{L^2_x} \le 2C_{str}\|\psi_0\|_{L^2_x}, \tag{6.415}$$

we have for all $n = 1, \dots$,

$$\|\psi_n(t)\|_{L_t^{8/3}L_x^4([0,T]\times\mathbb{R}^3)} \leq 2C_{str}\|\psi_0\|_{L_x^2} \tag{6.416}$$

if (6.412) is satisfied. Now we use standard contraction mapping argument to show ψ_n converges both in L_x^2 and $L_t^{8/3}([0,T])L_x^4$:

$$\begin{aligned}
\|\psi_{n+1}(t) - \psi_n(t)\|_{L_x^2} &\leq \int_0^t ds\|K_s([f * |\psi_n(s)|^2](x))\|_{L_x^2\to L_x^2}\|\psi_{n+1}(s) - \psi_n(s)\|_{L_x^2} + \\
&\int_0^t ds\|K_s([f * (|\psi_n(s)|^2 - |\psi_{n-1}(s)|^2)](x))\|_{L_x^2\to L_x^2}\|\psi_n(s)\|_{L_x^2} \\
&\leq CT^{1/4}\|f(x)\|_{L_x^2}(2C_{str}\|\psi_0\|_{L_x^2})^2 \sup_{t\in[0,T]}\|\psi_{n+1}(t) - \psi_n(t)\|_{L_x^2} + \\
CT^{1/4}\|f(x)\|_{L_x^2} \cdot 4C_{str}\|\psi_0\|_{L_x^2} &\cdot 2\|\psi_0\|_{L_x^2}\|\psi_n(t) - \psi_{n-1}(t)\|_{L_t^{8/3}([0,T])L_x^4}
\end{aligned} \tag{6.417}$$

where we use

$$\||\psi_n(t)| - |\psi_{n-1}(t)|\|_{L_t^{8/3}([0,T])L_x^4} \leq \|\psi_n(t) - \psi_{n-1}(t)\|_{L_t^{8/3}([0,T])L_x^4}. \tag{6.418}$$

Then we have

$$\begin{aligned}
\sup_{t\in[0,T]}\|\psi_{n+1}(t) - \psi_n(t)\|_{L_x^2} &\leq CT^{1/4}\|f(x)\|_{L_x^2}(2C_{str}\|\psi_0\|_{L_x^2})^2 \sup_{t\in[0,T]}\|\psi_{n+1}(t) - \psi_n(t)\|_{L_x^2} + \\
CT^{1/4}\|f(x)\|_{L_x^2} \cdot 4C_{str}\|\psi_0\|_{L_x^2} &\cdot 2\|\psi_0\|_{L_x^2}\|\psi_n(t) - \psi_{n-1}(t)\|_{L_t^{8/3}([0,T])L_x^4}.
\end{aligned} \tag{6.419}$$

Similarly, we have

$$\begin{aligned}
\|\psi_{n+1}(t) - \psi_n(t)\|_{L_t^{8/3}([0,T])L_x^4} &\leq C_{str}\int_0^t ds\|K_s([f * |\psi_n(s)|^2](x))\|_{L_x^2\to L_x^2}\|\psi_{n+1}(s) - \psi_n(s)\|_{L_x^2} + \\
&C_{str}\int_0^t ds\|K_s([f * (|\psi_n(s)|^2 - |\psi_{n-1}(s)|^2)](x))\|_{L_x^2\to L_x^2}\|\psi_n(s)\|_{L_x^2} \\
&\leq C_{str}CT^{1/4}\|f(x)\|_{L_x^2}(2C_{str}\|\psi_0\|_{L_x^2})^2 \sup_{t\in[0,T]}\|\psi_{n+1}(t) - \psi_n(t)\|_{L_x^2} + \\
C_{str}CT^{1/4}\|f(x)\|_{L_x^2} \cdot 4C_{str}\|\psi_0\|_{L_x^2} &\cdot 2\|\psi_0\|_{L_x^2}\|\psi_n(t) - \psi_{n-1}(t)\|_{L_t^{8/3}([0,T])L_x^4}.
\end{aligned} \tag{6.420}$$

Thus, by taking T small enough such that we get

$$\sup_{t\in[0,T]}\|\psi_{n+1}(t) - \psi_n(t)\|_{L_x^2} \leq \frac{1}{3}\sup_{t\in[0,T]}\|\psi_{n+1}(t) - \psi_n(t)\|_{L_x^2} + \frac{1}{3}\|\psi_n(t) - \psi_{n-1}(t)\|_{L_t^{8/3}([0,T])L_x^4} \tag{6.421}$$

from (6.419), and

$$\|\psi_{n+1}(t)-\psi_n(t)\|_{L_t^{8/3}([0,T])L_x^4} \le \frac{1}{3}\sup_{t\in[0,T]}\|\psi_{n+1}(t)-\psi_n(t)\|_{L_x^2}+\frac{1}{3}\|\psi_n(t)-\psi_{n-1}(t)\|_{L_t^{8/3}([0,T])L_x^4} \tag{6.422}$$

from (6.420). Hence, according to (6.421), (6.422), we get

$$\|\psi_{n+1}(t)-\psi_n(t)\|_{L_t^{8/3}([0,T])L_x^4} \le \frac{5}{6}\|\psi_n(t)-\psi_{n-1}(t)\|_{L_t^{8/3}([0,T])L_x^4} \tag{6.423}$$

and

$$\sup_{t\in[0,T]}\|\psi_{n+1}(t)-\psi_n(t)\|_{L_x^2} \le \frac{1}{2}\|\psi_n(t)-\psi_{n-1}(t)\|_{L_t^{8/3}([0,T])L_x^4}. \tag{6.424}$$

Thus, by contraction mapping argument, we get that $\psi_n(t)$ converges to $\psi(t)$ in $L_t^{8/3}([0,T])L_x^4$ and therefore converges to $\psi(t)$ in in $C_t([0,T])L_x^2$. Thus,

$$\|\psi(t)\|_{L_t^{8/3}([0,T])L_x^4} \le 2C_{str}\|\psi_0\|_{L_x^2} \tag{6.425}$$

due to (6.416). We finish the proof.

□

Proof of Theorem 6.1.4 part 2. Based on the proof of Lemma 6.6.1 and Lemma 6.6.2, we get the global wellposedness of (6.40) in L_x^2. (For L_x^2, local wellposedness is equivalent to global wellposedness) Its local Strichartz estimates follow by using conditions 1, 1 and 2. Here condition 1 is used to establish the local Strichartz estimates for $U_V(t,0)$ with $U_V(t,0)$, the semigroup generated by $H_0+V(x,t)$. We finish the proof of Theorem 6.1.4. □

Typical examples

Here are some typical examples:

Example 6.6.1 (Global wellposedness)**.** *When*

$$N(|\psi(t)|) = -\lambda[\frac{1}{|x|^{3/2-\delta}} * |\psi(t)|^2](x), \text{ for } \delta\in(0,\frac{3}{2}), \lambda>0, \tag{6.426}$$

$$i\partial_t\psi(t) = (H_0+V(x,t))\psi(t) + N(|\psi(t)|)\psi(t), \quad \psi(0)=\psi_0, \tag{6.427}$$

with $V(x,t)$, satisfying conditions 1, 2,has global wellposedness in L_x^2.

Proof. Compute its $\mathcal{F}L^1_x$ norm,

$$\begin{aligned}\|N(|\psi(t)|)\|_{\mathcal{F}L^1_x} &= \Big\|\frac{1}{|\xi|^{3/2+\delta}}\mathcal{F}[|\psi(t)|^2](\xi)\Big\|_{L^1_\xi}\\ &\leq \Big\|\frac{\chi(|\xi|\leq 1)}{|\xi|^{3/2+\delta}}\mathcal{F}[|\psi(t)|^2](\xi)\Big\|_{L^1_\xi} + \Big\|\frac{\chi(|\xi|>1)}{|\xi|^{3/2+\delta}}\mathcal{F}[|\psi(t)|^2](\xi)\Big\|_{L^1_\xi}\\ \text{(Hölder's inequality)} &\lesssim_\delta \|\psi(t)\|^2_{L^2_x} + \Big\|\frac{\chi(|\xi|>1)}{|\xi|^{3/2+\delta}}\Big\|_{L^2_\xi}\|\mathcal{F}[|\psi(t)|^2](\xi)\|_{L^2_\xi}\\ &\lesssim_\delta \|\psi(t)\|^2_{L^2_x} + \|\psi(t)\|^2_{L^4_x}. \end{aligned} \tag{6.428}$$

Take $k_1 = \frac{4}{3}$ and we have

$$\|N(|\psi(t)|)\|_{L^{4/3}_t([-T,T])\mathcal{F}L^1_x} \lesssim_\delta \|\psi(t)\|^2_{C_t([-T,T])L^2_x} + \|\psi(t)\|^2_{L^{8/3}_t([-T,T])L^4_x}. \tag{6.429}$$

So (6.42) is satisfied. Similarly,

$$\begin{aligned}\|N(|\psi(t)|) - N(|\phi(t)|)\|_{\mathcal{F}L^1_x} &= \Big\|\Big[\frac{1}{|x|^{3/2-\delta}} * (|\psi(t)| - |\phi(t)|)(|\psi(t)| + |\phi(t)|)\Big]\Big\|_{\mathcal{F}L^1_x}\\ &\lesssim \|(|\psi(t)| - |\phi(t)|)(|\psi(t)| + |\phi(t)|)\|_{L^1_x} + \|(|\psi(t)| - |\phi(t)|)(|\psi(t)| + |\phi(t)|)\|_{L^2_x}\\ &\lesssim \|\psi(t) - \phi(t)\|_{L^2_x}(\|\psi(t)\|_{L^2_x} + \|\phi(t)\|_{L^2_x}) + \|\psi(t) - \phi(t)\|_{L^4_x}(\|\psi(t)\|_{L^4_x} + \|\phi(t)\|_{L^4_x}).\end{aligned} \tag{6.430}$$

Then

$$\begin{aligned}\int_{-T}^{T} dt\|N(|\psi(t)|) - N(|\phi(t)|)\|_{\mathcal{F}L^1_x} &\lesssim T\|\psi(t) - \phi(t)\|_{C_t([-T,T])L^2_x}(\|\psi(t)\|_{C_t(-T,T)L^2_x} + \|\phi(t)\|_{C_t(-T,T)L^2_x})\\ &+ T^{1/4}\|\psi(t) - \phi(t)\|_{L^{8/3}_t([-T,T])L^4_x}(\|\psi(t)\|_{L^{8/3}_t([-T,T])L^4_x} + \|\phi(t)\|_{L^{8/3}_t([-T,T])L^4_x}).\end{aligned} \tag{6.431}$$

So (6.43) is satisfied. Thus, we have global wellposedness for (6.427). □

Example 6.6.2 (Global wellposedness and L^∞ boundedness)**.** *When*

$$N(|\psi(t)|) = -\lambda\Big[\frac{e^{-c|x|}}{|x|^{3/2-\delta}} * |\psi(t)|^2\Big](x), \text{ for } \delta \in (0, \frac{3}{2}), \lambda > 0, c > 0, \tag{6.432}$$

$$i\partial_t\psi(t) = (H_0 + V(x,t))\psi(t) + N(|\psi(t)|)\psi(t), \quad \psi(0) = \psi_0, \tag{6.433}$$

with $V(x,t)$, satisfying 1, 2, has global wellposedness in L^2_x and for any $c_0 > 0$,

$$\sup_{|t|\leq c_0}\|\psi(t)\|_{L^\infty_x} \lesssim_{c_0, \|\psi_0\|_{L^1_x \cap L^2_x}} 1. \tag{6.434}$$

Proof. Since

$$F[\frac{e^{-c|x|}}{|x|^{\frac{3}{2}-\delta}}](\xi) \sim \frac{1}{\langle\xi\rangle^{\frac{3}{2}+\delta}}, \tag{6.435}$$

similarly, following the same estimate for Example 6.6.1, (6.42), (6.43) are satisfied and we get global wellposedness in L^2_x. In this case, according to Hölder's inequality, we have

$$\begin{aligned}\|N(|\psi(t)|)\psi(t)\|_{L^\infty_x} &\lesssim \|[\frac{e^{-c|x|}}{|x|^{3/2-\delta}} * |\psi(t)|^2](x)\|_{L^2_x}\|\psi(t)\|_{L^2_x}\\ &\lesssim \|\frac{e^{-c|x|}}{|x|^{3/2-\delta}}\|_{L^2_x}\|\psi(t)\|^3_{C([-T,T])L^2_x}.\end{aligned} \tag{6.436}$$

So 3 is satisfied and we conclude (6.433) has global wellposedness in L^2_x and

$$\sup_{|t|\leq c_0}\|\psi(t)\|_{L^\infty_x} \lesssim_{c_0,\|\psi_0\|_{L^\infty_x\cap L^2_x}} 1. \tag{6.437}$$

□

6.6.2 Uniform L^p boundedness of wave operators for NLS equations for $2\leq p\leq\infty$

In this section, we prove Theorem 6.1.6 and Theorem 6.1.7.

L^∞ boundedness of $e^{itH_0}U(t,0)-1$

We show the L^∞_x boundedness of $e^{itH_0}U(t,0)-1$(uniformly in $t\in[-\infty,\infty]$) on $L^p_x\cap H^1_x$ for $6<p\leq\infty$ by using the method of ItT potential(ACL). If we only assume $\psi_0\in H^1_x$ instead of $\psi_0\in H^1_x\cap L^p_x$, then $(e^{itH_0}U(t,0)-1)\psi_0$ is in $L^\infty_x+\mathcal{F}L^{1+\epsilon}_x$ for any $\epsilon\in(0,1)$, see Lemma 6.6.4. As an application of Lemma 6.6.4, we prove a similar result for $U(t,0)-e^{-itH_0}$, see Corollary 6.6.1. As an application of Theorem 6.1.6, we are able to get similar result for $U(t,0)$, see Lemma 6.6.6.

Proof of Theorem 6.1.6. We consider the case of L^∞ boundedness and begin with $t=\infty$. Let $\psi_0(x)\in H^1_x$. Then due to (6.57), we have $\psi(t)\in H^1_x$ uniformly in t. In the following context of the proof, $\psi(t)\in H^1_x$ uniformly in $t\in\mathbb{R}$. We will give a proof for $\Omega-1$ and by replacing ∞ with t, we will get the same result for $e^{itH_0}U(t,0)-1$.

According to Duhamel's formula, we have

$$i(\Omega_{+} - 1)\psi_0(x) = \int_1^{\infty} dse^{isH_0} N(|\psi(s)|)\psi(s) + \int_0^1 dse^{isH_0}[\beta(|P| > \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}})N(|\psi(s)|)]\psi(s) +$$
$$\int_0^1 dse^{isH_0}[\beta(|P| \leq \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}})N(|\psi(s)|)]e^{-isH_0}\psi_0(x) + \int_0^1 dse^{isH_0}[\beta(|P| \leq \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}})N(|\psi(s)|)]e^{-isH_0}\psi_1(s)$$
$$=: i\,[\,J_1(\psi_0) + J_2(\psi_0) + J_3(\psi_0) + J_4(\psi_0)], \quad (6.438)$$

where

$$\psi_1(s) := (-i)\int_0^s due^{iuH_0} N(|\psi(s)|)\psi(u). \quad (6.439)$$

For $J_1(\psi_0)$, we have

$$\|J_1(\psi_0)\|_{L^\infty_x} \lesssim \int_1^\infty ds\frac{1}{s^{3/2}}\|N(|\psi(s)|)\|_{L^2_x}\|\psi(s)\|_{H^1_x} \lesssim C(\|\psi_0(x)\|_{H^1_x}). \quad (6.440)$$

In order to estimate

$$\int_0^1 dse^{isH_0} N(|\psi(s)|)\psi(s), \quad (6.441)$$

we break it into 3 parts($J_2(\psi_0)$, $J_3(\psi_0)$, $J_4(\psi_0)$) and estimate them separately.

For $J_2(\psi_0)$, we have

$$\|J_2(\psi_0)\|_{L^\infty_x} \lesssim \sum_{l=1}^{3}\int_0^1 ds\|e^{isH_0}\left[\frac{1}{P_l}\beta_l(|P| > \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}})P_l[N(|\psi(s)|)]\right]\psi(s)\|_{L^\infty_x}$$
$$\text{(Hölder's inequality)} \lesssim \sum_{l=1}^{3}\int_0^1 dss^{\frac{1}{2}+\frac{\epsilon}{2}}\frac{1}{s^{3/2}}\|P_l[N(|\psi(s)|)]\|_{L^{6/5}_x}\|\psi(s)\|_{L^6_x}$$
$$\text{(Since } \epsilon > 0) \lesssim_\epsilon C(\|\psi_0(x)\|_{H^1_x}) \quad (6.442)$$

where $\epsilon > 0$ will be chosen later(see (6.454), (6.471)), $\beta_l(|P| > \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}})(l = 1, 2, 3)$ is defined by

$$\begin{cases} \beta_1(P > \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}}) := \beta(P_1 > \frac{1}{100s^{\frac{1}{2}+\frac{\epsilon}{2}}})\beta(|P| > \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}}) \\ \beta_2(|P| > \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}}) := \beta(P_2 > \frac{1}{100s^{\frac{1}{2}+\frac{\epsilon}{2}}})\bar{\beta}(P_1 > \frac{1}{100s^{\frac{1}{2}+\frac{\epsilon}{2}}})\beta(|P| > \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}}) \\ \beta_3(P > \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}}) := \beta(P_3 > \frac{1}{100s^{\frac{1}{2}+\frac{\epsilon}{2}}})\bar{\beta}(P_2 > \frac{1}{100s^{\frac{1}{2}+\frac{\epsilon}{2}}})\bar{\beta}(P_1 > \frac{1}{100s^{\frac{1}{2}+\frac{\epsilon}{2}}})\beta(P > \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}}) \end{cases}. \quad (6.443)$$

Here we also use

$$\|\frac{1}{P_l}\beta(P_l > \frac{1}{100s^{\frac{1}{2}+\frac{\epsilon}{2}}})\|_{L^{6/5}_x \to L^{6/5}_x} \lesssim s^{\frac{1}{2}+\frac{\epsilon}{2}}, \quad (6.444)$$

see Lemma 6.3.2, and using (6.57),

$$\|P_l[N(|\psi(s)|)]\|_{L_x^{6/5}} \lesssim \|N'(|\psi(s)|) \quad |P_l[\psi(s)]|\|_{L_x^{6/5}} + \|N'(|\psi(s)|) \quad |P_l[\psi \ (s)]|\|_{L_x^{6/5}}$$
$$\lesssim C(\|\psi(s)\|_{H_x^1}) \lesssim C(\|\psi_0(x)\|_{H_x^1}). \quad (6.445)$$

For $J_3(\psi_0)$, we need the method of ItT.

Lemma 6.6.3 (ItT for NLS-1)**.** *If $\psi_0 \in H_x^1 \cap L_x^p$ for some $p \in (6, \infty]$, then*

$$\|J_3(\psi_0)\|_{L_x^\infty} \lesssim C(\|\psi_0(x)\|_{H_x^1}, \|\psi_0(x)\|_{L_x^p}). \quad (6.446)$$

Proof. According to the standard computation for tT potential, we have

$$J_3(\psi_0) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_0^1 ds \int d^3\xi \beta(|\xi| - \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}}) \hat{V}(\xi, s) e^{i(x\cdot\xi + s\xi^2)} \psi_0(x + 2s\xi) \quad (6.447)$$

where

$$V(x, s) := N(|\psi(s)|). \quad (6.448)$$

Control the L_x^∞ norm of $J_3(\psi_0)$ directly

$$\|J_3(\psi_0)\|_{L_x^\infty} \lesssim \sup_{x\in\mathbb{R}^3} \int_0^1 ds \int d^3\xi \beta(|\xi| - \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}}) |\hat{V}(\xi, s)| |\psi_0(x + 2s\xi)|$$
$$\text{(Hölder's inequality)} \lesssim \sup_{x\in\mathbb{R}^3} \int_0^1 ds \|\beta(|\xi| - \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}})\|_{L_\xi^q} \|\hat{V}(\xi, s)\|_{L_\xi^2} \|\psi_0(x + 2s\xi)\|_{L_\xi^p}$$
$$\lesssim \int_0^1 ds \frac{1}{s^{\frac{3}{2q}+\frac{3\epsilon}{2q}}} C(\|\psi(s)\|_{H_x^1}) \|\psi_0(x)\|_{L_x^p} \quad \frac{1}{s^{3/p}}$$
$$\lesssim_{\epsilon,p} C(\|\psi_0(x)\|_{H_x^1}) \|\psi_0(x)\|_{L_x^p} \quad (6.449)$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} \quad (6.450)$$

and we use that

$$\frac{3}{2q} + \frac{3\epsilon}{2q} + \frac{3}{p} = \frac{3}{2} - \frac{3}{2q} + \frac{3\epsilon}{2q} \quad (6.451)$$
$$= \frac{3}{2} - \frac{3}{2q}(1 - \epsilon) \quad (6.452)$$
$$< 1 \quad (6.453)$$

if we choose $\epsilon > 0$ small enough such that

$$\frac{3}{q}(1 - \epsilon) > 1 \quad (6.454)$$

and this can be achieved since $q < 3$ due to $p > 6$.

□

According to Lemma 6.6.3, we have

$$\| J_3(\psi_0)\|_{L^\infty_x} \lesssim C(\|\psi_0(x)\|_{H^1_x}, \|\psi_0(x)\|_{L^{p_0}_x}). \tag{6.455}$$

For $J_4(\psi_0)$, we need following lemma:

Lemma 6.6.4. *If $\psi_0 \in H^1_x$ and N satisfies (6.57), then in (6.56), for any $\epsilon_1 \in (0,1)$, $\psi_1(s) \in L^\infty_x + FL^{1+\epsilon_1}_x$ and its $L^\infty_x + FL^{1+\epsilon_1}_x$ norm is uniformly in $s \in \mathbb{R}$. To be precise,*

$$\sup_{s\in\mathbb{R}} \|\psi_1(s)\|_{L^\infty_x + FL^{1+\epsilon_1}_x} \lesssim_{\epsilon_1} C(\|\psi_0(x)\|_{H^1_x}), \tag{6.456}$$

that is,

$$\sup_{s\in\mathbb{R}} \|e^{isH_0}\psi(s) - \psi_0\|_{L^\infty_x + FL^{1+\epsilon_1}_x} \lesssim_{\epsilon_1} C(\|\psi_0(x)\|_{H^1_x}). \tag{6.457}$$

Proof. Choose $\psi_0 \in H^1_x$. Due to the assumptions on N, $\psi(t) \in H^1_x$ uniformly in $t \in \mathbb{R}$. It is sufficient to look at

$$J_{11}(\psi_0)(s) := \int_0^{\min\{1,s\}} du e^{iuH_0} N(|\psi(u)|)\psi(u) \tag{6.458}$$

since for $s \geq 1$, due to (6.57),

$$\begin{aligned} \|\int_1^s du e^{iuH_0} N(|\psi(u)|)\psi(u)\|_{L^\infty_x} &\lesssim \int_1^s du \frac{1}{u^{3/2}} \|N(|\psi(u)|)\|_{L^2_x}\|\psi(u)\|_{L^2_x} \\ &\lesssim \int_1^\infty du \frac{1}{u^{3/2}} C(\|\psi(u)\|_{H^1_x}) \lesssim C(\|\psi_0(x)\|_{H^1_x}). \end{aligned} \tag{6.459}$$

Break $J_{11}(\psi_0)$ into two parts

$$\begin{aligned} J_{11}(\psi_0)(s) = &\int_0^{\min\{1,s\}} du \beta(|P| > \frac{1}{u^{\frac{1}{2}+\frac{\epsilon_1}{2}}}) e^{iuH_0} N(|\psi(u)|)\psi(u) + \\ &\int_0^{\min\{1,s\}} du \beta(|P| \leq \frac{1}{u^{\frac{1}{2}+\frac{\epsilon_1}{2}}}) e^{iuH_0} N(|\psi(u)|)\psi(u) \\ &=: J_{1,11}(\psi_0)(s) + J_{1,12}(\psi_0)(s). \end{aligned} \tag{6.460}$$

For $J_{1,11}(\psi_0)(s)$, we break $\beta(|P| > \frac{1}{u^{\frac{1}{2}+\frac{\epsilon_1}{2}}})$ into 3 pieces

$$\beta(|P| > \frac{1}{u^{\frac{1}{2}+\frac{\epsilon_1}{2}}}) = \sum_{l=1}^{3} \beta_l(|P| > \frac{1}{u^{\frac{1}{2}+\frac{\epsilon_1}{2}}}), \tag{6.461}$$

where for β_l, see (6.443).

The L^1 estimate for $J_{1,11}(\psi_0)$ follows from (6.57),

$$\begin{aligned}\|J_{1,11}(\psi_0)\|_{L^1_x} &\lesssim \sum_{l=1}^{3}\int_0^1 du \|\frac{1}{P_l}\beta_l(|P| > \frac{1}{u^{\frac{1}{2}+\frac{\epsilon_1}{2}}})e^{iuH_0}P_l[N(|\psi(u)|)\psi(u)]\|_{L^1_x}\\ &\lesssim \sum_{l=1}^{3}\int_0^1 du u^{\frac{1}{2}+\frac{\epsilon_1}{2}}\frac{1}{u^{3/2}}\|P_l[N(|\psi(u)|)\psi(u)]\|_{L^1_x}\\ &\lesssim_{\epsilon_1} C(\|\psi_0(x)\|_{H^1_x})\end{aligned} \tag{6.462}$$

where we used

$$\|\frac{1}{P_l}\beta(P_l > \frac{1}{100u^{\frac{1}{2}+\frac{\epsilon_1}{2}}})\|_{L^1_x\to L^1_x} \lesssim u^{\frac{1}{2}+\frac{\epsilon_1}{2}}, \tag{6.463}$$

and by to (6.445)

$$\begin{aligned}\|P_l[N(|\psi(u)|)\psi(u)]\|_{L^1_x} &\lesssim \|N(|\psi(u)|)\|_{L^2_x}\|P_l[\psi(u)]\|_{L^2_x}+\|P_l[N'(|\psi(u)|)]\|_{L^{6/5}_x}\|\psi(u)\|_{L^6_x}\\ &\lesssim C(\|\psi_0(x)\|_{H^1_x}).\end{aligned} \tag{6.464}$$

For $J_{1,12}(\psi_0)$, compute its Fourier transform

$$\mathcal{F}[J_{1,12}(\psi_0)](\xi) = \int_0^{\min\{1,s\}} du\beta(|\xi| \leq \frac{1}{u^{\frac{1}{2}+\frac{\epsilon_1}{2}}})e^{iu\xi^2}\hat{\phi}(\xi,u) \tag{6.465}$$

with

$$\phi(x,u) := N(|\psi(u)|)\psi(u). \tag{6.466}$$

Then

$$\begin{aligned}|\mathcal{F}[J_{1,12}(\psi_0)](\xi)| &\lesssim \sum_{l=1}^{3}\int_0^1 du\beta(|\xi| \leq \frac{1}{u^{\frac{1}{2}+\frac{\epsilon_1}{2}}})\beta_l(\xi) \frac{1}{|\xi|}|\xi_l\hat{\phi}(\xi,u)|\\ &\lesssim \int_0^1 du\beta(|\xi| \leq \frac{1}{u^{\frac{1}{2}+\frac{\epsilon_1}{2}}}) \frac{1}{|\xi|}C(\|\psi_0(x)\|_{H^1_x}) \lesssim \frac{1}{|\xi|^{1+\frac{2}{1+\epsilon_1}}}C(\|\psi_0(x)\|_{H^1_x}) \in L^1_\xi + L^{1+\epsilon_1}_\xi\end{aligned} \tag{6.467}$$

where we used (6.464) and

$$|\xi_l\hat{\phi}(\xi,u)| \lesssim \|P_l[N(|\psi(u)|)\psi(u)]\|_{L^1_x}. \tag{6.468}$$

Thus, $J_{1,12}(\psi_0) \in L^\infty_x + \mathcal{F}L^{1+\epsilon_1}_x$ and finish the proof. □

Remark 40. *If in addition $\psi_0 \in L^p_x$ for some $p \in [1, \frac{6}{5})$, then based on Lemma 6.6.4, we have $\psi(t) \in L^{p'}_x$ ($p' > 6$ since $p < \frac{6}{5}$), which implies that in (6.468) $\xi_l \hat{\phi}(\xi, u) \in L^q_\xi$ with $1/q + 5/6 + \frac{1}{p'} = 1$. If we choose ϵ wisely, we are able to get $\mathcal{F}[\mathcal{J}_{1,12}(\psi_0)](\xi) \in L^1_\xi$ and have $\psi(t) - e^{-itH_0}\psi_0 \in L^\infty_x$. For detailed statement, see Lemma 6.6.6.*

Corollary 6.6.1. *If $\psi_0 \in H^1_x$ and $\mathcal{N}$ satisfies (6.57), then in (6.56), for any $\epsilon_1 \in (0,1)$, $\psi_1(s) \in L^\infty_x + \mathcal{F}L^{1+\epsilon_1}_x$ and its $L^\infty_x + \mathcal{F}L^{1+\epsilon_1}_x$ norm is uniformly bounded in $s \in \mathbb{R}$. To be precise,*

$$\sup_{s\in\mathbb{R}} \|\psi(s) - e^{-isH_0}\psi_0\|_{L^\infty_x + \mathcal{F}L^{1+\epsilon_1}_x} \lesssim_{\epsilon_1} C(\|\psi_0(x)\|_{H^1_x}). \tag{6.469}$$

According to Lemma 6.6.4and by interpolation we have $\psi_1(x,t) \in L^p_x$ for any $p \in [2, \infty)$ uniformly in t and we get the ItT potential estimate for $\mathcal{J}_4(\psi_0)$:

Lemma 6.6.5 (ItT for NLS-2)**.** *If $\psi_0 \in H^1_x$, then*

$$\|\mathcal{J}_4(\psi_0)\|_{L^\infty_x} \lesssim C(\|\psi_0(x)\|_{H^1_x}). \tag{6.470}$$

Proof. We have

$$\begin{aligned}
\|\mathcal{J}_4(\psi_0)\|_{L^\infty_x} &\lesssim \int_0^\infty ds \int d^3\xi\, \beta\Big(|\xi| - \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}}\Big)|\hat{V}(\xi,s)||\psi_1(x+2s\xi,s)| \\
\text{(Hölder's inequality)} &\lesssim \int_0^\infty ds \Big\|\beta\Big(|\xi| - \frac{1}{s^{\frac{1}{2}+\frac{\epsilon}{2}}}\Big)\Big\|_{L^2_\xi \cap L^{2+\epsilon_2}_\xi} \|\hat{V}(\xi,s)\|_{L^2_\xi} \|\psi_1(x+2s\xi,s)\|_{L_\xi^{\frac{1+\epsilon_1}{\epsilon}} + L^\infty_\xi} \\
\text{(Lemma 6.6.4)} &\lesssim \int_0^\infty ds \frac{1}{s^{\frac{3}{4}+\frac{3\epsilon}{4}}} \|\psi(s)\|^3_{L^6_x} C(\|\psi_0(x)\|_{H^1_x}) \frac{1}{s^{3\epsilon_1/(1+\epsilon_1)}} \\
\text{(Choosing } \epsilon, \epsilon_1 \text{ sufficiently small)} &\lesssim \int_0^\infty ds \frac{1}{s^{7/8}} C(\|\psi_0(x)\|_{H^1_x}) \\
&\lesssim C(\|\psi_0(x)\|_{H^1_x}),
\end{aligned} \tag{6.471}$$

where

$$\frac{1}{2+\epsilon_2} + \frac{\epsilon_1}{1+\epsilon_1} = \frac{1}{2}, \tag{6.472}$$

$\epsilon_1 \in (0, \frac{1}{4})$ and we also use that

$$\frac{1}{s^{3(\frac{1}{2}+\frac{\epsilon}{2})\frac{1}{2+\epsilon_2}}} \le \frac{1}{s^{\frac{9}{20}+\frac{9\epsilon}{20}}} \tag{6.473}$$

since

$$\frac{1}{2+\epsilon_2} = \frac{1}{2} - \frac{\epsilon_1}{1+\epsilon_1} > \frac{1}{2} - \frac{1/4}{5/4} = \frac{3}{10}. \tag{6.474}$$

□

According to (6.440), (6.442), (6.455) and Lemma 6.6.5, we get

$$\|(\Omega_- - 1)\psi_0(x)\|_{L^\infty_x} \lesssim C(\|\psi_0(x)\|_{H^1_x \cap L^p_x}). \tag{6.475}$$

The L^∞_x boundedness for $e^{itH_0}U(t,0) - 1$ with $t \in [-\infty, \infty)$ follows in the same argument. Since for $t \in \mathbb{R}$, $e^{itH_0}U(t,0) - 1 : H^1_x \to L^2_x$, is bounded, by using interpolation inequality, we get

$$\|(e^{itH_0}U(t,0) - 1)\psi_0(x)\|_{L^p_x} \le C(\|\psi_0\|_{H^1_x}) \tag{6.476}$$

for $p \in [2, \infty], t \in \mathbb{R}, \psi_0(x) \in H^1_x \cap L^{p'}_x$.

Now we consider the L^p estimate of Ω_- for $p > 6$ with an additional assumption, $\psi_0 \in L^p_x \cap L^1_x$. Due to Lemma 6.6.4, we have $\psi(t) \in L^\infty_x + FL^{1+\epsilon}_x$ for $|t| \ge 1$, any $\epsilon > 0$ if $\psi_0 \in L^1_x$. Then

$$\begin{aligned}
\Big\|\int_1^\infty ds e^{isH_0} N(|\psi(s)|)\psi(s)\Big\|_{L^p_x} &\lesssim \int_1^\infty ds\, s^{-3(\frac{1}{2}-\frac{1}{p})}\|N(|\psi(s)|)\|_{L^2_x}\|\psi(s)\|_{L^q_x} \\
&\lesssim C(p, \|\psi_0(x)\|_{H^1_x})\int_1^\infty ds\, s^{-3(\frac{1}{2}-\frac{1}{p})} \\
(\text{ use } p > 6 \text{ and } \psi(s) \in L^q \text{ due to interpolation}) &\lesssim C(p, \|\psi_0(x)\|_{H^1_x})
\end{aligned} \tag{6.477}$$

where q satisfies

$$\frac{1}{q} + \frac{1}{2} = \frac{1}{p'}. \tag{6.478}$$

Thus,

$$\|(\Omega_+ - e^{iH_0}U(1,0))\psi_0(x)\|_{L^p_x} \lesssim_p C(\|\psi_0\|_{H^1_x}) \tag{6.479}$$

which implies that for $p \in (6, \infty]$(Recall that this time we have $\psi_0 \in L^p_x$),

$$\|\Omega_+\psi_0(x)\|_{L^p_x} \lesssim_p C(\|\psi_0\|_{H^1_x}). \tag{6.480}$$

Similarly, we have the same result for Ω_- by using a similar argument and finish the proof of Theorem 6.1.6. □

Proof of Theorem 6.1.7. It follows directly from Lemma 6.6.4 since in Lemma 6.6.4, we have $(e^{itH_0}U(t,0) - 1)\psi_0 \in L^\infty + FL^{1+\epsilon}_x$ for any $\epsilon \in (0,1)$. □

We also have a similar result for $U(t,0) - e^{-itH_0}$:

Lemma 6.6.6. *If $\psi_0 \in H^1_x$ and N satisfies* (6.57), *then for any $\epsilon \in (0,1)$,*

$$\sup_{|t|\geq 1}\|\psi(t)-e^{-itH_0}\psi_0\|_{L^\infty_x+L^{1+\epsilon}_x}\leq C(\sup_{t\in\mathbb{R}}\|\psi(t)\|_{H^1_x},\epsilon). \tag{6.481}$$

Furthermore, if $\psi_0 \in L^p_x \cap H^1_x$ for some $p\in[1,\frac{6}{5})$ and

$$\sup_{t\in\mathbb{R}}\|\psi(t)\|_{H^1_x}\lesssim 1, \tag{6.482}$$

then

$$\sup_{|t|\geq 1}\|\psi(t)-e^{-itH_0}\psi_0\|_{L^\infty_x}\leq C(\sup_{t\in\mathbb{R}}\|\psi(t)\|_{H^1_x},\|\psi_0\|_{L^p_x},p'-6). \tag{6.483}$$

Proof of Lemma 6.6.6. For (6.481), it follows by using a similar argument as what we did in Lemma 6.6.4. For (6.483), by using Duhamel's formula, write $\psi(t)$

$$\psi(t)=e^{-itH_0}\psi_0+(-i)\int_0^t ds\, e^{-i(t-s)H_0}N(|\psi(s)|)\psi(s). \tag{6.484}$$

For L^∞_x estimate, it is sufficient to estimate

$$\psi_2(t):=(-i)\int_{t-\frac{1}{2}}^t ds\, e^{-i(t-s)H_0}N(|\psi(s)|)\psi(s). \tag{6.485}$$

Since $\psi_0\in L^p_x$ implies $e^{-itH_0}\psi_0\in L^{p'}_x$ for $p'>6, t\geq\frac{1}{2}$, by using a similar argument as what we did in the proof of Theorem 6.1.6 and due to Remark 40, we get (6.483).

□

Typical examples and remarks on the advanced cancelation lemma

Example 6.6.3 (L^∞ boundedness)**.** *When*

$$N(|\psi(t)|)=|\psi(t)|^3, \tag{6.486}$$

$$i\partial_t\psi(t)=H_0\psi(t)+N(|\psi(t)|)\psi(t),\quad \psi(0)=\psi_0\in L^1_x\cap H^1_x, \tag{6.487}$$

satisfies (6.57)*. Then*

$$\sup_{|t|\geq 1}\|\psi(t)\|_{L^\infty_x}\lesssim 1. \tag{6.488}$$

Proof. When

$$N(|\psi(t)|)=|\psi(t)|^3, \tag{6.489}$$

it is the defocusing case and if $\psi_0 \in H^1_x$, we have a global solution $\psi(t)$ with a uniform H^1_x norm. We also have

$$\|N(|\psi(t)|)\|_{L^2_x} = \||\psi(t)|^3\|_{L^2_x} = \|\psi(t)\|^3_{L^6_x} \lesssim \|\psi(t)\|^3_{H^1_x} \tag{6.490}$$

and

$$\|N'(|\psi(t)|)\|_{L^3_x} = 3\||\psi(t)|^2\|_{L^3_x} = 3\|\psi(t)\|^2_{L^6_x} \lesssim \|\psi(t)\|^2_{H^1_x}. \tag{6.491}$$

So (6.57) is satisfied and we have (6.488). □

Example 6.6.4 (L^∞ boundedness of mixed power nonlinearity)**.** *When*

$$N(|\psi(t)|) = -|\psi(t)|^2 + |\psi(t)|^3, \tag{6.492}$$

if $\psi(t) \in H^1_x$*, uniformly in* t*, then*

$$i\partial_t\psi(t) = H_0\psi(t) + N(|\psi(t)|)\psi(t), \quad \psi(0) = \psi_0 \in L^1_x \cap H^1_x, \tag{6.493}$$

satisfies (6.57)*. Then*

$$\sup_{|t|\geq 1} \|\psi(t)\|_{L^\infty_x} \lesssim 1. \tag{6.494}$$

Proof. When

$$N(|\psi(t)|) = -|\psi(t)|^2 + |\psi(t)|^3, \tag{6.495}$$

according to Lemma 6.1.2, we have

$$\|\psi(t)\|_{H^1_x} \lesssim \|\psi_0\|_{L^2_x} + \sup_{s\in[t-1,t+1]} \|\nabla\psi(t)\|_{L^2_x} \lesssim C(\|\psi_0\|_{H^1_x}). \tag{6.496}$$

We also have

$$\|N(|\psi(t)|)\|_{L^2_x} \lesssim \||\psi(t)|^2\|_{L^2_x} + \||\psi(t)|^3\|_{L^2_x} = \|\psi(t)\|^3_{L^6_x} + \|\psi(t)\|^2_{L^4_x} \lesssim C(\|\psi(t)\|_{H^1_x}) \tag{6.497}$$

and

$$\|N'(|\psi(t)|)\|_{L^3_x} \lesssim 2\||\psi(t)|\|_{L^3_x} + 3\||\psi(t)|^2\|_{L^3_x} \lesssim C(\|\psi(t)\|_{H^1_x}). \tag{6.498}$$

So (6.57) is satisfied and we have (6.494). □

6.7 Intertwining property

In time-independent case, there exists an intertwining operator between $f(H)$ and $f(H_0)$ with f measurable

$$f(H)P_c = \Omega_+ f(H_0)\Omega_+ \tag{6.499}$$

where P_c denotes the projection on the continuous spectrum of H, and this projection comes from the fact that Ω_+ is unitary from $L^2 \to Ran(\Omega_+)$, with the range of Ω_+ equal to the continuous spectrum of H.

When it comes to time-dependent case, (6.499) fails in most situations; $U(t+s,t)$ will not generally have a nice limit as $t \to \infty$, see [68]. In this section, we will introduce a new type of intertwining property based on new wave operators Ω_T (For Ω_T, see (6.315).)

$$U(T,0) = \Omega_T e^{-iTH_0}\Omega_+, \text{ on } \mathcal{R}(\Omega_+) \tag{6.500}$$

where $U(t,0)$ denotes the solution operator of a Schrödinger equation with a Hamiltonian $H(t)$, $\mathcal{R}(\Omega)$ is the range of Ω_+, a subspace equipped with L^p norm, $1 \le p \le 2$. It follows from

$$U(T,0) = U(T,T+s)U(T+s,0) = U(T,T+s)e^{-isH_0}e^{-iTH_0}e^{i(s+T)H_0}U(T+s,0), \text{ on } L^2 \tag{6.501}$$

and

$$\Omega_T = s\text{-}\lim_{s\to\infty} U(T,T+s)e^{-isH_0}, \text{ on } L^2, \tag{6.502}$$

$$\Omega_+ = s\text{-}\lim_{s\to\infty} U(0,s)e^{-isH_0}, \text{ on } L^2. \tag{6.503}$$

Based on Corollary 6.4.3 and Theorem 6.1.1, we have

$$\|\Omega_T e^{-iTH_0}\beta(|P| > M)\Omega_+\|_{L^p \to L^{p'}} \lesssim \frac{1}{T^{\frac{3}{2}(2-p)}}, \text{ in dimension 3} \tag{6.504}$$

with $\frac{1}{p}+\frac{1}{p'} = 1, 1 \le p \le 2$. The decay estimates follow if we make a low-frequency assumption:

Lemma 6.7.1. *If*

$$\|\Omega_T e^{-iTH_0}\beta(|P| \le M)\Omega_+\|_{L^p \to L^{p'}} \lesssim \frac{1}{T^{\frac{3}{2}(2-p)}} \tag{6.505}$$

for $1 \le p \le 2$, some sufficiently large M and $V(x,t)$ satisfies the condition in Theorem 6.1.1, then $U(T,0)$ satisfies decay estimates on $R(\Omega_+) \cap L^p_x$ for $T > 0$.

Proof. Based on Corollary 6.4.3 and Theorem 6.1.1, we have (6.504). Then combining (6.504) with assumption (6.505), we get

$$\|\Omega_T e^{-iTH_0}\Omega_+\|_{L^p \to L^{p'}} \lesssim \frac{1}{T^{\frac{3}{2}(2-p)}}. \tag{6.506}$$

Based on (6.500), we get

$$\sup_{T \ge 0} T^{3/2}\|U(T,0)\|_{R(\Omega_+)\cap L^1_x \to L^\infty_x} \lesssim 1. \tag{6.507}$$

Later by interpolation, we get L^p decay estimates on $R(\Omega_+) \cap L^p_x$.

□

More information about intertwining property will be discussed in our following paper.

References

[1] Werner O. Amrein, Anne Boutet de Monvel, and Vladimir Georgescu. *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*, volume 135 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1996.

[2] Galtbayar Artbazar and Kenji Yajima. The L^p-continuity of wave operators for one dimensional Schrödinger operators. *J. Math. Sci. Univ. Tokyo*, 7(2):221–240, 2000.

[3] Marius Beceanu. New estimates for a time-dependent Schrödinger equation. *Duke Math. J.*, 159(3):417–477, 2011.

[4] Marius Beceanu, Qingquan Deng, Avy Soffer, and Yifei Wu. Large global solutions for nonlinear schr*n*" odinger equations iii, energy-supercritical cases. *arXiv preprint arXiv:1901.07709*, 2019.

[5] Marius Beceanu and Avy Soffer. The schrödinger equation with potential in random motion. *arXiv preprint arXiv:1111.4584*, 2011.

[6] Marius Beceanu and Avy Soffer. The Schrödinger equation with a potential in rough motion. *Comm. Partial Di erential Equations*, 37(6):969–1000, 2012.

[7] Marius Beceanu and Avy Soffer. A semilinear schroedinger equation with random potential. *arXiv preprint arXiv:1903.03451*, 2019.

[8] Kaihua Cai. Fine properties of charge transfer models. *arXiv preprint math-ph/0311048*, 2003.

[9] Gong Chen. Strichartz estimates for wave equations with charge transfer Hamiltonians. *Mem. Amer. Math. Soc.*, 273(1339):v + 84, 2021.

[10] Gong Chen, Jiaqi Liu, and Bingying Lu. Long-time asymptotics and stability for the sine-gordon equation. *arXiv preprint arXiv:2009.04260*, 2020.

[11] Michael Christ, James Colliander, and Terrence Tao. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. *Amer. J. Math.*, 125(6):1235–1293, 2003.

[12] Ovidiu Costin, Joel L Lebowitz, and Alexander Rokhlenko. Ionization of a model atom: exact results and connection with experiment. *arXiv preprint physics/9905038*, 1999.

[13] R. Côte. On the soliton resolution for equivariant wave maps to the sphere. *Comm. Pure Appl. Math.*, 68(11):1946–2004, 2015.

[14] E. B. Davies. Time-dependent scattering theory. *Math. Ann.*, 210:149–162, 1974.

[15] E. B. Davies. On Enss' approach to scattering theory. *Duke Math. J.*, 47(1):171–185, 1980.

[16] Qingquan Deng, Avy Soffer, and Xiaohua Yao. Endpoint Strichartz estimates for charge transfer Hamiltonians. *Indiana Univ. Math. J.*, 67(6):2487–2522, 2018.

[17] Jan Dereziński. Asymptotic completeness of long-range N-body quantum systems. *Ann. of Math. (2)*, 138(2):427–476, 1993.

[18] Jan Dereziński and Christian Gérard. *Scattering theory of classical and quantum N-particle systems*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.

[19] Thomas Duyckaerts, Carlos Kenig, Yvan Martel, and Frank Merle. Soliton resolution for critical co-rotational wave maps and radial cubic wave equation. *Comm. Math. Phys.*, 391(2):779–871, 2022.

[20] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. Profiles of bounded radial solutions of the focusing, energy-critical wave equation. *Geom. Funct. Anal.*, 22(3):639–698, 2012.

[21] Volker Enss. Asymptotic completeness for quantum mechanical potential scattering. I. Short range potentials. *Comm. Math. Phys.*, 61(3):285–291, 1978.

[22] Richard Froese and Ira Herbst. A new proof of the Mourre estimate. *Duke Math. J.*, 49(4):1075–1085, 1982.

[23] Jesse Gell-Redman, Sean Gomes, and Andrew Hassell. Propagation of singularities and fredholm analysis of the time-dependent schr*n*" odinger equation. *arXiv preprint arXiv:2201.03140*, 2022.

[24] Christian Gérard. A proof of the abstract limiting absorption principle by energy estimates. *J. Funct. Anal.*, 254(11):2707–2724, 2008.

[25] Pierre Germain and Fabio Pusateri. Quadratic Klein-Gordon equations with a potential in one dimension. *Forum Math. Pi*, 10:Paper No. e17, 172, 2022.

[26] Pierre Germain, Fabio Pusateri, and Frédéric Rousset. The nonlinear Schrödinger equation with a potential. *Ann. Inst. H. Poincare C Anal. Non Lineaire*, 35(6):1477–1530, 2018.

[27] Pierre Germain, Fabio Pusateri, and Katherine Zhiyuan Zhang. On 1d Quadratic Klein-Gordon Equations with a Potential and Symmetries. *Arch. Ration. Mech. Anal.*, 247(2):17, 2023.

[28] J. Ginibre and G. Velo. Smoothing properties and retarded estimates for some dispersive evolution equations. *Comm. Math. Phys.*, 144(1):163–188, 1992.

[29] M. Goldberg. Dispersive bounds for the three-dimensional Schrödinger equation with almost critical potentials. *Geom. Funct. Anal.*, 16(3):517–536, 2006.

[30] M. Goldberg and W. Schlag. Dispersive estimates for Schrödinger operators in dimensions one and three. *Comm. Math. Phys.*, 251(1):157–178, 2004.

[31] Michael Goldberg. Strichartz estimates for the Schrödinger equation with time-periodic $L^{n/2}$ potentials. *J. Funct. Anal.*, 256(3):718–746, 2009.

[32] Gian Michele Graf. Asymptotic completeness for N-body short-range quantum systems: a new proof. *Comm. Math. Phys.*, 132(1):73–101, 1990.

[33] Gian Michele Graf. Phase space analysis of the charge transfer model. *Helv. Phys. Acta*, 63(1-2):107–138, 1990.

[34] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[35] B. Helffer and Sjöstrand J. *Schrödinger operators*. Springer, 1989.

[36] Lars Hörmander. *Lectures on nonlinear hyperbolic di erential equations*, volume 26 of *Mathematiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1997.

[37] James S. Howland. Stationary scattering theory for time-dependent Hamiltonians. *Math. Ann.*, 207:315–335, 1974.

[38] James S. Howland. Born series and scattering by time-dependent potentials. *Rocky Mountain J. Math.*, 10(3):521–531, 1980.

[39] James S Howland. Two problems with time-dependent hamiltonians. *Mathematical methods and applications of scattering theory*, 130:163–168, 1980.

[40] W. Hunziker and I. M. Sigal. The quantum N-body problem. *J. Math. Phys.*, 41(6):3448–3510, 2000.

[41] W. Hunziker and I. M. Sigal. Time-dependent scattering theory of N-body quantum systems. *Rev. Math. Phys.*, 12(8):1033–1084, 2000.

[42] W. Hunziker, I. M. Sigal, and A. Soffer. Minimal escape velocities. *Comm. Partial Di erential Equations*, 24(11-12):2279–2295, 1999.

[43] Mihaela Ifrim and Daniel Tataru. Global bounds for the cubic nonlinear Schrödinger equation (NLS) in one space dimension. *Nonlinearity*, 28(8):2661–2675, 2015.

[44] Robert Jenkins, Jiaqi Liu, Peter Perry, and Catherine Sulem. Soliton resolution for the derivative nonlinear Schrödinger equation. *Comm. Math. Phys.*, 363(3):1003–1049, 2018.

[45] Arne Jensen. Spectral properties of Schrödinger operators and time-decay of the wave functions results in $L^2(\mathbf{R}^m)$, m 5. *Duke Math. J.*, 47(1):57–80, 1980.

[46] Arne Jensen. Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in $L^2(\mathbf{R}^4)$. *J. Math. Anal. Appl.*, 101(2):397–422, 1984.

[47] Hao Jia, Baoping Liu, and Guixiang Xu. Long time dynamics of defocusing energy critical $3+1$ dimensional wave equation with potential in the radial case. *Comm. Math. Phys.*, 339(2):353–384, 2015.

[48] J.-L. Journé, A. Soffer, and C. D. Sogge. Decay estimates for Schrödinger operators. *Comm. Pure Appl. Math.*, 44(5):573–604, 1991.

[49] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[50] Alexander Komech and Elena Kopylova. *Dispersion decay and scattering theory*. John Wiley & Sons, Inc., Hoboken, NJ, 2012.

[51] MichałKowalczyk, Yvan Martel, Claudio Muñoz, and Hanne Van Den Bosch. A sufficient condition for asymptotic stability of kinks in general $(1+1)$-scalar field models. *Ann. PDE*, 7(1):Paper No. 10, 98, 2021.

[52] Hans Lindblad, Jonas Lührmann, and Avy Soffer. Decay and asymptotics for the one-dimensional Klein-Gordon equation with variable coefficient cubic nonlinearities. *SIAM J. Math. Anal.*, 52(6):6379–6411, 2020.

[53] Hans Lindblad, Jonas Lührmann, and Avy Soffer. Asymptotics for 1D Klein-Gordon equations with variable coefficient quadratic nonlinearities. *Arch. Ration. Mech. Anal.*, 241(3):1459–1527, 2021.

[54] Hans Lindblad and Avy Soffer. A remark on asymptotic completeness for the critical nonlinear Klein-Gordon equation. *Lett. Math. Phys.*, 73(3):249–258, 2005.

[55] Hans Lindblad and Avy Soffer. Scattering and small data completeness for the critical nonlinear Schrödinger equation. *Nonlinearity*, 19(2):345–353, 2006.

[56] Hans Lindblad and Avy Soffer. Scattering for the Klein-Gordon equation with quadratic and variable coefficient cubic nonlinearities. *Trans. Amer. Math. Soc.*, 367(12):8861–8909, 2015.

[57] B Liu and A Soffer. The large time asymptotics of nonlinear multichannel schrödinger equations". *Submitted*, 1(2):3, 2021.

[58] Baoping Liu and Avy Soffer. A general scattering theory for nonlinear and non-autonomous schrödinger type equations-a brief description. *arXiv preprint arXiv:2012.14382*, 2020.

[59] Jonas Luhrmann and Wilhelm Schlag. Asymptotic stability of the sine-gordon kink under odd perturbations. *arXiv preprint arXiv:2106.09605*, 2021.

[60] Frank Merle, Pierre Raphaël, and Jeremie Szeftel. On collapsing ring blow-up solutions to the mass supercritical nonlinear Schrödinger equation. *Duke Math. J.*, 163(2):369–431, 2014.

[61] E. Mourre. Absence of singular continuous spectrum for certain selfadjoint operators. *Comm. Math. Phys.*, 78(3):391–408, 1980/81.

[62] Eric Mourre. Link between the geometrical and the spectral transformation approaches in scattering theory. *Comm. Math. Phys.*, 68(1):91–94, 1979.

[63] Galina Perelman. Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations. *Comm. Partial Di erential Equations*, 29(7-8):1051–1095, 2004.

[64] P. Perry, I. M. Sigal, and B. Simon. Spectral analysis of N-body Schrödinger operators. *Ann. of Math. (2)*, 114(3):519–567, 1981.

[65] Jeffrey Rauch. Local decay of scattering solutions to Schrödinger's equation. *Comm. Math. Phys.*, 61(2):149–168, 1978.

[66] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972.

[67] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[68] Michael Reed and Barry Simon. Scattering theory. methods of modern mathematical physics, vol. _3. *Scattering theory. Methods of modern mathematical physics*, 1979.

[69] I Rodnianski, W Schlag, and A Soffer. Asymptotic stability of n-soliton states of nls. *arXiv preprint math/0309114*, 2003.

[70] I. Rodnianski and T. Tao. Longtime decay estimates for the Schrödinger equation on manifolds. In *Mathematical aspects of nonlinear dispersive equations*, volume 163 of *Ann. of Math. Stud.*, pages 223–253. Princeton Univ. Press, Princeton, NJ, 2007.

[71] Igor Rodnianski and Wilhelm Schlag. Time decay for solutions of Schrödinger equations with rough and time-dependent potentials. *Invent. Math.*, 155(3):451–513, 2004.

[72] Igor Rodnianski, Wilhelm Schlag, and Avraham Soffer. Dispersive analysis of charge transfer models. *Comm. Pure Appl. Math.*, 58(2):149–216, 2005.

[73] W Schlag. Intertwining wave operators, fourier restriction, and wiener theorems. *arXiv preprint arXiv:1802.01982*, 2018.

[74] I. M. Sigal. On long-range scattering. *Duke Math. J.*, 60(2):473–496, 1990.

[75] I. M. Sigal and A. Soffer. The N-particle scattering problem: asymptotic completeness for short-range systems. *Ann. of Math. (2)*, 126(1):35–108, 1987.

[76] I. M. Sigal and A. Soffer. Long-range many-body scattering. Asymptotic clustering for Coulomb-type potentials. *Invent. Math.*, 99(1):115–143, 1990.

[77] I. M. Sigal and A. Soffer. Asymptotic completeness for N 4 particle systems with the Coulomb-type interactions. *Duke Math. J.*, 71(1):243–298, 1993.

[78] I. M. Sigal and A. Soffer. Asymptotic completeness of N-particle long-range scattering. *J. Amer. Math. Soc.*, 7(2):307–334, 1994.

[79] Israel Michael Sigal and Avy Soffer. Local decay and propagation estimates for time-dependent and time-independent hamiltonians. *Preprint Princeton University*, 2(11):1, 1988.

[80] Erik Skibsted. Propagation estimates for N-body Schroedinger operators. *Comm. Math. Phys.*, 142(1):67–98, 1991.

[81] A. Soffer and M. I. Weinstein. Multichannel nonlinear scattering for nonintegrable equations. *Comm. Math. Phys.*, 133(1):119–146, 1990.

[82] A. Soffer and M. I. Weinstein. Nonautonomous Hamiltonians. *J. Statist. Phys.*, 93(1-2):359–391, 1998.

[83] A. Soffer and M. I. Weinstein. Ionization and scattering for short-lived potentials. *Lett. Math. Phys.*, 48(4):339–352, 1999.

[84] A. Soffer and M. I. Weinstein. Selection of the ground state for nonlinear Schrödinger equations. *Rev. Math. Phys.*, 16(8):977–1071, 2004.

[85] Avy Soffer. Soliton dynamics and scattering. In *International Congress of Mathematicians. Vol. III*, pages 459–471. Eur. Math. Soc., Zürich, 2006.

[86] Avy Soffer. Monotonic local decay estimates. *arXiv preprint arXiv:1110.6549*, 2011.

[87] Avy Soffer and Xiaoxu Wu. l^p boundedness of the scattering wave operators of schr*n*" odinger dynamics–part 2. *arXiv preprint arXiv:2202.03307*, 2022.

[88] Avy Soffer and Xiaoxu Wu. l^p boundedness of the scattering wave operators of schr*n*" odinger dynamics–part 2. *arXiv preprint arXiv:2202.03307*, 2022.

[89] Avy Soffer and Xiaoxu Wu. On the existence of self-similar solutions for some nonlinear schr*n*" odinger equations. *arXiv preprint arXiv:2205.14765*, 2022.

[90] Avy Soffer and Xiaoxu Wu. On the large time asymptotics of klein-gordon type equations with general data-i. *arXiv preprint arXiv:2204.11261*, 2022.

[91] Avy Soffer and Xiaoxu Wu. On the large time asymptotics of schr*n*" odinger type equations with general data. *arXiv preprint arXiv:2203.00724*, 2022.

[92] Gavin Stewart. Long time decay and asymptotics for the complex mkdv equation. *arXiv preprint arXiv:2111.00630*, 2021.

[93] Robert S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

[94] Terence Tao. On the asymptotic behavior of large radial data for a focusing non-linear Schrödinger equation. *Dyn. Partial Di er. Equ.*, 1(1):1–48, 2004.

[95] Terence Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.

[96] Terence Tao. A (concentration-)compact attractor for high-dimensional nonlinear Schrödinger equations. *Dyn. Partial Di er. Equ.*, 4(1):1–53, 2007.

[97] Terence Tao. A global compact attractor for high-dimensional defocusing nonlinear Schrödinger equations with potential. *Dyn. Partial Di er. Equ.*, 5(2):101–116, 2008.

[98] Terence Tao, Monica Visan, and Xiaoyi Zhang. The nonlinear Schrödinger equation with combined power-type nonlinearities. *Comm. Partial Di erential Equations*, 32(7-9):1281–1343, 2007.

[99] András Vasy. Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov). *Invent. Math.*, 194(2):381–513, 2013.

[100] Ricardo Weder. L^p-$L^{\underline{p}}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential. *J. Funct. Anal.*, 170(1):37–68, 2000.

[101] Ulrich Wüller. Geometric methods in scattering theory of the charge transfer model. *Duke Math. J.*, 62(2):273–313, 1991.

[102] K. Yajima. The L^p boundedness of wave operators for Schrödinger operators with threshold singularities. I. The odd dimensional case. *J. Math. Sci. Univ. Tokyo*, 13(1):43–93, 2006.

[103] Kenji Yajima. Scattering theory for Schrödinger equations with potentials periodic in time. *J. Math. Soc. Japan*, 29(4):729–743, 1977.

[104] Kenji Yajima. A multichannel scattering theory for some time dependent Hamiltonians, charge transfer problem. *Comm. Math. Phys.*, 75(2):153–178, 1980.

[105] Kenji Yajima. Existence of solutions for Schrödinger evolution equations. *Comm. Math. Phys.*, 110(3):415–426, 1987.

[106] Kenji Yajima. The $W^{k,p}$-continuity of wave operators for Schrödinger operators. *J. Math. Soc. Japan*, 47(3):551–581, 1995.

[107] Kenji Yajima. The $W^{k,p}$-continuity of wave operators for Schrödinger operators. III. Even-dimensional cases m 4. *J. Math. Sci. Univ. Tokyo*, 2(2):311–346, 1995.

[108] Kenji Yajima. L^p-boundedness of wave operators for two-dimensional Schrödinger operators. *Comm. Math. Phys.*, 208(1):125–152, 1999.